Nordhaus–Gaddum-type theorems for decompositions into many parts

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Abstract

A *k-decomposition* $(G_1, \ldots, G_k)$ of a graph G is a partition of its edge set to form k spanning subgraphs $G_1, \ldots, G_k$. The classical theorem of Nordhaus and Gaddum bounds $\chi(G_1)+\chi(G_2)$ and $\chi(G_1)\chi(G_2)$ over all 2-decompositions of K_n. For a graph parameter p, let $p(k;G)$ denote the maximum of $\sum_{i=1}^{k} p(G_i)$ over all k-decompositions of the graph G.

The clique number ω, chromatic number χ, list chromatic number χ_ℓ, and Szekeres–Wilf number σ satisfy $\omega(2;K_n) = \chi(2;K_n) = \chi_\ell(2;K_n) = \sigma(2;K_n) = n+1$. We obtain lower and upper bounds for $\omega(k;K_n)$, $\chi(k;K_n)$, $\chi_\ell(k;K_n)$, and $\sigma(k;K_n)$. The last three behave differently for large k. We also obtain lower and upper bounds for the maximum of $\chi(k;G)$ over all graphs embedded on a given surface.

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1 Introduction

A *k-decomposition* of a (hyper)graph G is a decomposition of G into k spanning sub(hyper)graphs $G_1, \dots, G_k$. That is, each G_i has the same vertices as G, and every edge of G belongs to exactly one of $G_1, \dots, G_k$. Such decompositions can be interpreted as unrestricted k-edge-colorings of G.

For a parameter p, a positive integer k, and a (hyper)graph G, let

$$p(k;G) = \max\{\textstyle\sum_{i=1}^{k} p(G_i)\colon\ (G_1, \dots, G_k) \text{ is a } k\text{-decomposition of } G\}.$$

Also, a *p-optimal k-decomposition* of G is a k-decomposition $(G_1, \dots, G_k)$ such that $p(k;G) = \sum_{i=1}^{k} p(G_i)$.

The parameters that interest us are the clique number ω, the chromatic number χ, the list-chromatic number χ_ℓ, and the Szekeres-Wilf number σ, where σ is defined by $\sigma(G) = 1 + \max_{H \subseteq G} \delta(H)$. Every graph G satisfies $\omega(G) \le \chi(G) \le \chi_\ell(G) \le \sigma(G)$. Therefore, for every k and G,

$$\omega(k;G) \le \chi(k;G) \le \chi_\ell(k;G) \le \sigma(k;G). \tag{1}$$

Clearly, (G_1, G_2) is a 2-decomposition of the complete graph K_n if and only if G_1 is the complement of G_2 and has n vertices. Thus the Nordhaus–Gaddum Theorem can be stated as follows.

Theorem 1 (Nordhaus–Gaddum [15]) *Let n be a positive integer. If (G_1, G_2) is a 2-decomposition of K_n, then the following statements hold:*

(a) $\lceil 2\sqrt{n}\,\rceil \le \chi(G_1) + \chi(G_2) \le n + 1$;

(b) $n \le \chi(G_1) \cdot \chi(G_2) \le \lfloor \left(\frac{n+1}{2}\right)^2 \rfloor$.

The proof of Theorem 1 implies that $\omega(2;K_n) = \chi(2;K_n) = \chi_\ell(2;K_n) = \sigma(2;K_n) = n+1$, achieving equalities in (1) (see also Section 5). Finck [9] described the χ-optimal 2-decompositions of K_n.

Plesník [16] studied $\chi(k;K_n)$ for $k > 2$ and proved that $\chi(k;K_n) \le n + 2^{\binom{k+1}{2}}$ for every n. He also proved (see Lemma 3 in [16]) that

$$n + \binom{k}{2} \le \omega(k;K_n) \le \chi(k;K_n) \quad \text{if} \quad n \ge \binom{k}{2} \tag{2}$$

and conjectured (see also Bosák [3]) that $\chi(k;K_n) = n + \binom{k}{2}$. Our first result is

Theorem 2 *If k and n are positive integers, then $\omega(k;K_n) \leq n + \binom{k}{2}$. If $n \geq \binom{k}{2}$, then $\omega(k;K_n) = n + \binom{k}{2}$.*

Watkinson [23] improved Plesník's upper bound to $\chi(k;K_n) \leq n + \frac{k!}{2}$. It follows that $\omega(3;K_n) = \chi(3;K_n) = n+3$ for every $n \geq 3$. Our second result improves Watkinson's bound for large k.

Theorem 3 *If k and n are positive integers, then $\chi(k;K_n) \leq n + 7^k$.*

The list chromatic number behaves somewhat differently for large k.

Theorem 4 *There exists a positive constant c such that, if $k = \binom{\ell+1}{2}$ and $n = \ell m$ where ℓ and m are integers greater than 1, then*

$$\chi_\ell(k;K_n) \geq n + ck\ln(\frac{n}{\sqrt{k}}).$$

On the other hand, for all positive integers k and n,

$$\chi_\ell(k;K_n) \leq n + 3k!\sqrt{1+8n\ln n}.$$

Thus, the leading behavior of $\chi_\ell(k,K_n)$ for fixed k as $n \to \infty$ is still n, but there is an additive term growing with n. The situation with $\sigma(k;K_n)$ is different.

Theorem 5 *If $k = p^2+p+1$ for some prime power p, and $n \equiv 0 \bmod k$, then*

$$\sigma(k;K_n) \geq (\sqrt{k}-1)n + k,$$

On the other hand, for all positive integers k and n,

$$\sigma(k;K_n) \leq \sqrt{k}n + k.$$

Furthermore, we determine $\sigma(k;K_n)$ exactly for $k \leq 4$.

We consider also decompositions of the complete r-uniform hypergraph K_n^r. Since $\chi(K_n^r) = \lceil \frac{n}{r-1} \rceil$, we have $\chi(k;K_n^r) \geq \lceil \frac{n}{r-1} \rceil + k - 1$. Although it is easy to improve the bound for large n, we will prove that up to a summand $c_{k,r}$ independent of n, this is the correct value of $\chi(k;K_n^r)$.

Theorem 6 *If k and r are positive integers with $r \geq 2$, then there exists an integer $c_{k,r}$ such that, for every positive integer n,*

$$\chi(k;K_n^r) \leq \frac{n}{r-1} + c_{k,r}.$$

Finally, we consider decompositions of graphs embedded on a given surface Σ. For a graph parameter p and positive integer k, let $p(k;\Sigma)$ denote the maximum of $p(k;G)$ over all graphs G embeddable on Σ. Let g be the Euler genus of Σ. We show that for fixed k and large g, the values of $\omega(k;\Sigma)$, $\chi(k;\Sigma)$, $\chi_\ell(k;\Sigma)$, and $\sigma(k;\Sigma)$ are asymptotically equal, unlike $\omega(k;K_n)$ and $\sigma(k;K_n)$.

Along with the maximum sum, we can also study the maximum product of the values of a parameter p over a decomposition. Let

$$\widetilde{p}(k;G) = \max\{\textstyle\prod_{i=1}^k p(G_i)\colon\ (G_1,\ldots,G_k) \text{ is a } k\text{-decomposition of } G\}.$$

Using the Cauchy-Schwarz Inequality, our bounds on $p(k;K_n)$ yield corresponding bounds on $\widetilde{p}(k;K_n)$. For fixed k,

$$\widetilde{\omega}(k;K_n) \le \widetilde{\chi}(k;K_n) \le \widetilde{\chi_\ell}(k;K_n) \le (1+o(1))(n/k)^k,$$

$$\widetilde{\chi}(k;K_n^r) \le (1+o(1))(n/k(r-1))^k, \quad \text{and}$$

$$\widetilde{\sigma}(k;K_n) \le (1+o(1))(n/\sqrt{k})^k.$$

Decomposing K_n and K_n^r into almost disjoint complete subgraphs of about the same size (pairwise sharing at most $r-1$ vertices) shows that except for the bound on $\widetilde{\sigma}$, these upper bounds are asymptotically tight for fixed k. The construction in Theorem 5 yields that $\widetilde{\sigma}(k;K_n) \ge e^{-\sqrt{k}}(n/\sqrt{k})^k$. Thus, for $\widetilde{\sigma}(k;K_n)$ we at least know the order of magnitude.

In the next section, we summarize notation and terminology. In subsequent sections we treat consecutively the clique number, the chromatic number, the list chromatic number, and the Szekeres-Wilf number of graphs. The last two sections are devoted to the chromatic number of r-uniform hypergraphs and to decompositions of graphs embedded on surfaces.

2 Preliminaries

Concepts and notation not defined in this paper are as in standard textbooks. Though the main objects of our study are graphs, we also consider the central concepts for hypergraphs.

A *hypergraph* G is a pair consisting of a finite set $V(G)$ of *vertices* and a set $E(G)$ of subsets of $V(G)$, called *edges*, each having size at least two.

A hypergraph G is *r-uniform* if $|e| = r$ for every $e \in E(G)$. A *graph* is a 2-uniform hypergraph.

Let G be a hypergraph. The *degree* $d_G(x)$ of a vertex $x \in V(G)$ is the number of edges in G that contain x. If $d_G(x) = s$ for every vertex $x \in V(G)$, then G is *s-regular*. Let $\delta(G)$ denote the minimum degree of G.

If H and G are hypergraphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subhypergraph* of G. In this case we write $H \subseteq G$.

Let G be a hypergraph. For $X \subseteq V(G)$, the subhypergraph of G *induced* by X, written $G[X]$, is defined by $V(G[X]) = X$ and $E(G[X]) = \{e \in E(G)\colon e \subseteq X\}$. Let $G - X$ denote $G[V(G) - X]$.

For an r-uniform hypergraph G, a set $X \subseteq V(G)$ is a *clique* or an *independent set* if $E(G[X])$ consists of all r-subsets of X or is empty, respectively. For an r-uniform hypergraph G, the *clique number* $\omega(G)$ is the maximum size of a clique in G. We write K_n^r for an r-uniform complete hypergraph on n vertices (the edges are all r-subsets of the vertices). Thus K_n^2 denotes simply a complete graph K_n.

For a (hyper)graph G, a *list assignment* L is a function that assigns to each vertex x of G a set $L(x)$ of colors (positive integers). An *L-coloring* of G is a function c that assigns a color to each vertex of G such that $c(x) \in L(x)$ for all $x \in V(G)$ and $|\{c(x)\colon x \in e\}| \geq 2$ for each $e \in E(G)$. If G admits an L-coloring, then G is *L-colorable*. When $L(x) = [k]$ for all $x \in V(G)$ (where $[k]$ denotes the set $\{1, \ldots, k\}$), the corresponding terms become *k-coloring* and *k-colorable*, respectively. G is *k-list-colorable* if G is L-colorable for every list assignment L satisfying $|L(x)| = k$ for all $x \in V(G)$.

The *chromatic number* of G, denoted by $\chi(G)$, is the least k such that G is k-colorable. The *list-chromatic number* of G, denoted by $\chi_\ell(G)$, is the least k such that G is k-list-colorable.

The *Szekeres-Wilf number* $\sigma(G)$ of a graph G, introduced by Szekeres and Wilf [20] in 1968, is equal to the *coloring number* introduced and studied by Erdös and Hajnal [7] in 1966. It is the smallest integer d such that in some linear ordering of $V(G)$ every vertex of G has at most $d-1$ neighbors following it. It has been observed repeatedly that $\sigma(G)$ can be easily computed by iteratively letting x_i be a vertex of minimum degree in the subgraph G_i obtained by deleting $\{x_j\colon j < i\}$. It follows that $\sigma(G) = 1 + \max_i\{\delta(G_i)\}$.

Let p be a (hyper)graph parameter. A graph G is *p-critical* if $p(H) < p(G)$ for every proper subgraph H of G. For $p \in \{\omega, \chi, \chi_\ell, \sigma\}$, every graph G contains a p-critical subgraph H satisfying $p(H) = p(G)$.

3 Clique number and chromatic number

In discussing $\omega(k;K_n)$, we noted that a lower bound of $n+\binom{k}{2}$ is proved in [16] for the case $n \ge \binom{k}{2}$. We include a simple construction for this before proving our upper bound.

Construction: *If $n \ge \binom{k}{2}$, then $\omega(k;K_n) \ge n+\binom{k}{2}$.* The construction for $n=\binom{k}{2}$ can be extended for each additional vertex by adding all the edges incident to the new vertex to one graph in the decomposition.

For $n=\binom{k}{2}$, we provide k cliques of order $k-1$ that are pairwise edge-disjoint and together use each vertex twice. The edges not covered can be added to one of these subgraphs to complete the decomposition of K_n. The sum of the orders of the cliques is $k(k-1)$, which equals $n+\binom{k}{2}$.

Let $V(K_n)=\{(i,j)\colon\ 1\le i<j\le k\}$. For $1\le r\le k$, let Q_r be the set of vertices whose names have r in one coordinate; note that $|Q_r|=k-1$. The only vertex shared by Q_i and Q_j with $i<j$ is (i,j), so the cliques are edge-disjoint. ■

Theorem 2 *If k and n are positive integers, then $\omega(k;K_n)\le n+\binom{k}{2}$. If $n\ge\binom{k}{2}$, then $\omega(k;K_n)=n+\binom{k}{2}$.*

Proof: By the construction above, it suffices to show that $\omega(k;K_n) \leq n+\binom{k}{2}$. We prove this by induction on $n+k$. For $n=1$, we have $\omega(k;K_1) = k \leq 1+\binom{k}{2}$. For $k=1$, we have $\omega(1;K_n) = n = n+\binom{1}{2}$.

For $\min\{k,n\} \geq 2$, let $(G_1,\ldots,G_k)$ be an ω-optimal k-decomposition of K_n. For $1 \leq i \leq k$, let X_i denote a maximum clique of G_i.

If $\omega(G_1) = |X_1| \leq k-1$, then since $(G_1 \cup G_2, G_3, \ldots, G_k)$ is a $(k-1)$-decomposition of K_n, it follows from the induction hypothesis that

$$\begin{aligned} \omega(k;K_n) &= \sum_{i=1}^{k} \omega(G_i) \leq (k-1) + \omega(k-1;K_n) \\ &\leq (k-1) + n + \binom{k-1}{2} = n + \binom{k}{2} \end{aligned}$$

Otherwise, since $|X_1 \cap X_j| \leq 1$ for $2 \leq j \leq k$, we can choose a vertex $x \in X_1 - \bigcup_{j=2}^{k} X_j$. Now $(G_1 - x, \ldots, G_k - x)$ is a k-decomposition of K_{n-1}. By the induction hypothesis, it follows that

$$\begin{aligned} \omega(k;K_n) &= \sum_{i=1}^{k} \omega(G_i) \leq 1 + \sum_{i=1}^{k} \omega(G_i - x) \\ &\leq 1 + \omega(k;K_{n-1}) \leq 1 + (n-1) + \binom{k}{2} = n + \binom{k}{2}. \quad \blacksquare \end{aligned}$$

To obtain our upper bound on $\chi(k;K_n)$, we need an upper bound on chromatic number in terms of Ramsey numbers. In our language, Ramsey's Theorem states that every k-decomposition of a sufficiently large r-uniform complete hypergraph has an r-uniform complete hypergraph of specified size in some factor.

Theorem 7 (Ramsey [19]) *For positive integers k, r, and $n_1,\ldots,n_k$ satisfying $k,r \geq 2$, there is a smallest integer R (written as $R_k^r(n_1,\ldots,n_k)$) such that if $(G_1,\ldots,G_k)$ is a k-decomposition of K_n^r with $n \geq R$, then for some $i \in [k]$ the subhypergraph G_i contains an r-uniform complete hypergraph with n_i vertices.*

Our bound uses the observation that proper colorings of the subgraphs induced by a partition of the vertices, with different colors, combine to form a proper coloring of the full graph. We call this *subadditivity* of χ.

Proposition 8 *For positive integers k, ℓ, and r with $r \geq 2$, let R be the Ramsey number $R_2^r(\ell, k)$. If G is an r-uniform n-vertex hypergraph containing no r-uniform complete hypergraph on ℓ vertices, then*

$$\chi(G) \leq \frac{n-R}{k} + R.$$

Proof: We use induction on n. If $n \leq R$, then

$$\chi(G) \leq n \;=\; \frac{n}{k} + n\frac{k-1}{k} \;\leq\; \frac{n}{k} + R\frac{k-1}{k} \;\leq\; \frac{n-R}{k} + R.$$

Now consider $n > R$. By the definition of the Ramsey number, G has an independent set X with $|X| \geq k$. By the induction hypothesis and subadditivity of χ,

$$\chi(G) \;\leq\; \chi(G-X)+1 \;\leq\; \frac{n-|X|-R}{k} + R + 1 \;\leq\; \frac{n-R}{k} + R. \qquad \blacksquare$$

In Theorem 3, our attention is on ordinary graphs, and we use only the case $r = 2$ of Proposition 8.

Theorem 3 *If k and n are positive integers, then $\chi(k; K_n) \leq n + 7^k$.*

Proof: The claim is trivial for $n = 1$ or $k = 1$. For $k > 1$, we use induction on n. For the induction step, consider $n \geq 2$. Let $(G_1, \ldots, G_k)$ be a k-decomposition of K_n. To show that $\sum_{i=1}^{k} \chi(G_i) \leq n + 7^k$, we distinguish two cases.

Case 1: *In some subgraph of the decomposition, there is a maximum clique whose deletion leaves a clique of size at least $2k-2$.* We may assume that this occurs in G_1. Thus G_1 has a maximum clique X such that $\omega(G_1 - X) \geq 2k-2$. Let Y be a maximum clique of $G_1 - X$, and let $Z = X \cup Y$. The subgraph $G_1[Z]$ is the complement of a bipartite graph, and each $G_i[Z]$ for $i \geq 2$ is bipartite. Hence all these subgraphs are perfect. Moreover, $\chi(G_1[Z]) = \omega(G_1[Z]) = |X|$, and $\chi(G_i[Z]) \leq 2$ for $2 \leq i \leq k$. Thus

$$\sum_{i=1}^{k} \chi(G_i[Z]) \;\leq\; |X| + 2(k-1) \;\leq\; |X| + |Y| \;=\; |Z|.$$

Furthermore, $(G_1 - Z, \ldots, G_k - Z)$ is a k-decomposition of $K_{n-|Z|}$. By the induction hypothesis, $\sum_{i=1}^k \chi(G_i - Z) \le n - |Z| + 7^k$. Using subadditivity,

$$\sum_{i=1}^k \chi(G_i) \;\le\; \sum_{i=1}^k \chi(G_i[Z]) + \sum_{i=1}^k \chi(G_i - Z) \;\le\; |Z| + n - |Z| + 7^k \;=\; n + 7^k.$$

Case 2: *For each i, if X is a maximum clique of G_i, then $\omega(G_i - X) \le 2k-3$.* Define a sequence of subsets of $V(K_n)$ as follows. Let $X_0 = \emptyset$. For $j \ge 1$, let X_j be a maximum clique among the graphs $G_1 - \bigcup_{i=0}^{j-1} X_i, \ldots, G_k - \bigcup_{i=0}^{j-1} X_i$.

Let ℓ_j be the index i such that X_j is a clique in G_i. For $i \ne \ell_j$, X_j is an independent set in G_i. For $j \ge 1$, this yields $\sum_{i=1}^k \chi(G_i[X_j]) \le |X_j| + k - 1$. Let p be the least index such that $\ell_p \in \{\ell_1, \ldots, \ell_{p-1}\}$. Let $X = X_1 \cup \cdots \cup X_{p-1}$, and let $m = |X|$. By subadditivity,

$$\sum_{i=1}^k \chi(G_i[X]) \;\le\; |X| + (p-1)(k-1) \;\le\; m + k(k-1).$$

Let $G_i' = G_i - X$ for each i. Since the second clique taken from G_p has size at most $2k-3$ and is a largest clique remaining in any subgraph when it is chosen, $\omega(G_i') \le 2k-3$ for all i. Let $R = R_2^2(k, 2k-2)$. By Proposition 8, $\chi(G_i') \le \frac{n-m-R}{k} + R$ for each i. Therefore,

$$\sum_{i=1}^k \chi(G_i') \;\le\; n - m - R + kR \;=\; n - m + (k-1)R.$$

By subadditivity,

$$\sum_{i=1}^k \chi(G_i) \;\le\; m + k(k-1) + n - m + (k-1)R \;=\; n + (k-1)(k+R).$$

It remains only to supply an upper bound on R. From the well-known bound $R_2^2(p,q) \le \binom{p+q-2}{p-1}$, we have $R = R_2^2(k, 2k-2) \le \binom{3k-4}{k-1}$. Let $t = k-1$. From Stirling's Formula for factorials, $\binom{3t-1}{t}$ is approximately $(3\pi t)^{-1/2}(\frac{27}{4})^t$. For all k, we have $(k-1)(k+R) < 7^k$, and the desired bound follows. ∎

The proof of Theorem 3 shows that in fact $n + O(\sqrt{k}(\frac{27}{4})^k)$ is an upper bound. As was mentioned in the introduction, $\omega(k; K_n) = \chi(k; K_n)$ for $k \le 3$. We do not know whether $\omega(k; K_n) = \chi(k; K_n)$ for any larger k.

4 The list-chromatic number

The list-chromatic number is not subadditive, but a weaker statement holds.

Lemma 9 *If H is a graph with at most n vertices, and $X_1, \ldots, X_k$ are disjoint sets with union $V(H)$, then*

$$\chi_\ell(H) \le \sum_{i=1}^{k} \chi_\ell(H[X_i]) + k\sqrt{8n\ln n}.$$

Proof: If any X_i is empty, then we can delete it, reduce k, and obtain a better upper bound. Hence we may assume that $X_1, \ldots, X_k$ are nonempty, which requires $k \le n$. Since $\chi_\ell(H) \le n$, the statement is trivial if $n \le \sqrt{8n\ln n}$, so we may also assume that $n > \sqrt{8n\ln n}$. Let $q = k\sqrt{8n\ln n}$, so $n > q/k$.

Let L be a list assignment for H such that $|L(x)| = m$ for all $x \in V(H)$, where m is the ceiling of the claimed upper bound. We show that H is L-colorable. To prove this, we use a probabilistic argument due to Alon [2]. Let $S = \bigcup_{x \in V(H)} L(x)$. The idea is to restrict each color in S to be used on only one $H[X_i]$ and show that some such restriction yields an L-coloring.

For $1 \le i \le k$, let $m_i = \chi_\ell(H[X_i])$, and let $p_i = \frac{m_i + q/k}{m}$. Thus $0 \le p_i \le 1$ and $p_1 + \ldots + p_k = 1$. For each $s \in S$, independently, place s into one of $S_1, \ldots, S_k$, letting the choice be S_i with probability p_i.

For each graph $H[X_i]$, define a list assignment L_i by letting $L_i(x) = L(x) \cap S_i$ for each $x \in X_i$. The list assignment L_i is *good* if $|L_i(x)| \ge m_i$ for all $x \in X_i$. If L_i is good, then $H[X_i]$ is L_i-colorable. If each $H[X_i]$ is L_i-colorable, then H is L-colorable. Therefore, it suffices to show that for some outcome of the experiment each L_i is good.

Fix i and $x \in X_i$, and let $\lambda = \mathbf{E}(|L_i(x)|)$. Note that $\lambda = mp_i = m_i + q/k$. By Chernoff's inequality (see Theorem 2.1 in [12]), we obtain

$$\mathbf{Pr}(|L_i(x)| < m_i) \;=\; \mathbf{Pr}(|L_i(x)| < \lambda - q/k) \;\le\; e^{-(q/k)^2/2\lambda}.$$

Since $n > q/k$, we have $\lambda = m_i + q/k \le n + q/k < 2n$. Therefore,

$$e^{-(q/k)^2/2\lambda} \;<\; e^{-8n\ln n/4n} \;=\; \frac{1}{n^2} \;\le\; \frac{1}{kn}.$$

Since $\mathbf{Pr}(|L_i(x)| < m_i) < 1/(kn)$, and we want to avoid kn such events, there is an outcome of the experiment such that each L_i is good. ■

Now we are ready to prove Theorem 4.

Theorem 4 *There exists a positive constant c such that, if $k = \binom{\ell+1}{2}$ and $n = \ell m$ with ℓ and m being integers greater than 1, then*

$$\chi_\ell(k; K_n) \ge n + ck\ln(n/\sqrt{k}).$$

On the other hand, for all positive integers k and n,

$$\chi_\ell(k; K_n) \le n + 3k!\sqrt{1 + 8n\ln n}.$$

Proof: We provide a construction for the lower bound. For k and n of the given form, K_n has a k-decomposition using ℓ graphs isomorphic to K_m and $\binom{\ell}{2}$ graphs isomorphic to the complete bipartite graph $K_{m,m}$. By a result of Alon [2], $\chi_\ell(K_{m,m}) \ge c_1 \ln m$ for some positive constant c_1. Also $\chi_\ell(K_m) = m$, so

$$\chi_\ell(k; K_n) \;\ge\; \ell m + c_1\binom{\ell}{2}\ln m \;\ge\; n + c_1(k-l)\ln\left(\frac{n}{l}\right) \;\ge\; n + 0.5c_1 k\ln(\frac{n}{\sqrt{k}}).$$

For the upper bound, we first define a_k for $k \ge 1$ by

$$a_k = \begin{cases} 3 - e & \text{if } k = 1,\\ k!\big(a_1 + \sum_{j=0}^{k-2}\frac{1}{j!}\big) & \text{if } k \ge 2.\end{cases}$$

For $k \ge 2$, we have $a_k = k(k-1) + ka_{k-1}$ and $k! \le a_k \le 3k!$. Therefore, it suffices to show that $\chi_\ell(k; K_n) \le n + a_k\sqrt{1 + 8n\ln n}$, which we prove by induction on k. The inequality is trivial for $k = 1$.

Suppose that $k \ge 2$. Let $(G_1, \ldots, G_k)$ be a χ-optimal k-decomposition of K_n. For each i, let H_i be a χ_ℓ-critical subgraph of G_i with $\chi_\ell(H_i) = \chi_\ell(G_i)$; note that $\delta(H_i) \ge \chi_\ell(H_i) - 1$. If some vertex x belongs to every H_i, then

$$\chi_\ell(k; K_n) \;=\; \sum_{i=1}^{k}\chi_\ell(H_i) \;\le\; \sum_{i=1}^{k}(d_{H_i}(x) + 1) \;\le\; n - 1 + k \;\le\; n + a_k\sqrt{1 + 8n\ln n}.$$

Otherwise, each vertex avoids some H_j, so we can choose disjoint $X^1, \ldots, X^k$ with union $V(K_n)$ such that $X^j \cap V(H_j) = \emptyset$ for all j. Now let $K^j = K_n[X^j]$ and $H_i^j = H_i[X^j]$. Since X^j avoids H_j and $H_1, \ldots, H_k$ are pairwise edge-disjoint, $(H_1^j, \ldots, H_{j-1}^j, H_{j+1}^j, \ldots, H_k^j)$ is a $(k-1)$-decomposition of K^j. With $n_j = |X^j|$, the induction hypothesis yields

$$\begin{aligned}\sum_i \chi_\ell(H_i^j) &\leq \chi_\ell(k-1; K_{n_j}) \leq n_j + a_{k-1}\sqrt{1 + 8n_j \ln n_j} \\ &\leq n_j + a_{k-1}\sqrt{1 + 8n \ln n}.\end{aligned}$$

For fixed i, on the other hand, Lemma 9 implies that

$$\chi_\ell(H_i) \leq \sum_j \chi_\ell(H_i^j) + (k-1)\sqrt{1 + 8n \ln n}.$$

Consequently,

$$\begin{aligned}\chi_\ell(k; K_n) &= \sum_{i=1}^{k} \chi_\ell(H_i) \leq \sum_{j=1}^{k} n_j + k(a_{k-1} + (k-1))\sqrt{1 + 8n \ln n} \\ &\leq n + a_k\sqrt{1 + 8n \ln n}.\end{aligned}$$

■

5 The Szekeres-Wilf number

Recall that $\sigma(G) = 1 + \max_{H \subseteq G} \delta(H)$.

Theorem 5 *If $k = p^2 + p + 1$ for some prime power p, and $n \equiv 0 \bmod k$, then*

$$\sigma(k; K_n) \geq (\sqrt{k} - 1)n + k,$$

On the other hand, for all positive integers k and n,

$$\sigma(k; K_n) \leq \sqrt{k}n + k.$$

Proof: For the upper bound, let $(G_1, \ldots, G_k)$ be a k-decomposition of K_n. Let $d_i = \sigma(G_i)$ and $D = \sum_{i=1}^{k} d_i$. We show that $D \leq \sqrt{k}n + k$. Each G_i has a subgraph H_i such that $d_i = \delta(H_i) - 1$. Thus $|E(G_i)| \geq |E(H_i)| \geq \binom{d_i}{2}$. Since $G_1, \ldots, G_k$ are edge-disjoint subgraphs of K_n, we obtain

$$\frac{n^2}{2} \;\geq\; \binom{n}{2} \;\geq\; \sum_{i=1}^{k} \binom{d_i}{2} \;=\; \frac{1}{2}\sum_{i=1}^{k}(d_i^2 - d_i) \;\geq\; \frac{1}{2}\left(\frac{D^2}{k} - D\right).$$

Consequently, $D^2 - kD - kn^2 \leq 0$, and thus $D \leq \frac{k}{2} + \sqrt{\frac{k^2}{4} + kn^2} \leq k + \sqrt{k}n$.

We provide a construction for the lower bound. Let p be a prime power, and let $k = p^2 + p + 1$ and $n = mk$ for some integer $m \geq 1$. There is a projective plane with points $[k]$ and lines $\{g_1, \ldots, g_k\}$. Partition $V(K_n)$ into sets $X_1, \ldots, X_k$ of size m. Each line g_i is a subset of $[k]$; let H_i be the complete $(p+1)$-partite graph whose color classes are the elements of $\{X_1, \ldots, X_k\}$ indexed by g_i. The graphs $H_1, \ldots, H_k$ are edge-disjoint subgraphs of K_n. Thus there is a k-decomposition $(G_1, \ldots, G_k)$ of K_n such that $H_i \subseteq G_i$ for each i. We have $\sigma(G_i) \geq \delta(H_i) + 1 \geq pm + 1$. Hence,

$$\sigma(k; K_n) \;\geq\; k(pm+1) \;=\; pn + k \;\geq\; (\sqrt{k} - 1)n + k. \qquad \blacksquare$$

The construction in the proof of Theorem 5 works only for special values of k. For small values of k, there are other natural candidates for σ-optimal k-decompositions of K_n.

Construction: Consider $n = m(k-1) + 1$, with $k \geq 2$ and $m \geq 1$. Let $V_1 \cup \ldots \cup V_{k-1} \cup \{v\}$ be a partition of an n-set into m-sets and a singleton. For $1 \leq i \leq k-1$, let G_i be the complete graph with vertex set $V_i \cup \{v\}$. Let G_k be the complete $(k-1)$-partite graph with color classes $V_1, \ldots, V_{k-1}$. The k-decomposition $(G_1, \ldots, G_k)$ of K_n yields $\sigma(k; K_n) \;\geq\; (k-1)m + (k-2)m + k$.

For general n, let $m = \left\lfloor \frac{n-1}{k-1} \right\rfloor$ and $r = n - 1 - m(k-1)$. Form $G_1, \ldots, G_k$ as above, except enlarge each of $V_1, \ldots, V_r$ by one vertex. Since $m = \frac{n-1-r}{k-1}$, this yields $\sigma(k; K_n) \;\geq\; (2k-3)\frac{n-1-r}{k-1} + 2r - 2 + k \;=\; (2k-3)\frac{n-1}{k-1} - \frac{k-2}{k-1} + k$. Since $\sigma(k; K_n)$ is an integer, we obtain

$$\sigma(k; K_n) \geq \left\lfloor \frac{2k-3}{k-1}(n-1) \right\rfloor + k.$$

In particular, for $n \geq 1$ we have $\sigma(2; K_n) \geq n + 1$, $\sigma(3; K_n) \geq \lfloor (3n+3)/2 \rfloor$, and $\sigma(4; K_n) \geq \lfloor (5n+7)/3 \rfloor$. $\blacksquare$

The following upper bound on $\sigma(k; K_n)$ shows that this construction is optimal for $k \leq 4$.

Theorem 10 *If $k \geq 2$ and $n \geq 2$, then $\sigma(k; K_n) \leq f(k,n)$, where*

$$f(k,n) = \begin{cases} n+1, & \text{if } k=2, \\ (3n+3)/2, & \text{if } k=3, \\ (5n+7)/3, & \text{if } k=4, \\ (k-1)(n+1)/2, & \text{if } k \geq 5. \end{cases}$$

Proof: The proof is by induction on n. For $n=2$ and $k \geq 2$, we have $\sigma(k; K_2) = k+1 \leq f(k,2)$.

For $n \geq 3$, consider a σ-optimal k-decomposition $(G_1, \ldots, G_k)$ of K_n. Let H_i be a σ-critical subgraph of G_i with $\sigma(H_i) = \sigma(G_i)$, so $\sigma(G_i) = \delta(H_i)+1$. Let $n_i = |V(H_i)|$ and $\delta_i = \delta(H_i)$.

First we prove for distinct i and j in $[k]$ that

$$\delta_i + \delta_j \leq n-1. \tag{3}$$

If H_i and H_j are disjoint, then $\delta_i + \delta_j \leq (n_i - 1) + (n_j - 1) \leq n-2$. If they have a common vertex x, then $\delta_i + \delta_j \leq d_{H_i}(x) + d_{H_j}(x) \leq n-1$. This proves (3) and the case $k=2$. Now consider $k \geq 3$.

Case 1 *Some vertex x of K_n lies in at most one of $H_1, \ldots, H_k$.* Here criticality of H_i and the induction hypothesis yield

$$\begin{aligned} \sigma(k; K_n) &= \sum_{i=1}^{k} \delta(H_i) + k \ \leq \ \sum_{i=1}^{k} \delta(H_i - x) + k + 1 \\ &\leq \ \sigma(k; K_{n-1}) + 1 \ \leq \ f(k, n-1) + 1 \ < \ f(k,n). \end{aligned}$$

Case 2 *Every vertex of K_n belongs to exactly two of $H_1, \ldots, H_k$.* Consider distinct i and j in $[k]$. Let $X_{ij} = V(H_i) \cap V(H_j)$ and $n_{ij} = n_{ji} = |X_{ij}|$. Furthermore, let $\delta_{ij} = \delta(H_i[X_{ij}])$ if $n_{ij} \neq 0$ and $\delta_{ij} = -1$ otherwise. Since $H_i[X_{ij}]$ and $H_j[X_{ij}]$ decompose the subgraph induced by X_{ij}, the case $k=2$ yields $\delta_{ij} + \delta_{ji} \leq n_{ij} - 1$. For $i \neq j$ and $S_{ij} = [k] - \{i,j\}$, a vertex x of minimum degree in $H_i[X_{ij}]$ yields $\delta_i \leq d_{H_i}(x) \leq \delta_{ij} + \sum_{\ell \in S_{ij}} n_{i\ell}$. Summing over all ordered pairs (i,j) yields

$$(k-1)\sum_{i=1}^{k} \delta_i \ \leq \sum_{1 \leq i < j \leq k} (n_{ij} - 1) + 2(k-2) \sum_{1 \leq i < j \leq k} n_{ij} \ = \ (2k-3)n - \binom{k}{2},$$

and hence

$$\sigma(k; K_n) = k + \sum_{i=1}^{k} \delta_i \leq k + \frac{2k-3}{k-1} n - \frac{k}{2} \leq \frac{2k-3}{k-1} n + \frac{k}{2}.$$

It is easily checked that $\left\lfloor \frac{2k-3}{k-1}n + \frac{k}{2} \right\rfloor \le f(k,n)$ for $k \ge 3$ and $n \ge 2$.

Case 3 *Some vertex x of K_n belongs to at least three of $H_1, \ldots, H_k$.* Indexing the decomposition so that $\{i\colon x \in V(H_i)\} = [m]$, we have $\delta_1 + \delta_2 + \ldots + \delta_m \le n-1$. If $m = k$, we are done, so we may assume that $k \ge 4$. If $m \le k-2$, then applying (3) to $G - x$ states for $i, j \in [k] - [m]$ that

$$\delta_i + \delta_j \le n - 2.$$

Summing over all such pairs yields $\sum_{i=m+1}^{k} \delta_i \le (k-m)(n-2)/2$. Since $m \ge 3$ and $k \ge 4$, this implies that

$$\begin{aligned} \sigma(k; K_n) &\le k + (n-1) + \frac{(k-m)(n-2)}{2} \\ &\le k + n - 1 + \frac{(k-3)(n-2)}{2} = \frac{1}{2}(k-1)n + 2 \le f(k,n). \end{aligned}$$

Finally, if $m = k-1$ and $\delta_1 \le \ldots \le \delta_m$, then $\delta_1 + \ldots + \delta_{m-1} \le \frac{m-1}{m}(n-1)$. By (3), $\delta_m + \delta_k \le n-1$. Hence,

$$\sigma(k; K_n) \le k + \frac{m-1}{m}(n-1) + n - 1 = k + \frac{2k-3}{k-1}(n-1).$$

It is easily checked that $\left\lfloor \frac{2k-3}{k-1}(n-1) + k \right\rfloor \le f(k,n)$ for $k \ge 4$ and $n \ge 2$. ■

The Construction and Theorem 10 together complete the computation of $\sigma(k; K_n)$ for $k \le 4$.

Corollary 11 *If n is a positive integer, then $\sigma(2; K_n) = n+1$, $\sigma(3; K_n) = \left\lfloor \frac{3}{2}n + \frac{3}{2} \right\rfloor$ and $\sigma(4; K_n) = \left\lfloor \frac{5}{3}n + \frac{7}{3} \right\rfloor$.* ■

6 Chromatic number of r-uniform hypergraphs

We next obtain an upper bound for $\chi(k; K_n^r)$. Again we need an auxiliary result.

Proposition 12 *If G is an r-uniform hypergraph with n vertices, then*

$$\chi(G) \le 1 + \frac{\omega(G)}{r-1} + \frac{n - \omega(G)}{r}.$$

Proof: We produce a proper coloring. If $V(G)$ is not a clique, then it has an independent set of size r; choose color classes of size r until the vertex set X that remains is a clique. Let $s = (n - |X|)/r$.

Since X is a clique, we have $|X| \leq \omega(G)$. Since sets of size $r-1$ contain no edge, $G[X]$ is $q+1$-colorable, where $q = \lfloor \frac{|X|}{r-1} \rfloor$. Thus $\chi(G) \leq s+q+1$.

Since $\frac{q}{r} \leq \frac{\omega(G)}{r(r-1)} = \omega(G)(\frac{1}{r-1} - \frac{1}{r})$ and $rs + (r-1)q \leq n$, we have

$$s+q \;\leq\; \frac{n}{r} - \frac{r-1}{r}q + q \;\leq\; \frac{n+q}{r} \;\leq\; \frac{n-\omega(G)}{r} + \frac{\omega(G)}{r-1}. \qquad \blacksquare$$

Now we are ready to prove Theorem 6.

Theorem 6 *If k and r are positive integers with $r \geq 2$, then there exists an integer $c_{k,r}$ such that, for every positive integer n,*

$$\chi(k; K_n^r) \leq \frac{n}{r-1} + c_{k,r}.$$

Proof: Let $(G_1, \ldots, G_k)$ be an arbitrary k-decomposition of the r-uniform complete hypergraph K_n^r. As in Theorem 3, we define a sequence of vertex subsets. Let $X_0 = \emptyset$. For $j \geq 1$, let X_j be a maximum clique among the induced subhypergraphs $G_1 - \bigcup_{i=0}^{j-1} X_i, \ldots, G_k - \bigcup_{i=0}^{j-1} X_i$.

If $|X_1| < kr(r-1)$, then $n < R$, where R is the Ramsey number $R_k^r(kr(r-1), \ldots, kr(r-1))$. Since $\sum_{i=1}^k \chi(G_i) \leq kn$, we have $\sum_{i=1}^k \chi(G_i) < kR$, and it suffices to have $c_{k,r} \geq kR$ in this case.

Otherwise, there is a largest positive integer s such that $|X_j| \geq kr(r-1)$ for $1 \leq j \leq s$. Let $X = \bigcup_{j=1}^s X_j$. For $1 \leq i \leq k$, let $Y_i = \bigcup\{X_j\colon X_j$ is a clique in G_i and $j \leq s\}$. Let $n_i = |Y_i|$, $G_i' = G_i[Y_i]$, $\omega_i = \omega(G_i')$, and $\omega^* = \sum_{i=1}^k \omega_i$. The sets $Y_1, \ldots, Y_k$ are pairwise disjoint and have union X. Furthermore, $\omega_i = |X_j|$ for some $X_j \subseteq Y_i$. Since $|X_j| \geq kr(r-1)$ for $j \leq s$, we conclude that $|X| - \omega^* \geq (s-k)kr(r-1)$.

By Proposition 12, $\chi(G_i') \leq \frac{\omega_i}{r-1} + \frac{n_i - \omega_i}{r} + 1$. We rewrite the upper bound as $\frac{n_i}{r-1} - \frac{n_i - \omega_i}{r(r-1)} + 1$. Summing the upper bounds yields

$$\sum_{i=1}^k \chi(G_i') \;\leq\; \frac{|X|}{r-1} - \frac{|X| - \omega^*}{r(r-1)} + k \;\leq\; \frac{n}{r-1} - k(s-k) + k.$$

A clique in G_i is an independent set in G_j for $j \neq i$. Therefore, the inequality above implies

$$\sum_{i=1}^{k} \chi(G_i[X]) \;\leq\; \sum_{i=1}^{k} \chi(G'_i) + (k-1)s \;\leq\; \frac{n}{r-1} + k^2 + k.$$

By construction, $\omega(G_i - X) < kr(r-1)$. Therefore, $\chi(G_i - X) \leq n - |X| < R$. By the subadditivity of χ, this yields

$$\sum_{i=1}^{k} \chi(G_i) \leq \frac{n}{r-1} + k^2 + k + kR,$$

which proves the claim with $c_{k,r} = k^2 + k + kR$. ■

7 Graphs on surfaces

This section considers graphs embedded on a surface Σ. For a graph parameter p and a positive integer k, let $p(k;\Sigma) = \max\{p(k;G)$: G embeds on $\Sigma\}$.

Surfaces can be classified by their genus and orientability. For $h \geq 0$, the *orientable surface* Σ_h is obtained by adding h handles to a sphere. For $h \geq 1$, the *non-orientable surface* Π_h is obtained from a sphere with h holes by attaching h Möbius bands along their boundaries to the boundaries of the holes. For example, Π_1 is the projective plane, Π_2 is the Klein bottle, etc. The *Euler genus* $g(\Sigma)$ of the surface Σ is $2h$ if $\Sigma = \Sigma_h$ and is h if $\Sigma = \Pi_h$. The *Euler characteristic* of Σ is $2 - g(\Sigma)$.

For a simple graph G with vertex set V and edge set E embedded on a surface Σ of Euler genus g, Euler's Formula states that $|V|-|E|+|F| \geq 2-g$, where F is the set of faces, with equality holding if and only if every face is a 2-cell. When $|V| \geq 3$, this yields $|E| \leq 3|V| - 6 + 3g$. For $g \geq 1$, this implies that every subgraph of G has a vertex of degree at most $H(g) - 1$, where

$$H(g) = \left\lfloor \frac{7 + \sqrt{24g+1}}{2} \right\rfloor.$$

In particular, $\sigma(G) \leq H(g)$. Consequently, if $g \geq 1$, then

$$\omega(G) \leq \chi(G) \leq \chi_\ell(G) \leq \sigma(G) \leq H(g).$$

For every surface Σ other than the Klein bottle, the Heawood number $H(g)$ is, in fact, the maximum chromatic number of graphs embeddable on Σ, attained by $K_{H(g)}$. This landmark result conjectured by Heawood [11] was proved by Ringel [18] and Ringel–Youngs [17]. Furthermore, every graph with chromatic number $H(g)$ embedded on Σ contains a complete graph on $H(g)$ vertices as a subgraph. This was proved by Dirac [4, 5] for the torus and for $g \geq 4$ and was proved by Albertson and Hutchinson [1] for $g \in \{1, 3\}$.

Although $H(2) = 7$, Franklin [10] proved that the maximum chromatic number for the Klein bottle is 6. Furthermore, there are 6-chromatic graphs on the Klein bottle not containing K_6. Such a graph appears in [1].

The version of Brooks' Theorem for list-chromatic number implies that if G is a graph on the Klein bottle, then also $\chi_\ell(G) \leq 6$. For graphs on the sphere, the maximum chromatic number is 4, but the maximum list-chromatic number is 5 (upper bound by Thomassen [21], lower bound by Voigt [22]).

Further results about the chromatic number of graphs embedded on given surfaces appear in the book of Jensen and Toft [13].

For a surface Σ and a positive integer k, we have the familiar inequalities

$$\omega(k; \Sigma) \leq \chi(k; \Sigma) \leq \chi_\ell(k; \Sigma) \leq \sigma(k; \Sigma).$$

When a graph embeds on the sphere Σ_0, the disjoint union of k copies of G also embeds on Σ_0. Hence $\sigma(k; \Sigma_0) = 6k$ and $\omega(k; \Sigma_0) = \chi(k; \Sigma_0) = 4k$. For all other surfaces, this is not true. We begin by establishing a lower bound for $\omega(k; \Sigma)$.

Theorem 13 *Let Σ be a surface with positive Euler genus g.*

(*a*) *If Σ is orientable, then $\omega(k; \Sigma) \geq k\, H(2\lfloor g/2k \rfloor)$.*

(*b*) *If Σ is non-orientable and $\lfloor g/k \rfloor \geq 3$, then $\omega(k; \Sigma) \geq k\, H(\lfloor g/k \rfloor)$.*

Proof: If Σ is orientable, then let $g' = 2\lfloor g/2k \rfloor$ and $m = H(g')$; since g' is even, K_m embeds on an orientable surface with Euler genus g'. If Σ is non-orientable, then let $g' = \lfloor g/k \rfloor$ and $m = H(g')$; since $g' \geq 3$, K_m embeds on a non-orientable surface with Euler genus g'.

In either case, let G be the disjoint union of k copies of K_m. Since $kg' \leq g$, it then follows (see [14]) that G embeds on Σ. Thus $\omega(k; G) \geq km$. ■

For a surface Σ of Euler genus g, this lower bound on $\omega(k;\Sigma)$ is approximately $(7k+\sqrt{24gk+k^2})/2$. We next establish an upper bound on $\sigma(k;\Sigma)$ that is asymptotic to this for fixed k and large g.

Theorem 14 *If Σ is a surface with positive Euler genus g, then*

$$\sigma(k;\Sigma) \leq \left\lfloor \frac{7k+\sqrt{24kg+49k^2-48k}}{2} \right\rfloor.$$

Proof: Given a graph G embedded on Σ, let $(G_1,\dots,G_k)$ be a k-decomposition of G. For each i, let H_i be a σ-critical subgraph of G_i with $\sigma(H_i)=\sigma(G_i)$, and let $d_i=\delta(H_i)$, so $\sigma(G_i)=d_i+1$.

We may assume that $d_1 \geq \ldots \geq d_k$. Let s be the unique nonnegative integer such that $d_i > 5$ if and only if $i \leq s$. If $s=0$, then

$$\sum_{i=1}^{k} \sigma(G_i) \leq 5k \leq \left\lfloor \frac{7k+\sqrt{24kg+49k^2-48k}}{2} \right\rfloor.$$

If $s \geq 1$, then let $H=\bigcup_{j\in[s]} H_j$. Let $n=|V(H)|$ and $e=|E(H)|$. For $I\subseteq[s]$, denote by V_I the set of all vertices of H that belong to each graph H_i with $i\in I$ and to no graph H_i with $i\in[s]-I$. Let $n_I=|V_I|$ and $n_i=n_{\{i\}}$. Thus,

$$|V(H_i)| = \sum_{i\in I\subseteq[s]} n_I \quad \text{and} \quad n=\sum_{I\subseteq[s]} n_I.$$

Since $s\geq 1$, we have $n\geq 7$. Since $H\subseteq G$, also H embeds on Σ. By Euler's Formula, $6n+6g-12\geq 2e$. Every vertex of V_I has degree at least $\sum_{i\in I} d_i$. Thus,

$$6\sum_{I\subseteq[s]} n_I + 6g-12 \geq 2e \geq \sum_{I\subseteq[s]} n_I \sum_{i\in I} d_i.$$

By rearranging the inequality and interchanging the order of summation (subtracting more copies of 6 when $|I|>1$), we obtain

$$6g-12 \;\geq\; \sum_{I\subseteq[s]} n_I(-6+\sum_{i\in I} d_i) \;\geq\; \sum_{i=1}^{s}\left((d_i-6)\sum_{i\in I\subseteq[s]} n_I\right) \;=\; \sum_{i=1}^{s}(d_i-6)|V(H_i)|.$$

Since $|V(H_i)| \geq d_i + 1$ and $d_i \geq 6$ for $i \in [s]$, we have

$$6g - 12 \;\geq\; \sum_{i=1}^{s}(d_i - 6)(d_i + 1) \;=\; \sum_{i=1}^{s}((d_i - \tfrac{5}{2})^2 - \tfrac{49}{4}).$$

Consequently,

$$6g - 12 + \tfrac{49}{4}s \geq \sum_{i=1}^{s}(d_i - \tfrac{5}{2})^2 \geq \tfrac{1}{s}\left(\sum_{i=1}^{s}(d_i - \tfrac{5}{2})\right)^2 = \tfrac{1}{s}\left(\sum_{i=1}^{s} d_i - s\,\tfrac{5}{2}\right)^2,$$

and therefore

$$\sum_{i=1}^{s} d_i \leq \tfrac{1}{2}(5\,s + \sqrt{24\,s\,g + 49s^2 - 48s}).$$

For $i > s$, we have $d_i \leq 5$. Thus we conclude that

$$\sum_{i=1}^{k}\sigma(G_i) = k + \sum_{i=1}^{k} d_i \leq 6k - 5s + \sum_{i=1}^{s} d_i \leq \tfrac{1}{2}(12k - 5s + \sqrt{24\,s\,g + 49\,s^2 - 48s}).$$

This upper bound increases with s in the domain $s \geq 0$. Since $s \leq k$, we thus set $s = k$ to obtain

$$\sum_{i=1}^{k}\sigma(G_i) \leq \left\lfloor \frac{7k + \sqrt{24kg + 49k^2 - 48k}}{2} \right\rfloor.$$

This completes the proof. ∎

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