

Polychromatic Cliques

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Abstract

The sub-Ramsey number $sr(K_n, k)$ is the smallest integer m such that in any edge-colouring of K_m which uses every colour at most k times some subgraph K_n has all edges of different colours. It was known that, for a fixed k , the function $sr(K_n, k)$ is $O(n^3)$ and $\Omega(n)$. We improve these bounds to $O(n^2)$ and $\Omega(n^{3/2})$ (slightly less for small values of k).

1 Background and Notation

An edge-colouring of a graph G (i.e., a mapping of $E(G)$ to a set of ‘colours’) is k -bounded if each colour is used at most k times. If $k = 1$, i.e., if all edges of G have different colours, we say that G is *polychromatic*. The *sub-Ramsey number* $sr(G, k)$ is the smallest integer m such that each k -bounded edge-colouring of K_m contains a polychromatic subgraph isomorphic to G . Recall that the *Ramsey number* $r(G, k)$ is the smallest integer m such that each edge-colouring of K_m with k colours contains a monochromatic (i.e., all edges coloured with the same colour) subgraph isomorphic to G . Thus sub-Ramsey numbers are in this sense dual to Ramsey numbers, and it is easy to see that each $sr(G, k) \leq r(G, k)$, and hence each $sr(G, k)$ is guaranteed to be finite [1].

F. Galvin appears to be the first person to suggest investigating $sr(G, k)$ [8]. In [1] it is shown that $sr(K_n, k)$ is $O(kn^3)$ and $\Omega(kn)$. In [11] the authors show that $sr(P_n, k) = sr(C_n, k) = n$ when n is large enough with respect to k . (P_n and C_n denote respectively the path and the cycle with n vertices.)

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Results on sub-Ramsey numbers of stars can be found in [9, 10, 4]. Related questions on k -bounded edge-colourings without polychromatic subgraphs which are not necessarily of fixed size (such as polychromatic Hamilton paths or cycles) are investigated in [7, 11, 12, 3].

In this note we improve both the upper bound and the lower bound for $sr(K_n, k)$. In particular we prove that $sr(K_n, k)$ is $O(kn^2)$ (better than $O(kn^3)$) and, for k greater than or equal to 15, $sr(K_n, k)$ is $\Omega(n^{3/2})$ (better than $\Omega(kn)$ for any fixed k). The lower bound for k between 3 and 15 is $\Omega(n^{4/3})$. (We have no improvement of the lower bound when $k = 2$.)

2 The Upper Bound

Theorem 1 *Let $n \geq 3$ and $k \geq 2$ be positive integers. Then*

$$sr(K_n, k) \leq (2n - 3)(n - 2)(k - 1) + 3.$$

Proof. Assume that K_m admits a *good* edge-colouring, i.e., a k -bounded edge-colouring in which no subgraph isomorphic to K_n is polychromatic. We shall show that $m \leq (2n - 3)(n - 2)(k - 1) + 2$.

Fix a good edge-colouring of K_m , with colours $1, 2, \dots, r$, and denote by d_i the number of edges coloured i . Note that each $d_i \leq k$. Let W denote the number of unordered pairs of edges of K_m coloured by the same colour. We claim that

$$W \leq \binom{m}{2} \frac{k - 1}{2}. \tag{1}$$

Indeed, $\binom{m}{2} = |E(K_m)| = \sum_{i=1}^r d_i$ and $W = \sum_{i=1}^r \binom{d_i}{2}$. Thus

$$W = \sum_{i=1}^r d_i \frac{d_i - 1}{2} \leq \sum_{i=1}^r d_i \frac{k - 1}{2} = \binom{m}{2} \frac{k - 1}{2}.$$

Consider now a subgraph Q of K_m isomorphic to K_q , for some integer $q > n$. We claim that Q must have many pairs of edges of the same colour, otherwise some subgraph of Q isomorphic to K_n would be polychromatic. Specifically, denote by $T(q, n)$ the maximum number of edges of a graph with q vertices which does not have a complete subgraph of n vertices. (The value of $T(q, n)$ is known by Turan's theorem [2].) If Q obtained more than $T(q, n)$ colours then taking one edge of each colour would result in a

graph G with q vertices and more than $T(q, n)$ edges, which therefore would have to contain a subgraph isomorphic to K_n . This contradicts the fact that we have a good edge-colouring. Therefore, Q is coloured by at most $T(q, n)$ colours, say, the colours $1, 2, \dots, t$, where $t \leq T(q, n)$. Denote by $w(Q)$ the number of unordered pairs of edges of Q which obtain the same colour, and by $d_i(Q)$ the number of edges of Q of colour $i, i \leq t$. We claim that

$$w(Q) \geq \binom{q}{2} - T(q, n). \quad (2)$$

Indeed,

$$\begin{aligned} w(Q) &= \sum_{i=1}^t \binom{d_i(Q)}{2} = \sum_{i=1}^t (d_i(Q) - 1) \frac{d_i(Q)}{2} \geq \sum_{i=1}^t (d_i(Q) - 1) = \\ &= \left(\sum_{i=1}^t d_i(Q) \right) - t = \binom{q}{2} - t \geq \binom{q}{2} - T(q, n). \end{aligned}$$

Let \mathcal{Q} denote the set of all subgraphs of K_m isomorphic to K_q . Since each pair of edges of K_m belongs to $\binom{m-3}{q-3}$ or $\binom{m-4}{q-4}$ graphs in \mathcal{Q} , depending on whether the two edges are adjacent or not, we see that $MW \geq \sum_{Q \in \mathcal{Q}} w(Q)$,

where $M = \max\{\binom{m-3}{q-3}, \binom{m-4}{q-4}\}$.

Suppose first that $\binom{m-3}{q-3} \leq \binom{m-4}{q-4}$. This inequality implies that $m-3 \leq q-3$ and thus $m \leq q$. Therefore our desired conclusion $m \leq (2n-3)(n-2)(k-1) + 2$ will follow as long as we choose q not too big. Let us in fact choose $q = 2n-2$.

In the remaining case, $\binom{m-3}{q-3} > \binom{m-4}{q-4}$, we have $\binom{m-3}{q-3}W > \sum_{Q \in \mathcal{Q}} w(Q)$ which by (1) and (2) implies that

$$\binom{m-3}{q-3} \binom{m}{2} \frac{k-1}{2} \geq \binom{m-3}{q-3} W > \binom{m}{q} \left(\binom{q}{2} - T(q, n) \right). \quad (3)$$

Using

$$T(q, n) = \frac{n-2}{2n-2} q^2$$

[2], and $q = 2n-2$, we have

$$\binom{q}{2} - T(q, n) = \binom{q}{2} - \frac{n-2}{2n-2} q^2 = \frac{q}{2}.$$

Therefore, we obtain from (3) that

$$\frac{(m-3)!}{(q-3)!(m-q)!} \cdot \frac{m(m-1)}{2} \cdot \frac{(k-1)}{2} \geq \frac{m!}{(m-q)!q!} \cdot \frac{q}{2}.$$

Simplifying the last inequality we reach the conclusion that $m-2 \leq (k-1)(q-1)(q-2)/2$ which is equivalent to $m \leq (2n-3)(n-2)(k-1) + 2$. \square

3 The Lower Bound

Let m and t be positive integers.

Let $U(t)$ denote the maximum number of ways a set can be written as a union of two t -element subsets. For instance, $U(2) = 3$ because the set $\{1, 2, 3, 4\}$ can be written as $\{1, 2\} \cup \{3, 4\}$, or $\{1, 3\} \cup \{2, 4\}$, or $\{1, 4\} \cup \{2, 3\}$, and no set can be written as the union of two 2-element subsets in four different ways. It is easy to check also that $U(3) = 15$.

Let $F(m, t)$ be the maximum size of a union free family of t -element subsets of $\{1, 2, \dots, m\}$. (A family of sets $S_i, i \in I$ is *union free* if all the $\binom{|I|}{2}$ unions $S_i \cup S_j, S_i, S_j \in F$, are distinct.)

The following result of Frankl and Füredi gives the asymptotic behaviour of $F(m, t)$:

Theorem 2 [5] *Let t be a positive integer.*

There exist positive constants c_t, c'_t so that

$$c'_t m^{\lceil \frac{4t}{3} \rceil / 2} \leq F(m, t) \leq c_t m^{\lceil \frac{4t}{3} \rceil / 2}.$$

The relevance of the functions $U(t)$ and $F(m, t)$ to our problem arises from the following fact:

Theorem 3 *If $n > F(m, t)$ and $k \geq U(t)$, then*

$$sr(K_n, k) \geq \binom{m}{t}.$$

Proof. Consider the complete graph $K(m, t)$ whose vertices are all the t -element subsets of $\{1, 2, \dots, m\}$, in which the edge SS' has colour $S \cup S'$. Then this colouring of $K(m, t)$ is k -bounded, since $k \geq U(t)$. At the same time, there is no polychromatic K_n , since $n > F(m, t)$. \square

Corollary 1 *If $k \geq 15$, then (for a positive constant c)*

$$sr(K_n, k) \geq cn^{3/2},$$

and, if $3 \leq k < 15$, then

$$sr(K_n, k) \geq cn^{4/3}.$$

Proof. If $k \geq 15 = U(3)$, then we can choose $t = 3$ in Theorem 3. For $t = 3$, any choice of m with $n > c_t m^{\lceil \frac{4t}{3} \rceil / 2} = c_3 m^2$ assures that we have $n > F(m, 3)$. We may choose $m = \lceil (c_3^{-1/2} - \epsilon)n^{1/2} \rceil$, for a small positive ϵ . Such a value of m yields

$\binom{m}{3} \geq cn^{3/2}$, thus $sr(K_n, k) \geq cn^{3/2}$ (for a positive constant c). For $k \geq 3 = U(2)$, we proceed analogously, taking $t = 2$ and m satisfying $n > c_2 m^{3/2}$, yielding $sr(K_n, k) \geq \binom{m}{3} \geq cn^{4/3}$. \square

We remark that using [6], in which $F(m, 3)$ is evaluated exactly as $F(m, 3) = \lfloor m(m-1)/6 \rfloor$, we find, for $k \geq 15$, that

$$sr(K_n, k) \geq (\sqrt{6} - \epsilon)n^{3/2}.$$

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