A sufficient condition for the existence of large empty convex polygons*

Pavel Valtr

Department of Applied Mathematics, Charles University

and

Institute for Theoretical Computer Science (ITI), Charles University

Malostranské nám. 25, 118 00 Praha 1, Czech Republic

valtr@kam.mff.cuni.cz

Abstract

Let P be a set of points in general position in the plane. We say that P is k-convex, if no triangle determined by P contains more than k points of P in the interior. We say that a subset A of P in convex position forms an *empty polygon* (in P), if no point of $P \setminus A$ lies in the convex hull of A. We show that for any k, n there is an $N = N(k, n)$ such that any k-convex set of at least N points in general position in the plane contains an empty n-gon. We also prove an analogous statement in $\mathbb{R}^d$ for each odd $d \geq 3$.

1 Introduction

A set P of points in the plane is in *general position*, if it contains no three points on a line. A finite set X of points is *in convex position*, if the points of X are the vertices of a convex polygon.

In 1935 Erdős and Szekeres proved the following classical result:

*This research was supported by project LN00A056 of The Ministry of Education of the Czech Republic, by Charles University grants No. 99/158 and 99/159 and by Czech Republic Grant GAČR 201/99/0242.

Theorem 1 (Erdős and Szekeres [4]) *For any $n \geq 3$, there is an integer $f(n)$ such that any set of at least $f(n)$ points in general position in the plane contains n points in convex position.*

We say that a subset A of P in convex position forms an *empty polygon*, if no point of $P \setminus A$ lies in the convex hull of A. In 1975 Erdős [3] asked whether the following sharpening of the Erdős–Szekeres theorem is true: Is there a (smallest) finite $g(n)$ such that any set of at least $g(n)$ points in general position in the plane contains an empty n-gon? Somewhat surprisingly, in 1983 Horton [7] gave a negative answer to this question for $n \geq 7$. It is easy to see that $g(3) = 3$, $g(4) = 5$. Harborth [6] proved that $g(5) = 10$. So far, the existence of $g(6)$ was neither proved nor disproved. In this paper we show that Erdős' question has an affirmative answer if we restrict ourselves to so-called k-convex sets.

We say that a finite planar point set P in general position is *k-convex*, if no triangle determined by P contains more than k points of P in the interior. Thus, 0-convex sets are just sets in convex position. Károlyi, Pach, and Tóth [8] proved that for any n there is an integer $N(n)$ such that any 1-convex set of at least N points in general position in the plane contains an empty n-gon. As far as we know, the 1-convexity is the first non-trivial condition which was proved to guarantee the existence of an empty n-gon in a large set of points in general position. Here we prove that for any fixed $k \geq 1$ the k-convexity is also sufficient:

Theorem 2 *For any k, n there is an $N = N(k, n)$ such that any k-convex set of at least N points in general position in the plane contains an empty n-gon.*

We conjecture that a statement analogous to Theorem 2 is true also in $\mathbb{R}^d$ for any $d \geq 2$. Besides $d = 2$, we can prove it for all odd $d \geq 3$ (see Theorem 3 below).

Let $d \geq 2$. We say that a set P of points in $\mathbb{R}^d$ is in *general position*, if there are no $\leq d+1$ affinely dependent points in P. We say that a finite set of points in $\mathbb{R}^d$ is *in convex position*, if none of the points lies in the convex hull of the others. We say that a finite set P of points in general position in $\mathbb{R}^d$ is *k-convex*, if no simplex determined by $d + 1$ points of P contains more than k points of P in the interior.

A notion analogous to empty polygons is the notion of holes. Let P be a set of points in $\mathbb{R}^d$, $d \geq 2$. An *n-hole* of P is any set X of n points of P

in convex position such that the convex hull of X contains no other points of P.

Theorem 3 *For any odd $d \geq 3$ and for any k, n, there is an $N = N_d(k,n)$ such that any k-convex set of at least N points in general position in $\mathbb{R}^d$ contains an n-hole.*

We remark that for any fixed $d \geq 2$ (similarly as for $d = 2$), if we define $g_d(n)$ as the smallest number such that any set of at least $g_d(n)$ points in general position in $\mathbb{R}^d$ contains an n-hole, then $g_d(n)$ does not exist for n sufficiently large (see [9]).

Organization of the paper. The proof of Theorem 2 is given in Section 2. The proof of Theorem 3 is given in Section 3. An alternative proof of Lemma 4 (applied in the proof of Theorem 2) is given in Section 4. Section 5 contains concluding remarks and open problems.

2 Proof of Theorem 2

Suppose $k \geq 0$ is fixed. Let P be a k-convex set of N points in general position in the plane, where N is very large.

Let $Q \subseteq P$ be the set of vertices of $\operatorname{conv} P$. Any triangulation of $\operatorname{conv} P = \operatorname{conv} Q$ partitions the interior of $\operatorname{conv} P$ into $|Q| - 2$ triangles. Since P is k-convex, it follows that $|P| = N \leq |Q| + (|Q| - 2)k$. Consequently, $|Q| = \Omega(N)$.

We color $\binom{Q}{3}$ by the $k+1$ colors $0, 1, \ldots, k$ so that a triple $\{q_1, q_2, q_3\} \in \binom{Q}{3}$ is colored by color i if the interior of the triangle $q_1q_2q_3$ contains exactly i points of P. By Ramsey's theorem, Q contains a monochromatic subset R of size $\Omega(\log\log |Q|) = \Omega(\log\log N) = \Omega(\log^{[2]} N)$ ($\log^{[t]}$ denotes t iterations of log). Every triangle determined by R contains a constant number, c, of points of P in the interior.

Let $r_1, r_2, \ldots, r_{|R|}$ be the points of R listed in the counterclockwise order. We could start the numbering from any vertex, but from now on the numbering is fixed. Consider the following coloring of $\binom{R}{4}$ by $c+1$ colors $0 \ldots, c$. A 4-tuple $\{r_i, r_j, r_k, r_l\}, 1 \leq i < j < k < l \leq |R|$, is colored by color q if the triangle $r_ir_jr_{ijkl}$ contains exactly q points of P in the interior (where r_{ijkl} denotes the intersection point of the lines r_ir_k, r_jr_l, see Fig. 1).

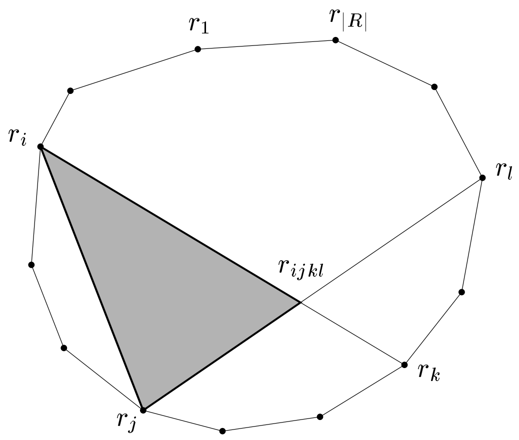

Figure 1: The triangle $r_i r_j r_{ijkl}$.

Another application of Ramsey's theorem gives a monochromatic subset S of R of size $\Omega(\log^{[3]}|R|) = \Omega(\log^{[5]} N)$. Let $s_1, s_2, \ldots, s_{|S|}$ be the points of S listed in the same order as in the sequence $r_1, r_2, \ldots, r_{|R|}$. Then there are two integers $a, b \geq 0$, $a + b = c$, such that the following holds. If $s_i, s_j, s_k, s_l, 1 \leq i < j < k < l \leq |S|$ are 4 points of S then the $2c = 2a + 2b$ points of P inside the quadrilateral $s_i s_j s_k s_l$ are partitioned by the diagonals into four subsets of sizes a, a, b, b in the way shown in Fig. 2 (here we use that S is monochromatic and that each triangle determined by $R \supseteq S$ contains exactly c points in the interior).

For $i = 2, \ldots, |S| - 1$, let T_i denote the triangle determined by the lines $s_i s_1$, $s_i s_{|S|}$, $s_{i-1} s_{i+1}$ (see Fig. 3). Here is the key lemma:

Lemma 4 *If $|S| \geq 6$ then $b = 0$ and the interior of each of the triangles T_i contains exactly $a = c$ points of P.*

Before proving Lemma 4, we derive Theorem 2 from it. Since $\operatorname{conv} S$ can be triangulated into $|S| - 2$ triangles, its interior contains exactly $(|S| - 2)c$ points of P. Lemma 4 now shows that $\operatorname{conv} S$ contains no point of P outside of $T_2 \cup T_3 \cup \cdots \cup T_{|S|-1}$. It follows that if we choose a point $p_i \in P$ inside T_i closest to the line $s_{i-1}s_{i+1}$, then $s_1 p_2 p_3 \ldots p_{|S|-1} s_{|S|}$ is an empty $|S|$-gon in P. Since $|S| = \Omega(\log^{[5]} N)$ is at least n for N sufficiently large, Theorem 2 follows. It remains to prove Lemma 4.

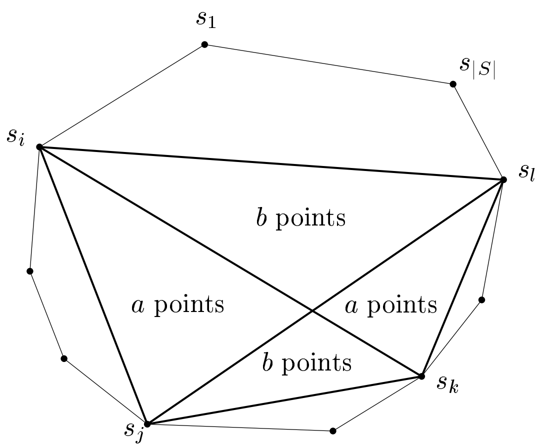

Figure 2: Distribution of the points in every quadrilateral $s_i s_j s_k s_l, 1 \leq i < j < k < l \leq |S|$.

We remark that Lemma 4 does not hold for $|S| = 5$ — Fig. 4 shows a possible placement of points of P inside a pentagon for any $a \geq b \geq 0, a + b = c$.

Proof of Lemma 4. We first prove that $b = 0$. Fig. 5 shows the two non-degenerate cases differing by the way how the diagonals partition the interior of the hexagon $s_1 s_2 s_3 s_4 s_5 s_6$ (the remaining, degenerate case when the diagonals $s_1 s_4, s_2 s_5, s_3 s_6$ have a common intersection point can be seen as a special case of any of them). We treat the two cases at the same time – they are very similar.

For each picture in Fig. 5, consider the regions R_1 through R_{12} (or R_{13}). For a region X, let $|X|$ denote the number of points of P lying in the interior of X.

We have $|R_1| = a$ (due to the partition of the points in the quadrilateral $s_1 s_2 s_3 s_6$) and $|R_4| = b$ (quadrilateral $s_1 s_2 s_5 s_6$). Consequently, we get $|R_3| = |R_5| = |R_{10}| = 0$ from $|R_3| + |R_4| + |R_5| + |R_{10}| \leq b$ (quadrilateral $s_1 s_3 s_4 s_6$) and also $(|R_{13}| =)|R_2| = |R_8| = |R_9| = 0$ from $(|R_{13}|+)|R_1| + |R_2| + |R_8| + |R_9| = a$ (quadrilateral $s_1 s_2 s_4 s_5$).

Analogously, $|R_7| = a$ and thus $|R_6| = |R_{11}| = |R_{12}| = 0$ follows from $(|R_{13}|+)|R_6| + |R_7| + |R_{11}| + |R_{12}| = a$. Thus, $(|R_{13}|+)|R_8| + |R_9| + |R_{10}| +$

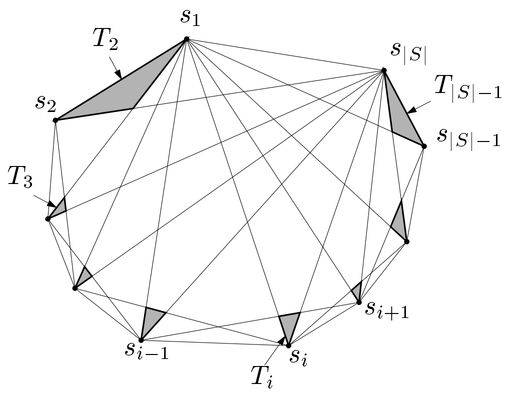

Figure 3: The triangles T_i.

$|R_{11}| + |R_{12}| = 0$ but this quantity also equals b (quadrilateral $s_1s_2s_5s_6$). Thus, $b = 0$ (and $a = c$).

We now easily show that $|T_i| = a$ for each i. We have $|T_2| = a$ (quadrilateral $s_1s_2s_3s_{|S|}$) and $|T_{|S|-1}| = a$ (quadrilateral $s_1s_{|S|-2}s_{|S|-1}s_{|S|}$). For $i = 3, \ldots, |S| - 2$, let U_i denote the triangle $s_{i-1}s_ix_i$, where x_i is the intersection point of the diagonals s_1s_i, $s_{i-1}s_{i+1}$ (see Fig. 6). Then $|U_i| = b = 0$ (quadrilateral $s_1s_{i-1}s_is_{i+1}$) and $|U_i \cup T_i| = a$ (quadrilateral $s_{i-1}s_is_{i+1}s_{|S|}$). Thus, $|T_i| = a$ for each i. $\square$

3 Proof of Theorem 3

Let $d \geq 3$ be odd. We may suppose that $n \geq d+2$. Let P be a k-convex set of N points in general position in $\mathbb{R}^d$, where $N = N_d(k, n)$ is very large. By a generalization of the Erdős–Szekeres theorem to higher dimensions (e.g., see [5, Exercise 7.3.6] or [2, Corollary 3.8]), P contains a subset Q of N' vertices of a cyclic polytope in $\mathbb{R}^d$, where $N' = N'_d(k, n)$ is still large. Since P is k-convex, we can color $\binom{Q}{d+1}$ by the $k + 1$ colors $0, 1, \ldots, k$ so that a $(d + 1)$-tuple $\{q_1, \ldots, q_{d+1}\} \in \binom{Q}{d+1}$ is colored by color i if the interior of the simplex with the vertex set $\{q_1, \ldots, q_{d+1}\}$ contains exactly i points of P. If N' is sufficiently large, then, by Ramsey's theorem, Q contains a

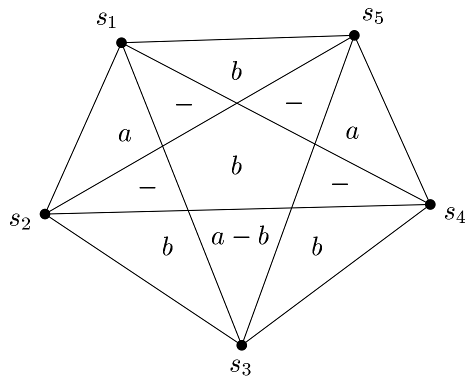

Figure 4: Possible distribution of points inside the pentagon $s_1s_2s_3s_4s_5$, if $|S|=5$, $a \geq b \geq 0$, $a+b=c$.

monochromatic subset R of size n. Every simplex determined by R contains a constant number, c, of points in the interior. We will show that $c=0$.

Let $r_1, \ldots, r_n$ be the points of R listed in the "cyclic-polytope" order. Set
$C := \mathrm{conv}\{r_1, \ldots, r_{d+2}\}$,
$C_i := \mathrm{conv}\{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{d+2}\}$, for $i = 1, \ldots, d+2$.

Lemma 5 *The cyclic polytope C can be triangulated into simplices in the following two ways: $C = C_1 \cup C_3 \cup \cdots \cup C_{d+2}$ and $C = C_2 \cup C_4 \cup \cdots \cup C_{d+1}$.*

Proof. If $i \neq j$ are of the same parity then the interiors of C_i, C_j are disjoint because C_i, C_j lie on different sides of their common facet $\mathrm{conv}\{r_\alpha : 1 \leq \alpha \leq d+2, \alpha \neq i, j\}$. It remains to show that $C \subseteq C_1 \cup C_3 \cup \cdots \cup C_{d+2}$ and $C \subseteq C_2 \cup C_4 \cup \cdots \cup C_{d+1}$.

The unique Radon partition of the set $R' := \{r_1, \ldots, r_{d+2}\}$ is $R' = R'_{\mathrm{odd}} \cup R'_{\mathrm{even}}$, where $R'_{\mathrm{odd}} = \{r_1, r_3, \ldots, r_{d+2}\}, R'_{\mathrm{even}} = \{r_2, r_4, \ldots, r_{d+1}\}$ (this is well-known and follows from the fact that any other pair of disjoint subsets of R' is separated by a hyperplane). Algebraicly,

$$\sum_{i \text{ odd}} \beta_i r_i = \sum_{i \text{ even}} \beta_i r_i, \text{ where } \sum_{i \text{ odd}} \beta_i = \sum_{i \text{ even}} \beta_i = 1,\ \beta_i \geq 0\ (i = 1, \ldots, d+2).$$

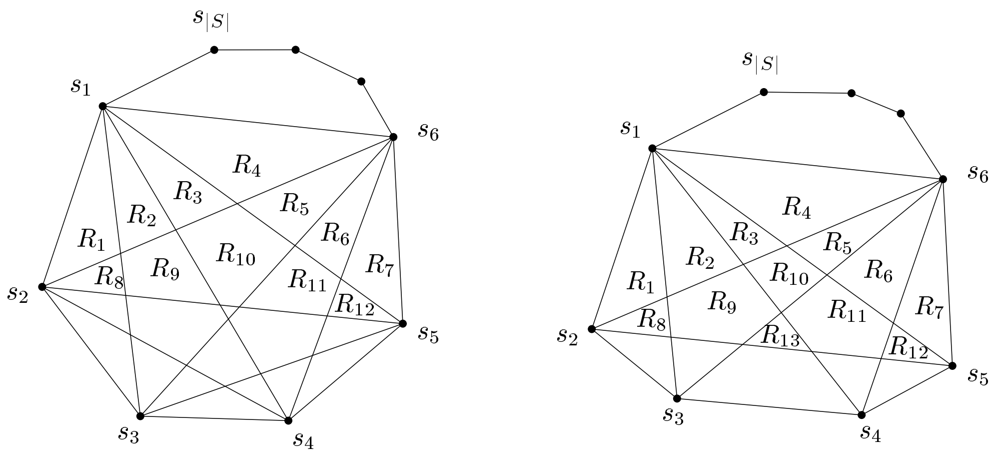

Figure 5: Possible partitions of the interior of the hexagon $s_1s_2s_3s_4s_5s_6$ and the regions R_i.

Let $x \in C$. Then

$$x = \sum_{i=1}^{d+2} \alpha_i r_i, \text{ where } \sum_{i=1}^{d+2} \alpha_i = 1, \ \alpha_i \geq 0 \ (i = 1, \ldots, d+2).$$

If we sum up the equality $x = \sum_{i=1}^{d+2} \alpha_i r_i$ and a proper non-negative multiple of the equality

$$0 = \sum_{i \text{ even}} \beta_i r_i - \sum_{i \text{ odd}} \beta_i r_i,$$

we obtain an equality

$$x = \sum_{i=1}^{d+2} \alpha_i' r_i, \text{ where } \sum_{i=1}^{d+2} \alpha_i' = 1, \ \alpha_i' \geq 0 \ (i = 1, \ldots, d+2),$$

such that $\alpha_i' = 0$ holds for at least one odd index i. It follows that $x \in C_i$ and thus $C \subseteq C_1 \cup C_3 \cup \cdots \cup C_{d+2}$. Similarly we can prove that $C \subseteq C_2 \cup C_4 \cup \cdots \cup C_{d+1}$. $\square$

Since the interior of each of the simplices C_i contains the same number, c, of points of P, Lemma 5 gives $\frac{d+3}{2}c = \frac{d+1}{2}c$ and thus $c = 0$. Thus, the

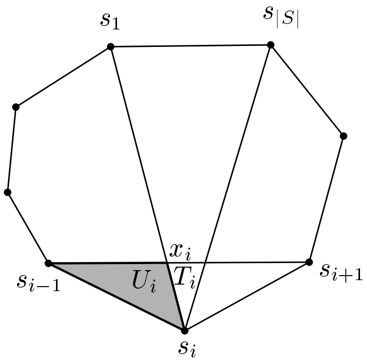

Figure 6: The region U_i.

interior of the convex hull of R contains no point of P. This concludes the proof of Theorem 3. $\square$

We remark that if d is even then every triangulation of a cyclic polytope in $\mathbb{R}^d$ partitions the polytope into the same number, $\binom{n-d/2-1}{d/2}$, of simplices (because any two triangulations in this situation are connected by a finite number of flips that preserve the triangulation size). Therefore the above proof does not work if d is even.

4 Second proof of Lemma 4

Here we give an alternative proof of Lemma 4. We believe that this proof (rather than the proof given in Section 2) might have a generalization in even dimensions which would consequently give an analogue of Theorems 2 and 3 in even dimensions.

We may suppose that $|S| = 2m$ is even. For $i = 1, 2, \ldots, 2m-1$, let v_i be the point of intersection of the main diagonals $s_i s_{i\pm m}$, $s_{i+1} s_{i+1\pm m}$ (we always choose "+" or "−" so that the obtained index lies in the range $\{1, 2, \ldots, 2m\}$) and let V_i be the triangle $s_i s_{i+1} v_i$. Further, put $v_{2m} := v_m$ and let V_{2m} be the triangle $s_{2m} s_1 v_{2m}$.

We first show that the triangles $V_1, \ldots, V_{2m}$ cover $\operatorname{conv} S$. This is clearly true for $m = 2$ (we have $v_1 = v_2 = v_3 = v_4$ in this case). We further proceed by induction on m. Let $m \geq 3$. We denote by x the point of intersection of the diagonals s_1s_{m+1}, s_3s_{m+3} (see Fig. 7). By the induction

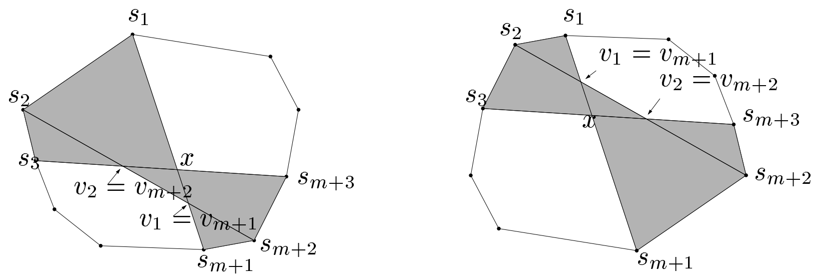

Figure 7: The point x and the main diagonals $s_is_{m+i}, i = 1, 2, 3$ (two combinatorially different cases). The shaded region is covered by $V_1 \cup V_2 \cup V_{m+1} \cup V_{m+2}$.

assumption, $\operatorname{conv}(S \setminus \{s_2, s_{m+2}\})$ is covered by the $2m-2$ triangles $\triangle s_1s_3x$, V_3, $V_4, \ldots, V_m$, $\triangle s_{m+1}s_{m+3}x$, $V_{m+3}, V_{m+4}, \ldots, V_{2m}$. The statement now follows from the following three facts (see Fig. 7): (a) $\operatorname{conv} S = \operatorname{conv}(S \setminus \{s_2, s_{m+2}\}) \cup \triangle s_1s_2s_3 \cup \triangle s_{m+1}s_{m+2}s_{m+3}$, (b) $\triangle s_1s_3x \cup \triangle s_1s_2s_3$ is covered by $V_1 \cup V_2$, (c) $\triangle s_{m+1}s_{m+3}x \cup \triangle s_{m+1}s_{m+2}s_{m+3}$ is covered by $V_{m+1} \cup V_{m+2}$.

For each triangle V_i, we now find a smaller triangle V_i' contained in V_i such that the interiors of the triangles V_i, V_i' contain the same number of points of P. Then $V_i \setminus V_i'$ will contain no point of P. Since $\operatorname{conv} S = V_1 \cup \cdots \cup V_{2m}$, points of P inside $\operatorname{conv} S$ will lie in $V_1' \cup \cdots \cup V_{2m}'$.

For $i = 1, \ldots, m-1$, we define V_i' as the intersection of the triangles $s_is_{i+1}s_{2m}$ and $s_is_{i+1}s_{i+2}$. For $i = m+1, \ldots, 2m-1$, we define V_i' as the intersection of the triangles $s_1s_is_{i+1}$ and $s_{i-1}s_is_{i+1}$. Further, we define V_m' as the intersection of the triangles $s_{m-1}s_ms_{m+1}$ and $s_ms_{m+1}s_{m+2}$, and we define V_{2m}' as the intersection of the triangles $s_{2m}s_1s_2$ and $s_{2m-1}s_{2m}s_1$ (see Fig. 8).

Clearly, if $i \in \{1, \ldots, 2m\}$, then $V_i' \subset V_i$ and, by the choice of S, the interiors of V_i', V_i contain the same number (a or b) of points of P. Thus, as argued above, all points of P inside $\operatorname{conv} S$ lie in $V_1' \cup \cdots \cup V_{2m}'$. By the choice of S, the interiors of V_m' and V_{2m}' contain b points each, and the interior of any other triangle V_i' contains a points of P. Since the triangle $s_1s_2s_{2m-1}$ contains V_1' and is disjoint from the interiors of all other triangles V_i', its interior contains a points of P. At the same time, it contains c points of P. It follows that $a = c$ and thus $b = 0$. The interior of V_{2m}' contains $b = 0$

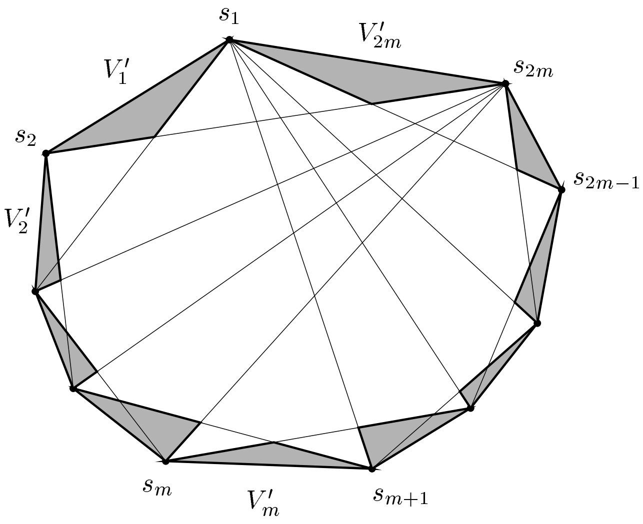

Figure 8: The triangles V'_i.

points of P. Therefore, each triangle $s_1 s_i s_{2m}, i = 2, \ldots, 2m-1$, contains its $a = c$ interior points belonging to P in its intersection with V'_i, which is the triangle T_i. □

5 Concluding remarks and open problems

5.1 The modular version of the Erdős-Szekeres theorem

Károlyi, Pach, and Tóth [8] studied 1-convex sets in relation to the following conjecture:

Conjecture 1 (Bialostocki, Dierker, and Voxman [1]) *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B = B(n,p)$ such that every set P of at least B points in general position in the plane determines a convex n-gon such that the number of points of P in its interior is divisible by p.*

Conjecture 1 was proved in [1] for every $n \geq p+2$, Károlyi et al. [8] proved it for every $n \geq 5p/6 + O(1)$. In their proof, they applied the special case $k = 1$ of Theorem 2. In fact, Theorem 2 gives Conjecture 1 for all $n \geq 3p/4 + O(1)$. This is shown in [10]. In [10] we also discuss a d-dimensional version of Conjecture 1 and prove it if $d \geq 3$ is odd or $n \geq n_0(p, d)$.

5.2 A cap or an (empty) cup

We believe that point sets with no large holes have a very special structure. Therefore, it is interesting to seek conditions ensuring the existence of a hole of any fixed size in every sufficiently large point set in $\mathbb{R}^d$. One such condition (k-convexity) is given in Theorems 2 and 3.

We say that a set X of n points in the plane, no two on a vertical line, is an *n-cap* (*n-cup*, respectively), if it is in convex position and all points of X lie on and above (on and below, respectively) the line connecting the leftmost point of X with the rightmost point of X (see Fig. 9).

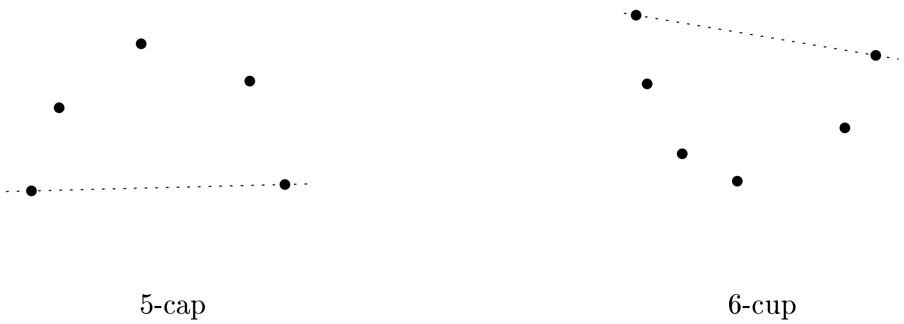

Figure 9: A cap and a cup.

In their classical paper [4], Erdős and Szekeres proved that any set of at least $\binom{2n-4}{n-2} + 1$ points in general position in the plane, no two of them on a vertical line, contains an n-cap or an n-cup. From this they concluded that Theorem 1 holds with $f(n) := \binom{2n-4}{n-2} + 1$. We conjecture that actually the following stronger result holds:

Conjecture 2 *For any $n \geq 3$, there is an integer $f'(n)$ such that any set of at least $f'(n)$ points in general position in the plane, no two of them on* *a vertical line, contains an n-cap or an empty n-cup. (An empty n-cup is an n-cup which forms an empty n-gon.)*

Conjecture 2, if true, would imply that for any k and n any sufficiently large set of points in the plane with no k-cap contains an empty n-gon.

5.3 Strengthenings of Theorems 2 and 3

Theorems 2 and 3 can be generalized in the following two ways:

- the general position is not necessary,
- the statement is true also if the considered set is a union of a fixed number of k-convex sets.

These generalizations may be combined in one theorem:

Theorem 6 *If $d = 2$ or $d \geq 3$ is odd, then, for any three positive integers k, l, n, there is an $N = N(d, k, l, n)$ with the following property. Let $P = P_1 \cup P_2 \cup \cdots \cup P_l$ be a set of N points in $\mathbb{R}^d$ such that for each i the convex hull of any $d+1$ points of P_i contains at most k other points of P_i. Then the set P has an n-hole.*

Proof. Suppose that $d = 2$, that k, l, n are three positive integers, and that N is very large. Let $P = P_1 \cup P_2 \cup \cdots \cup P_l$ be as in the theorem. Without loss of generality, we may suppose that the sets P_i are pairwise disjoint and that $|P_1| \geq \frac{N}{l}$. The set P_1 cannot contain more than $k+3$ points on a line (otherwise the convex hull of three of them would contain more than k other points of P_1). A greedy algorithm gives a set P_1' of N_1 points of P_1 in general position such that no segment determined by two points of P_1' contains other points of P_1 and N_1 is still very large. By the same proof as for Theorem 2, we can show that P_1' has a subset P_1'' forming an empty N_1'-gon in P_1, where N_1' is still very large. If $\operatorname{conv} P_1''$ contains fewer than $\frac{N_1'}{n}$ of points of $P_2 \cup \cdots \cup P_l$, then P_1'' contains n points forming an empty n-gon in $P_1 \cup \cdots \cup P_l$. Otherwise, we consider the points of $P \setminus P_1$ lying in $\operatorname{conv} P_1''$. Their number is at least $\frac{N_1'}{n}$, which is still a very large number. If we repeat the same argument (at most) $l-1$ times, we obtain an empty n-gon in $P_1 \cup \cdots \cup P_l$ formed by n points of one of the sets $P_2, \ldots, P_l$.

If $d \geq 3$ is odd, we can proceed similarly, applying Theorem 3 instead of Theorem 2. $\square$

5.4 A density statement does not hold

It does not suffice that 99% of the simplices determined by P are empty:

Claim 7 *For any $d \geq 2$ and for any $\varepsilon > 0$, there is an integer $c = c(d,\varepsilon) \geq 3$ such that for any $N \geq d+1$ there is a set P of N points in general position in $\mathbb{R}^d$ such that more than $(1-\varepsilon)\binom{|P|}{d+1}$ simplices determined by P are $(d+1)$-holes (i.e., empty simplices), but the set P does not contain a c-hole.*

Proof. First, let $d = 2$. Suppose for simplicity that $\frac{3}{\varepsilon}$ is an integer and that N is divisible by $\frac{3}{\varepsilon}$. Let H be a set of $\frac{N}{3/\varepsilon} = \frac{\varepsilon N}{3}$ points in general position in the plane with no empty 7-gon (arbitrarily large sets with no empty 7-gon were constructed by Horton [7]). We shrink the second coordinates of points in H (without changing the combinatorial structure of H) so that all lines determined by H will be almost horizontal. Further, we take $\frac{3}{\varepsilon}$ copies of H rotated along a large circle (see Fig. 10). The union of these copies will

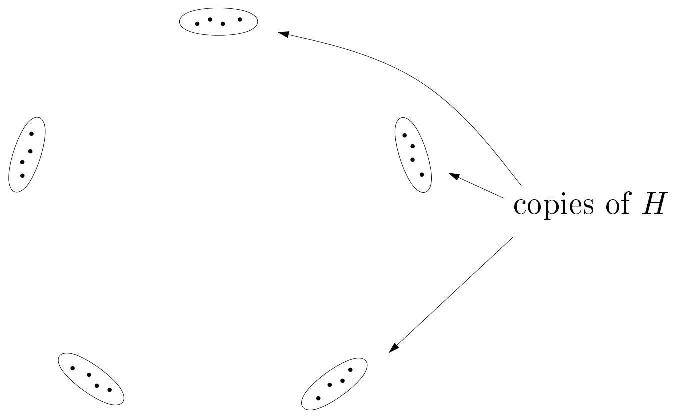

Figure 10: $\frac{3}{\varepsilon}$ copies of H along a circle.

be our set P. It contains no empty convex $(\frac{18}{\varepsilon}+1)$-gon (an empty polygon cannot contain 7 points from a copy of H). Moreover, each triple of points from distinct copies of H is empty. There are more than

$$\frac{\frac{3}{\varepsilon}-1}{\frac{3}{\varepsilon}} \cdot \frac{\frac{3}{\varepsilon}-2}{\frac{3}{\varepsilon}} \binom{n}{3} > (1-\varepsilon)\binom{n}{3}$$

such triples. This completes the case $d = 2$. If $d > 2$, we proceed similarly, using the construction of [9] of arbitrarily large sets of points in general position in $\mathbb{R}^d$ with no c_d-hole, where c_d is a constant depending on d only. $\square$

Acknowledgment. I am indebted to Gyula Károlyi for interesting and motivating discussions. Alex Below and Jesus De Loera explained me some properties of the cyclic polytope triangulations. I also thank the referees for their helpful suggestions.

References

[1] A. Bialostocki, P. Dierker, and B. Voxman, Some notes on the Erdős–Szekeres theorem, *Discrete Mathematics* 91 (1991), 231–238.

[2] R. Cordovil and P. Duchet, Cyclic polytopes and oriented matroids, *Europ. J. Combinatorics* 21 (2000), 49–64.

[3] P. Erdős, On some problems of elementary and combinatorial geometry, *Ann. Mat. Pura. Appl.* (4) 103 (1975), 99–108.

[4] P. Erdős and Gy. Szekeres, A combinatorial problem in geometry, *Compositio Math.* 2 (1935), 464-470.

[5] B. Grünbaum, *Convex polytopes*, Pure and Applied Mathematics, Vol. 16, John Wiley & Sons, New York, 1967.

[6] H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, *Elem. Math.* 33 (1978), 116-118.

[7] J. D. Horton, Sets with no empty convex 7-gons, *Canadian Math. Bull.* 26 (1983), 482-484.

[8] Gy. Károlyi, J. Pach, and G. Tóth, A modular version of the Erdős–Szekeres theorem, *Studia Sc. Math. Hungar.* 38 (2001), 245-259.

[9] P. Valtr, Sets in $\mathbb{R}^d$ with no large empty convex subsets, *Discrete Mathematics* 108 (1992), 115-124.

[10] P. Valtr, On the modular version of the Erdős–Szekeres theorem, in preparation.

Note added in proof. An alternative proof of Theorem 2 is given in the very recent manuscript "G. Kun and G. Lippner, Large convex empty polygons in k-convex sets". The proof gives $N(k,n) \leq 2^{(k+2)^{n+1}}$.