

# A sufficient condition for the existence of large empty convex polygons\*

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## Abstract

Let  $P$  be a set of points in general position in the plane. We say that  $P$  is  $k$ -convex, if no triangle determined by  $P$  contains more than  $k$  points of  $P$  in the interior. We say that a subset  $A$  of  $P$  in convex position forms an *empty polygon* (in  $P$ ), if no point of  $P \setminus A$  lies in the convex hull of  $A$ . We show that for any  $k, n$  there is an  $N = N(k, n)$  such that any  $k$ -convex set of at least  $N$  points in general position in the plane contains an empty  $n$ -gon. We also prove an analogous statement in  $\mathbb{R}^d$  for each odd  $d \geq 3$ .

## 1 Introduction

A set  $P$  of points in the plane is in *general position*, if it contains no three points on a line. A finite set  $X$  of points is *in convex position*, if the points of  $X$  are the vertices of a convex polygon.

In 1935 Erdős and Szekeres proved the following classical result:

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\*This research was supported by project LN00A056 of The Ministry of Education of the Czech Republic, by Charles University grants No. 99/158 and 99/159 and by Czech Republic Grant GAČR 201/99/0242.

**Theorem 1 (Erdős and Szekeres [4])** *For any  $n \geq 3$ , there is an integer  $f(n)$  such that any set of at least  $f(n)$  points in general position in the plane contains  $n$  points in convex position.*

We say that a subset  $A$  of  $P$  in convex position forms an *empty polygon*, if no point of  $P \setminus A$  lies in the convex hull of  $A$ . In 1975 Erdős [3] asked whether the following sharpening of the Erdős–Szekeres theorem is true: Is there a (smallest) finite  $g(n)$  such that any set of at least  $g(n)$  points in general position in the plane contains an empty  $n$ -gon? Somewhat surprisingly, in 1983 Horton [7] gave a negative answer to this question for  $n \geq 7$ . It is easy to see that  $g(3) = 3$ ,  $g(4) = 5$ . Harborth [6] proved that  $g(5) = 10$ . So far, the existence of  $g(6)$  was neither proved nor disproved. In this paper we show that Erdős’ question has an affirmative answer if we restrict ourselves to so-called  $k$ -convex sets.

We say that a finite planar point set  $P$  in general position is  *$k$ -convex*, if no triangle determined by  $P$  contains more than  $k$  points of  $P$  in the interior. Thus, 0-convex sets are just sets in convex position. Károlyi, Pach, and Tóth [8] proved that for any  $n$  there is an integer  $N(n)$  such that any 1-convex set of at least  $N$  points in general position in the plane contains an empty  $n$ -gon. As far as we know, the 1-convexity is the first non-trivial condition which was proved to guarantee the existence of an empty  $n$ -gon in a large set of points in general position. Here we prove that for any fixed  $k \geq 1$  the  $k$ -convexity is also sufficient:

**Theorem 2** *For any  $k, n$  there is an  $N = N(k, n)$  such that any  $k$ -convex set of at least  $N$  points in general position in the plane contains an empty  $n$ -gon.*

We conjecture that a statement analogous to Theorem 2 is true also in  $\mathbb{R}^d$  for any  $d \geq 2$ . Besides  $d = 2$ , we can prove it for all odd  $d \geq 3$  (see Theorem 3 below).

Let  $d \geq 2$ . We say that a set  $P$  of points in  $\mathbb{R}^d$  is in *general position*, if there are no  $\leq d + 1$  affinely dependent points in  $P$ . We say that a finite set of points in  $\mathbb{R}^d$  is in *convex position*, if none of the points lies in the convex hull of the others. We say that a finite set  $P$  of points in general position in  $\mathbb{R}^d$  is  *$k$ -convex*, if no simplex determined by  $d + 1$  points of  $P$  contains more than  $k$  points of  $P$  in the interior.

A notion analogous to empty polygons is the notion of holes. Let  $P$  be a set of points in  $\mathbb{R}^d$ ,  $d \geq 2$ . An  *$n$ -hole* of  $P$  is any set  $X$  of  $n$  points of  $P$

in convex position such that the convex hull of  $X$  contains no other points of  $P$ .

**Theorem 3** *For any odd  $d \geq 3$  and for any  $k, n$ , there is an  $N = N_d(k, n)$  such that any  $k$ -convex set of at least  $N$  points in general position in  $\mathbb{R}^d$  contains an  $n$ -hole.*

We remark that for any fixed  $d \geq 2$  (similarly as for  $d = 2$ ), if we define  $g_d(n)$  as the smallest number such that any set of at least  $g_d(n)$  points in general position in  $\mathbb{R}^d$  contains an  $n$ -hole, then  $g_d(n)$  does not exist for  $n$  sufficiently large (see [9]).

**Organization of the paper.** The proof of Theorem 2 is given in Section 2. The proof of Theorem 3 is given in Section 3. An alternative proof of Lemma 4 (applied in the proof of Theorem 2) is given in Section 4. Section 5 contains concluding remarks and open problems.

## 2 Proof of Theorem 2

Suppose  $k \geq 0$  is fixed. Let  $P$  be a  $k$ -convex set of  $N$  points in general position in the plane, where  $N$  is very large.

Let  $Q \subseteq P$  be the set of vertices of  $\text{conv} P$ . Any triangulation of  $\text{conv} P = \text{conv} Q$  partitions the interior of  $\text{conv} P$  into  $|Q| - 2$  triangles. Since  $P$  is  $k$ -convex, it follows that  $|P| = N \leq |Q| + (|Q| - 2)k$ . Consequently,  $|Q| = \Omega(N)$ .

We color  $\binom{Q}{3}$  by the  $k + 1$  colors  $0, 1, \dots, k$  so that a triple  $\{q_1, q_2, q_3\} \in \binom{Q}{3}$  is colored by color  $i$  if the interior of the triangle  $q_1q_2q_3$  contains exactly  $i$  points of  $P$ . By Ramsey's theorem,  $Q$  contains a monochromatic subset  $R$  of size  $\Omega(\log \log |Q|) = \Omega(\log \log N) = \Omega(\log^{[2]} N)$  ( $\log^{[t]}$  denotes  $t$  iterations of  $\log$ ). Every triangle determined by  $R$  contains a constant number,  $c$ , of points of  $P$  in the interior.

Let  $r_1, r_2, \dots, r_{|R|}$  be the points of  $R$  listed in the counterclockwise order. We could start the numbering from any vertex, but from now on the numbering is fixed. Consider the following coloring of  $\binom{R}{4}$  by  $c + 1$  colors  $0, \dots, c$ . A 4-tuple  $\{r_i, r_j, r_k, r_l\}, 1 \leq i < j < k < l \leq |R|$ , is colored by color  $q$  if the triangle  $r_i r_j r_{ijkl}$  contains exactly  $q$  points of  $P$  in the interior (where  $r_{ijkl}$  denotes the intersection point of the lines  $r_i r_k, r_j r_l$ , see Fig. 1).

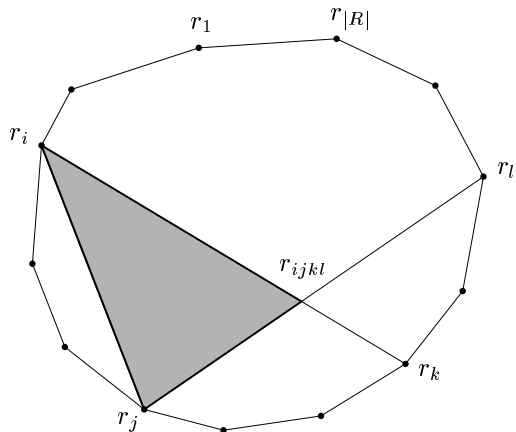


Figure 1: The triangle  $r_i r_j r_{ijkl}$ .

Another application of Ramsey's theorem gives a monochromatic subset  $S$  of  $R$  of size  $\Omega(\log^{[3]} |R|) = \Omega(\log^{[5]} N)$ . Let  $s_1, s_2, \dots, s_{|S|}$  be the points of  $S$  listed in the same order as in the sequence  $r_1, r_2, \dots, r_{|R|}$ . Then there are two integers  $a, b \geq 0$ ,  $a + b = c$ , such that the following holds. If  $s_i, s_j, s_k, s_l, 1 \leq i < j < k < l \leq |S|$  are 4 points of  $S$  then the  $2c = 2a + 2b$  points of  $P$  inside the quadrilateral  $s_i s_j s_k s_l$  are partitioned by the diagonals into four subsets of sizes  $a, a, b, b$  in the way shown in Fig. 2 (here we use that  $S$  is monochromatic and that each triangle determined by  $R \supseteq S$  contains exactly  $c$  points in the interior).

For  $i = 2, \dots, |S| - 1$ , let  $T_i$  denote the triangle determined by the lines  $s_i s_1, s_i s_{|S|}, s_{i-1} s_{i+1}$  (see Fig. 3). Here is the key lemma:

**Lemma 4** *If  $|S| \geq 6$  then  $b = 0$  and the interior of each of the triangles  $T_i$  contains exactly  $a = c$  points of  $P$ .*

Before proving Lemma 4, we derive Theorem 2 from it. Since  $\text{conv } S$  can be triangulated into  $|S| - 2$  triangles, its interior contains exactly  $(|S| - 2)c$  points of  $P$ . Lemma 4 now shows that  $\text{conv } S$  contains no point of  $P$  outside of  $T_2 \cup T_3 \cup \dots \cup T_{|S|-1}$ . It follows that if we choose a point  $p_i \in P$  inside  $T_i$  closest to the line  $s_{i-1} s_{i+1}$ , then  $s_1 p_2 p_3 \dots p_{|S|-1} s_{|S|}$  is an empty  $|S|$ -gon in  $P$ . Since  $|S| = \Omega(\log^{[5]} N)$  is at least  $n$  for  $N$  sufficiently large, Theorem 2 follows. It remains to prove Lemma 4.

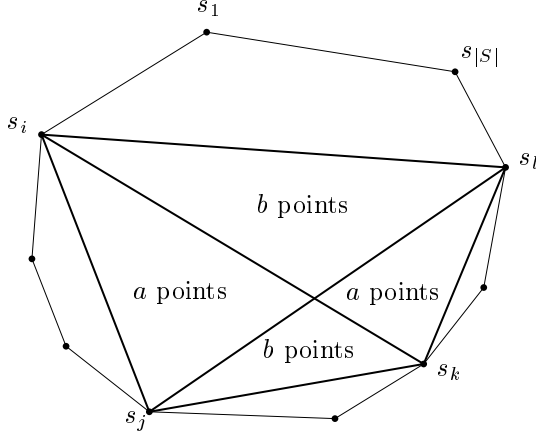


Figure 2: Distribution of the points in every quadrilateral  $s_i s_j s_k s_l, 1 \leq i < j < k < l \leq |S|$ .

We remark that Lemma 4 does not hold for  $|S| = 5$  — Fig. 4 shows a possible placement of points of  $P$  inside a pentagon for any  $a \geq b \geq 0, a + b = c$ .

**Proof of Lemma 4.** We first prove that  $b = 0$ . Fig. 5 shows the two non-degenerate cases differing by the way how the diagonals partition the interior of the hexagon  $s_1 s_2 s_3 s_4 s_5 s_6$  (the remaining, degenerate case when the diagonals  $s_1 s_4, s_2 s_5, s_3 s_6$  have a common intersection point can be seen as a special case of any of them). We treat the two cases at the same time — they are very similar.

For each picture in Fig. 5, consider the regions  $R_1$  through  $R_{12}$  (or  $R_{13}$ ). For a region  $X$ , let  $|X|$  denote the number of points of  $P$  lying in the interior of  $X$ .

We have  $|R_1| = a$  (due to the partition of the points in the quadrilateral  $s_1 s_2 s_3 s_6$ ) and  $|R_4| = b$  (quadrilateral  $s_1 s_2 s_5 s_6$ ). Consequently, we get  $|R_3| = |R_5| = |R_{10}| = 0$  from  $|R_3| + |R_4| + |R_5| + |R_{10}| \leq b$  (quadrilateral  $s_1 s_3 s_4 s_6$ ) and also  $(|R_{13}| =) |R_2| = |R_8| = |R_9| = 0$  from  $(|R_{13}| +) |R_1| + |R_2| + |R_8| + |R_9| = a$  (quadrilateral  $s_1 s_2 s_4 s_5$ ).

Analogously,  $|R_7| = a$  and thus  $|R_6| = |R_{11}| = |R_{12}| = 0$  follows from  $(|R_{13}| +) |R_6| + |R_7| + |R_{11}| + |R_{12}| = a$ . Thus,  $(|R_{13}| +) |R_8| + |R_9| + |R_{10}| +$

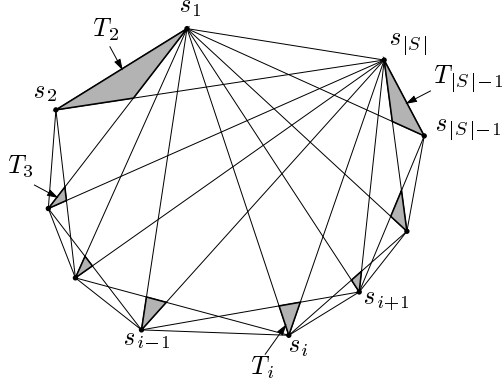


Figure 3: The triangles  $T_i$ .

$|R_{11}| + |R_{12}| = 0$  but this quantity also equals  $b$  (quadrilateral  $s_1 s_2 s_5 s_6$ ). Thus,  $b = 0$  (and  $a = c$ ).

We now easily show that  $|T_i| = a$  for each  $i$ . We have  $|T_2| = a$  (quadrilateral  $s_1 s_2 s_3 s_{|S|}$ ) and  $|T_{|S|-1}| = a$  (quadrilateral  $s_1 s_{|S|-2} s_{|S|-1} s_{|S|}$ ). For  $i = 3, \dots, |S| - 2$ , let  $U_i$  denote the triangle  $s_{i-1} s_i x_i$ , where  $x_i$  is the intersection point of the diagonals  $s_1 s_i, s_{i-1} s_{i+1}$  (see Fig. 6). Then  $|U_i| = b = 0$  (quadrilateral  $s_1 s_{i-1} s_i s_{i+1}$ ) and  $|U_i \cup T_i| = a$  (quadrilateral  $s_{i-1} s_i s_{i+1} s_{|S|}$ ). Thus,  $|T_i| = a$  for each  $i$ .  $\square$

### 3 Proof of Theorem 3

Let  $d \geq 3$  be odd. We may suppose that  $n \geq d + 2$ . Let  $P$  be a  $k$ -convex set of  $N$  points in general position in  $\mathbb{R}^d$ , where  $N = N_d(k, n)$  is very large. By a generalization of the Erdős–Szeker theorem to higher dimensions (e.g., see [5, Exercise 7.3.6] or [2, Corollary 3.8]),  $P$  contains a subset  $Q$  of  $N'$  vertices of a cyclic polytope in  $\mathbb{R}^d$ , where  $N' = N'_d(k, n)$  is still large. Since  $P$  is  $k$ -convex, we can color  $\binom{Q}{d+1}$  by the  $k + 1$  colors  $0, 1, \dots, k$  so that a  $(d + 1)$ -tuple  $\{q_1, \dots, q_{d+1}\} \in \binom{Q}{d+1}$  is colored by color  $i$  if the interior of the simplex with the vertex set  $\{q_1, \dots, q_{d+1}\}$  contains exactly  $i$  points of  $P$ . If  $N'$  is sufficiently large, then, by Ramsey's theorem,  $Q$  contains a

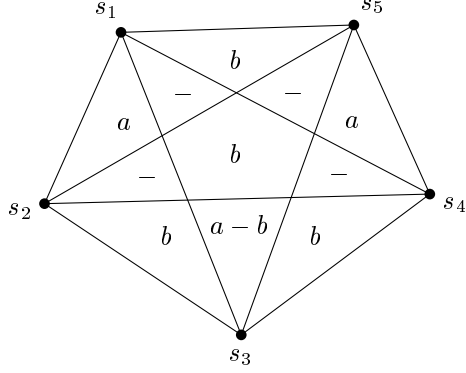


Figure 4: Possible distribution of points inside the pentagon  $s_1s_2s_3s_4s_5$ , if  $|S| = 5$ ,  $a \geq b \geq 0$ ,  $a + b = c$ .

monochromatic subset  $R$  of size  $n$ . Every simplex determined by  $R$  contains a constant number,  $c$ , of points in the interior. We will show that  $c = 0$ .

Let  $r_1, \dots, r_n$  be the points of  $R$  listed in the “cyclic-polytope” order. Set

$$C := \text{conv}\{r_1, \dots, r_{d+2}\},$$

$$C_i := \text{conv}\{r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_{d+2}\}, \text{ for } i = 1, \dots, d+2.$$

**Lemma 5** *The cyclic polytope  $C$  can be triangulated into simplices in the following two ways:  $C = C_1 \cup C_3 \cup \dots \cup C_{d+2}$  and  $C = C_2 \cup C_4 \cup \dots \cup C_{d+1}$ .*

*Proof.* If  $i \neq j$  are of the same parity then the interiors of  $C_i, C_j$  are disjoint because  $C_i, C_j$  lie on different sides of their common facet  $\text{conv}\{r_\alpha : 1 \leq \alpha \leq d+2, \alpha \neq i, j\}$ . It remains to show that  $C \subseteq C_1 \cup C_3 \cup \dots \cup C_{d+2}$  and  $C \subseteq C_2 \cup C_4 \cup \dots \cup C_{d+1}$ .

The unique Radon partition of the set  $R' := \{r_1, \dots, r_{d+2}\}$  is  $R' = R'_{\text{odd}} \cup R'_{\text{even}}$ , where  $R'_{\text{odd}} = \{r_1, r_3, \dots, r_{d+2}\}, R'_{\text{even}} = \{r_2, r_4, \dots, r_{d+1}\}$  (this is well-known and follows from the fact that any other pair of disjoint subsets of  $R'$  is separated by a hyperplane). Algebraically,

$$\sum_{i \text{ odd}} \beta_i r_i = \sum_{i \text{ even}} \beta_i r_i, \text{ where } \sum_{i \text{ odd}} \beta_i = \sum_{i \text{ even}} \beta_i = 1, \beta_i \geq 0 \text{ (} i = 1, \dots, d+2\text{)}.$$

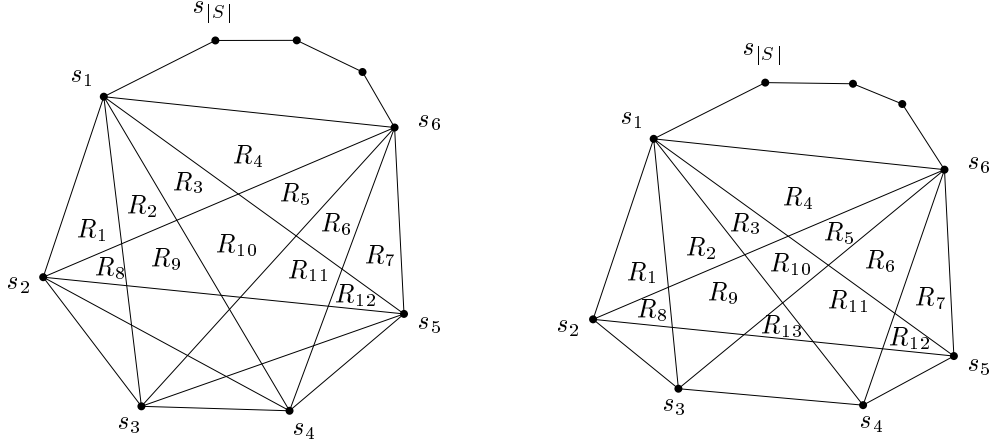


Figure 5: Possible partitions of the interior of the hexagon  $s_1s_2s_3s_4s_5s_6$  and the regions  $R_i$ .

Let  $x \in C$ . Then

$$x = \sum_{i=1}^{d+2} \alpha_i r_i, \text{ where } \sum_{i=1}^{d+2} \alpha_i = 1, \alpha_i \geq 0 \ (i = 1, \dots, d+2).$$

If we sum up the equality  $x = \sum_{i=1}^{d+2} \alpha_i r_i$  and a proper non-negative multiple of the equality

$$0 = \sum_{i \text{ even}} \beta_i r_i - \sum_{i \text{ odd}} \beta_i r_i,$$

we obtain an equality

$$x = \sum_{i=1}^{d+2} \alpha'_i r_i, \text{ where } \sum_{i=1}^{d+2} \alpha'_i = 1, \alpha'_i \geq 0 \ (i = 1, \dots, d+2),$$

such that  $\alpha'_i = 0$  holds for at least one odd index  $i$ . It follows that  $x \in C_i$  and thus  $C \subseteq C_1 \cup C_3 \cup \dots \cup C_{d+2}$ . Similarly we can prove that  $C \subseteq C_2 \cup C_4 \cup \dots \cup C_{d+1}$ .  $\square$

Since the interior of each of the simplices  $C_i$  contains the same number,  $c$ , of points of  $P$ , Lemma 5 gives  $\frac{d+3}{2}c = \frac{d+1}{2}c$  and thus  $c = 0$ . Thus, the

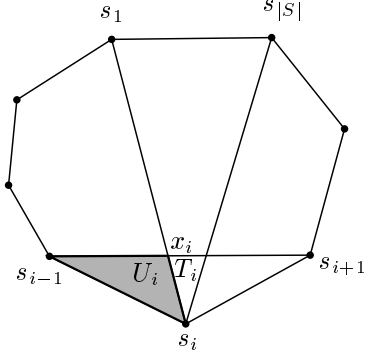


Figure 6: The region  $U_i$ .

interior of the convex hull of  $R$  contains no point of  $P$ . This concludes the proof of Theorem 3.  $\square$

We remark that if  $d$  is even then every triangulation of a cyclic polytope in  $\mathbb{R}^d$  partitions the polytope into the same number,  $\binom{n-d/2-1}{d/2}$ , of simplices (because any two triangulations in this situation are connected by a finite number of flips that preserve the triangulation size). Therefore the above proof does not work if  $d$  is even.

## 4 Second proof of Lemma 4

Here we give an alternative proof of Lemma 4. We believe that this proof (rather than the proof given in Section 2) might have a generalization in even dimensions which would consequently give an analogue of Theorems 2 and 3 in even dimensions.

We may suppose that  $|S| = 2m$  is even. For  $i = 1, 2, \dots, 2m - 1$ , let  $v_i$  be the point of intersection of the main diagonals  $s_i s_{i \pm m}$ ,  $s_{i+1} s_{i+1 \pm m}$  (we always choose “+” or “-” so that the obtained index lies in the range  $\{1, 2, \dots, 2m\}$ ) and let  $V_i$  be the triangle  $s_i s_{i+1} v_i$ . Further, put  $v_{2m} := v_m$  and let  $V_{2m}$  be the triangle  $s_{2m} s_1 v_{2m}$ .

We first show that the triangles  $V_1, \dots, V_{2m}$  cover  $\text{conv } S$ . This is clearly true for  $m = 2$  (we have  $v_1 = v_2 = v_3 = v_4$  in this case). We further proceed by induction on  $m$ . Let  $m \geq 3$ . We denote by  $x$  the point of

intersection of the diagonals  $s_1s_{m+1}, s_3s_{m+3}$  (see Fig. 7). By the induction

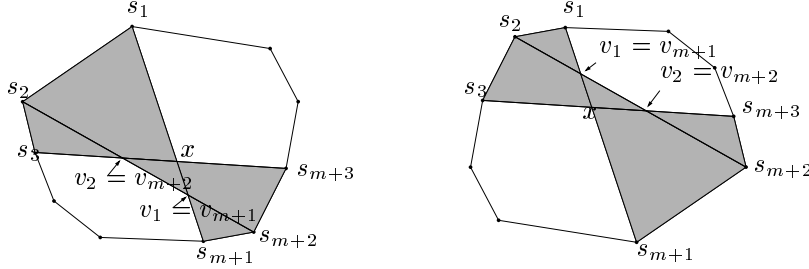


Figure 7: The point  $x$  and the main diagonals  $s_i s_{m+i}, i = 1, 2, 3$  (two combinatorially different cases). The shaded region is covered by  $V_1 \cup V_2 \cup V_{m+1} \cup V_{m+2}$ .

assumption,  $\text{conv}(S \setminus \{s_2, s_{m+2}\})$  is covered by the  $2m - 2$  triangles  $\Delta s_1 s_3 x, V_3, V_4, \dots, V_m, \Delta s_{m+1} s_{m+3} x, V_{m+3}, V_{m+4}, \dots, V_{2m}$ . The statement now follows from the following three facts (see Fig. 7): (a)  $\text{conv} S = \text{conv}(S \setminus \{s_2, s_{m+2}\}) \cup \Delta s_1 s_2 s_3 \cup \Delta s_{m+1} s_{m+2} s_{m+3}$ , (b)  $\Delta s_1 s_3 x \cup \Delta s_1 s_2 s_3$  is covered by  $V_1 \cup V_2$ , (c)  $\Delta s_{m+1} s_{m+3} x \cup \Delta s_{m+1} s_{m+2} s_{m+3}$  is covered by  $V_{m+1} \cup V_{m+2}$ .

For each triangle  $V_i$ , we now find a smaller triangle  $V'_i$  contained in  $V_i$  such that the interiors of the triangles  $V_i, V'_i$  contain the same number of points of  $P$ . Then  $V_i \setminus V'_i$  will contain no point of  $P$ . Since  $\text{conv} S = V_1 \cup \dots \cup V_{2m}$ , points of  $P$  inside  $\text{conv} S$  will lie in  $V'_1 \cup \dots \cup V'_{2m}$ .

For  $i = 1, \dots, m - 1$ , we define  $V'_i$  as the intersection of the triangles  $s_i s_{i+1} s_{2m}$  and  $s_i s_{i+1} s_{i+2}$ . For  $i = m + 1, \dots, 2m - 1$ , we define  $V'_i$  as the intersection of the triangles  $s_1 s_i s_{i+1}$  and  $s_{i-1} s_i s_{i+1}$ . Further, we define  $V'_m$  as the intersection of the triangles  $s_{m-1} s_m s_{m+1}$  and  $s_m s_{m+1} s_{m+2}$ , and we define  $V'_{2m}$  as the intersection of the triangles  $s_{2m} s_1 s_2$  and  $s_{2m-1} s_{2m} s_1$  (see Fig. 8).

Clearly, if  $i \in \{1, \dots, 2m\}$ , then  $V'_i \subset V_i$  and, by the choice of  $S$ , the interiors of  $V'_i, V_i$  contain the same number ( $a$  or  $b$ ) of points of  $P$ . Thus, as argued above, all points of  $P$  inside  $\text{conv} S$  lie in  $V'_1 \cup \dots \cup V'_{2m}$ . By the choice of  $S$ , the interiors of  $V'_m$  and  $V'_{2m}$  contain  $b$  points each, and the interior of any other triangle  $V'_i$  contains  $a$  points of  $P$ . Since the triangle  $s_1 s_2 s_{2m-1}$  contains  $V'_1$  and is disjoint from the interiors of all other triangles  $V'_i$ , its interior contains  $a$  points of  $P$ . At the same time, it contains  $c$  points of  $P$ . It follows that  $a = c$  and thus  $b = 0$ . The interior of  $V'_{2m}$  contains  $b = 0$

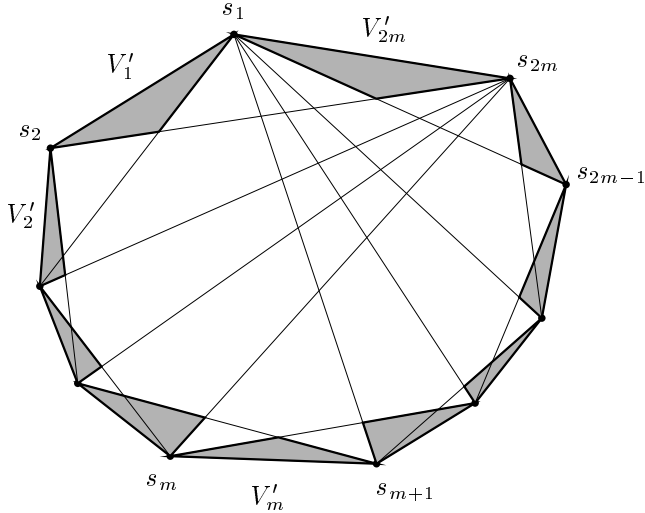


Figure 8: The triangles  $V'_i$ .

points of  $P$ . Therefore, each triangle  $s_1s_i s_{2m}, i = 2, \dots, 2m - 1$ , contains its  $a = c$  interior points belonging to  $P$  in its intersection with  $V'_i$ , which is the triangle  $T_i$ .  $\square$

## 5 Concluding remarks and open problems

### 5.1 The modular version of the Erdős-Szekeres theorem

Károlyi, Pach, and Tóth [8] studied 1-convex sets in relation to the following conjecture:

**Conjecture 1 (Bialostocki, Dierker, and Voxman [1])** *For any  $n \geq 3$  and  $p \geq 2$ , there exists an integer  $B = B(n, p)$  such that every set  $P$  of at least  $B$  points in general position in the plane determines a convex  $n$ -gon such that the number of points of  $P$  in its interior is divisible by  $p$ .*

Conjecture 1 was proved in [1] for every  $n \geq p + 2$ , Károlyi et al. [8] proved it for every  $n \geq 5p/6 + O(1)$ . In their proof, they applied the special case  $k = 1$  of Theorem 2. In fact, Theorem 2 gives Conjecture 1 for all  $n \geq 3p/4 + O(1)$ . This is shown in [10]. In [10] we also discuss a  $d$ -dimensional version of Conjecture 1 and prove it if  $d \geq 3$  is odd or  $n \geq n_0(p, d)$ .

## 5.2 A cap or an (empty) cup

We believe that point sets with no large holes have a very special structure. Therefore, it is interesting to seek conditions ensuring the existence of a hole of any fixed size in every sufficiently large point set in  $\mathbb{R}^d$ . One such condition ( $k$ -convexity) is given in Theorems 2 and 3.

We say that a set  $X$  of  $n$  points in the plane, no two on a vertical line, is an  $n$ -cap ( $n$ -cup, respectively), if it is in convex position and all points of  $X$  lie on and above (on and below, respectively) the line connecting the leftmost point of  $X$  with the rightmost point of  $X$  (see Fig. 9).



Figure 9: A cap and a cup.

In their classical paper [4], Erdős and Szekeres proved that any set of at least  $\binom{2n-4}{n-2} + 1$  points in general position in the plane, no two of them on a vertical line, contains an  $n$ -cap or an  $n$ -cup. From this they concluded that Theorem 1 holds with  $f(n) := \binom{2n-4}{n-2} + 1$ . We conjecture that actually the following stronger result holds:

**Conjecture 2** *For any  $n \geq 3$ , there is an integer  $f'(n)$  such that any set of at least  $f'(n)$  points in general position in the plane, no two of them on*

a vertical line, contains an  $n$ -cup or an empty  $n$ -cup. (An empty  $n$ -cup is an  $n$ -cup which forms an empty  $n$ -gon.)

Conjecture 2, if true, would imply that for any  $k$  and  $n$  any sufficiently large set of points in the plane with no  $k$ -cap contains an empty  $n$ -gon.

### 5.3 Strengthenings of Theorems 2 and 3

Theorems 2 and 3 can be generalized in the following two ways:

- the general position is not necessary,
- the statement is true also if the considered set is a union of a fixed number of  $k$ -convex sets.

These generalizations may be combined in one theorem:

**Theorem 6** *If  $d = 2$  or  $d \geq 3$  is odd, then, for any three positive integers  $k, l, n$ , there is an  $N = N(d, k, l, n)$  with the following property. Let  $P = P_1 \cup P_2 \cup \dots \cup P_l$  be a set of  $N$  points in  $\mathbb{R}^d$  such that for each  $i$  the convex hull of any  $d + 1$  points of  $P_i$  contains at most  $k$  other points of  $P_i$ . Then the set  $P$  has an  $n$ -hole.*

*Proof.* Suppose that  $d = 2$ , that  $k, l, n$  are three positive integers, and that  $N$  is very large. Let  $P = P_1 \cup P_2 \cup \dots \cup P_l$  be as in the theorem. Without loss of generality, we may suppose that the sets  $P_i$  are pairwise disjoint and that  $|P_1| \geq \frac{N}{l}$ . The set  $P_1$  cannot contain more than  $k + 3$  points on a line (otherwise the convex hull of three of them would contain more than  $k$  other points of  $P_1$ ). A greedy algorithm gives a set  $P'_1$  of  $N_1$  points of  $P_1$  in general position such that no segment determined by two points of  $P'_1$  contains other points of  $P_1$  and  $N_1$  is still very large. By the same proof as for Theorem 2, we can show that  $P'_1$  has a subset  $P''_1$  forming an empty  $N'_1$ -gon in  $P_1$ , where  $N'_1$  is still very large. If  $\text{conv } P''_1$  contains fewer than  $\frac{N'_1}{n}$  of points of  $P_2 \cup \dots \cup P_l$ , then  $P''_1$  contains  $n$  points forming an empty  $n$ -gon in  $P_1 \cup \dots \cup P_l$ . Otherwise, we consider the points of  $P \setminus P_1$  lying in  $\text{conv } P''_1$ . Their number is at least  $\frac{N'_1}{n}$ , which is still a very large number. If we repeat the same argument (at most)  $l - 1$  times, we obtain an empty  $n$ -gon in  $P_1 \cup \dots \cup P_l$  formed by  $n$  points of one of the sets  $P_2, \dots, P_l$ .

If  $d \geq 3$  is odd, we can proceed similarly, applying Theorem 3 instead of Theorem 2. □

## 5.4 A density statement does not hold

It does not suffice that 99% of the simplices determined by  $P$  are empty:

**Claim 7** *For any  $d \geq 2$  and for any  $\varepsilon > 0$ , there is an integer  $c = c(d, \varepsilon) \geq 3$  such that for any  $N \geq d + 1$  there is a set  $P$  of  $N$  points in general position in  $\mathbb{R}^d$  such that more than  $(1 - \varepsilon) \binom{|P|}{d+1}$  simplices determined by  $P$  are  $(d + 1)$ -holes (i.e., empty simplices), but the set  $P$  does not contain a  $c$ -hole.*

*Proof.* First, let  $d = 2$ . Suppose for simplicity that  $\frac{3}{\varepsilon}$  is an integer and that  $N$  is divisible by  $\frac{3}{\varepsilon}$ . Let  $H$  be a set of  $\frac{N}{3/\varepsilon} = \frac{\varepsilon N}{3}$  points in general position in the plane with no empty 7-gon (arbitrarily large sets with no empty 7-gon were constructed by Horton [7]). We shrink the second coordinates of points in  $H$  (without changing the combinatorial structure of  $H$ ) so that all lines determined by  $H$  will be almost horizontal. Further, we take  $\frac{3}{\varepsilon}$  copies of  $H$  rotated along a large circle (see Fig. 10). The union of these copies will

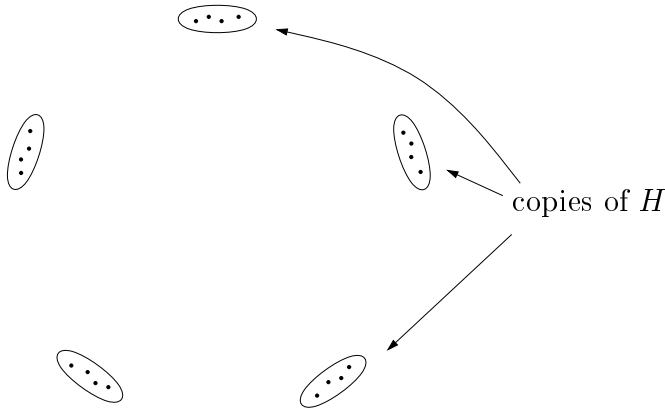


Figure 10:  $\frac{3}{\varepsilon}$  copies of  $H$  along a circle.

be our set  $P$ . It contains no empty convex  $(\frac{18}{\varepsilon} + 1)$ -gon (an empty polygon cannot contain 7 points from a copy of  $H$ ). Moreover, each triple of points from distinct copies of  $H$  is empty. There are more than

$$\frac{\frac{3}{\varepsilon} - 1}{\frac{3}{\varepsilon}} \cdot \frac{\frac{3}{\varepsilon} - 2}{\frac{3}{\varepsilon}} \binom{n}{3} > (1 - \varepsilon) \binom{n}{3}$$

such triples. This completes the case  $d = 2$ . If  $d > 2$ , we proceed similarly, using the construction of [9] of arbitrarily large sets of points in general position in  $\mathbb{R}^d$  with no  $c_d$ -hole, where  $c_d$  is a constant depending on  $d$  only.  $\square$

**Acknowledgment.** I am indebted to Gyula Károlyi for interesting and motivating discussions. Alex Below and Jesus De Loera explained me some properties of the cyclic polytope triangulations. I also thank the referees for their helpful suggestions.

## References

- [1] A. Bialostocki, P. Dierker, and B. Voxman, Some notes on the Erdős–Szekeres theorem, *Discrete Mathematics* 91 (1991), 231–238.
- [2] R. Cordovil and P. Duchet, Cyclic polytopes and oriented matroids, *Europ. J. Combinatorics* 21 (2000), 49–64.
- [3] P. Erdős, On some problems of elementary and combinatorial geometry, *Ann. Mat. Pura. Appl.* (4) 103 (1975), 99–108.
- [4] P. Erdős and Gy. Szekeres, A combinatorial problem in geometry, *Compositio Math.* 2 (1935), 464–470.
- [5] B. Grünbaum, *Convex polytopes*, Pure and Applied Mathematics, Vol. 16, John Wiley & Sons, New York, 1967.
- [6] H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, *Elem. Math.* 33 (1978), 116–118.
- [7] J. D. Horton, Sets with no empty convex 7-gons, *Canadian Math. Bull.* 26 (1983), 482–484.
- [8] Gy. Károlyi, J. Pach, and G. Tóth, A modular version of the Erdős–Szekeres theorem, *Studia Sc. Math. Hungar.* 38 (2001), 245–259.
- [9] P. Valtr, Sets in  $\mathbb{R}^d$  with no large empty convex subsets, *Discrete Mathematics* 108 (1992), 115–124.
- [10] P. Valtr, On the modular version of the Erdős–Szekeres theorem, in preparation.

**Note added in proof.** An alternative proof of Theorem 2 is given in the very recent manuscript “G. Kun and G. Lippner, Large convex empty polygons in  $k$ -convex sets”. The proof gives  $N(k, n) \leq 2^{(k+2)^{n+1}}$ .