A canonical Ramsey-type theorem for finite subsets of $\mathbb{N}$

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December 20, 2002

Abstract

T. Brown proved that whenever we color $\mathcal{P}_f(\mathbb{N})$ (the set of finite subsets of natural numbers) with finitely many colors, we find a monochromatic structure, called an arithmetic copy of an ω-forest.

In this paper we show a canonical extension of this theorem; i.e. whenever we color $\mathcal{P}_f(\mathbb{N})$ with arbitrarily many colors, we find a canonically colored arithmetic copy of an ω-forest. The five types of the canonical coloring are determined. This solves a problem of T. Brown.

1 INTRODUCTION

In [Br–00] T. Brown generalized the well known van der Waerden's theorem, for finite coloring of finite subsets of natural numbers, which we shall

[1] The autor aknowleges the support of ITI, the Project LN00A056 of the Czech Ministry of education.
keywords : Canonical coloring, forests, van der Waerden's theorem, arithmetic progression
2000 Mathematical Subject Classification. 05C55

state after introducing some definitions. For a formulation and a proof of the van der Waerden's theorem see [Ne–95].

Convention and notation: In this article we understand rooted trees and forests as partially ordered sets, where the roots are the minimal elements and the leaves, the maximal. By predecessor of a vertex x we understand any vertex y for which $x > y$. Then vertex x is a successor of vertex y. $x \wedge y$ is the biggest vertex for which x and y are the successors. If such vertex does not exist, it is not defined. By chain we understand any sequence of vertices $x_1 < x_2 < \ldots < x_k$.

Definition: (ω-trees, ω-forest) An *ω-tree* of height n is an infinite rooted tree such that any maximal chain has $n+1$ vertices and any vertex that is not a leaf has degree ω. An *ω-forest* of height n is a disjoint union of ω many ω-trees of height n.

Definition: (Arithmetic Copy) Let $\mathcal{F}$ be a finite (resp. infinite) rooted forest. An *arithmetic copy* of $\mathcal{F}$ in $\mathcal{P}(\{1,\ldots,n\})$ (resp. in $\mathcal{P}_f(\mathbb{N})$) is a subset $\mathcal{S}$ of $\mathcal{P}(\{1,\ldots,n\})$ (resp. of $\mathcal{P}_f(\mathbb{N})$) for which there exist natural numbers a, d and a bijection $\phi : V(\mathcal{F}) \to \mathcal{S}$ such that for all vertices $x, y \in V(\mathcal{F})$ hold:
1. $x \leq y \Leftrightarrow \phi(x) \subseteq \phi(y)$,
2. $\exists\, x \wedge y \Rightarrow \phi(x \wedge y) = \phi(x) \cap \phi(y)$,
3. $x \in F_1,\, y \in F_2,\, F_1 \neq F_2 \in \mathcal{F} \Rightarrow \phi(x) \cap \phi(y) = \emptyset$,
4. $(\forall\, x < y)\,(\nexists z,\ x < z < y) \Rightarrow |\phi(y) - \phi(x)| = d$,
5. x is a root of a tree in $\mathcal{F} \Rightarrow |\phi(x)| = a$.

Theorem 1.1 (Brown) *If we color $\mathcal{P}_f(\mathbb{N})$ by finitely many colors, then for every natural number n there exists a monochromatic arithmetic copy of an ω-forest of height n in $\mathcal{P}_f(\mathbb{N})$.*

The purpose of this paper is to show a generalization of the canonical van der Waerden's theorem in the same spirit as Brown generalized van der Waerden's theorem. This solves an open problem stated in [Br–00]. For the canonical van der Waerden's theorem see [ErGr–80].

In Section 2, we shall give some technical tools for the proof of the generalization of the canonical van der Waerden's theorem. Section 3 is devoted to this generalization. In section 4 we shall prove that the mentioned canonical coloring cannot be reduced to less than the five stated types.

2 TECHNICAL TOOLS

Definition: (Under-diagonal mapping) An *under-diagonal mapping* is any mapping $f : X \to X$ (the range and the domain are the same) for which $f(x) \leq x$ for all $x \in X$.

Definition: (Stair mapping) A *stair mapping* induced by an under-diagonal mapping $f : X \to X$ and a set $B \subseteq X$ is a mapping $g_{fB} : B \to B$ for which $g_{fB}(x) = \min\{y \in B;\ f(x) \leq y\}$.

Remark 2.1 *A stair mapping g_{fB} induced by a under-diagonal mapping f and a set B is itself an under-diagonal mapping and $f(x) \leq g_{fB}(x) \leq x$.*

Lemma 2.2 *For any natural number n there exists a natural number m such that for any under-diagonal mapping $f : \{0, \ldots, m\} \to \{0, \ldots, m\}$ there exists an arithmetic progression B of length n, such that the stair mapping g_{fB} induced by f and the arithmetic progression B is either a constant mapping or an identity on B.*

Proof of Lemma 2.2: First we remark that if g_{fA} is a constant mapping on A, then $g_{fA}(x) = minA$ for all $x \in A$. Therefore we may suppose that for every arithmetic progression B in $\{0, \ldots, m\}$ of length n $\quad g_{fB}$ is not a constant mapping on B. Now we want to find an arithmetic progression A of length n such that $g_{fA}(x) = x$ for all $x \in A$.

From our assumption, for every $k \in \{0, \ldots, m\}$ and for every $B = \{k + d; d \in \{0, \ldots, n\}\}$ there exists $x \in B \quad f(x) > k$; otherwise $g_{fB}(B) \equiv k = minB$. Set $r = \lfloor \frac{m}{n} \rfloor$. Now we build a sequence $(B_i)_{i=0}^{r}$, $B_i = \{a_i + d; d \in \{0, \ldots, n\}\}$,
$a_i = i(n + 1)$. We get that for every i there exists $x_i \in B_i$ such that $f(x_i) \in B_i$.

Let us define a coloring $\chi : \{0, \ldots, r-1\} \to \{0, \ldots, n\}$ such that $\chi(i) = t$ for $x_i = a_i + t$, where x_i is an arbitrary element of B_i, for which holds $f(x_i) \in B_i$. By using van der Waerden's theorem we get an arithmetic progression $\tilde{A} \subset \{0, \ldots, r - 1\}$ of length n, $\tilde{A} = \{p, p + d, \ldots, p + nd\}$, which is monochromatic. It means that we have an arithmetic progression $A \subset \{1,\dots,m\}$, $A = a_p+t, a_{p+d}+t, \dots, a_{p+nd}+t$ for which $g_{fA}(x_i) = min\{y \in A | f(x_i) \leq y\} = x_i$ for all $x_i \in A$, which we wanted to prove. $\square$

Proposition 2.3 *Let $\mathcal{L}$ be an ω-forest of height n and let $\chi : V(\mathcal{L}) \to \omega$ be a coloring of its vertices, such that no two vertices of the same height have the same color. Then there exists an ω-forest of height n $\mathcal{F} \subseteq \mathcal{L}$, such that the coloring χ restricted to $V(\mathcal{F})$ is injective.*

Proof: Let us first introduce some notation. For $s \in \{0,\dots,n\}$ let L_s denote the set of vertices of height s. We partition L_s into ω many classes l_{sk} of size ω such that two vertices are in the same partition class iff they have the same direct ancestor i.e. for $x \in l_{sk}$, $y \in l_{sk'}$ $\quad k = k'$ iff $x \wedge y \in L_{s-1}$. If $s = 0$ then we have only one class of partition and $l_0 = L_0$. Set

$$X_{sl} = \{i < \omega : \chi^{-1}(i) \cap L_s \neq \emptyset \wedge \chi^{-1}(i) \cap L_s \neq \emptyset\}.$$

For a fixed pair $i < j \in \{0,\dots,n\}$ we shall delete vertices of height i and j in such a way that we shall still have an ω-forest of height n and the colors of vertices of height i will not occur between the colors of the vertices of height j. Set

$$\begin{aligned} Y_h &= \chi^{-1}(X_{ij}) \cap l_{ih} \quad & Y_{hr} &= \{x \in Y_h : \exists y \in l_{jr}\ \chi(x) = \chi(y)\} \\ Z_k &= \chi^{-1}(X_{ij}) \cap l_{jk} \quad & Z_{kr} &= \{y \in Z_k : \exists x \in l_{ir}\ \chi(y) = \chi(x)\} \end{aligned}$$

Note that for $s, m < \omega$ $|Y_{sm}| = |Z_{ms}|$. We define a bijection $\varphi_{sm} : Y_{sm} \to Z_{ms}$ by $\varphi_{sm}(x) = y$, where y is the vertex in L_j having the same color as x.

For every pair (s,m), $s,m \in \omega$, we partition Y_{sm} into two classes Y^1_{sm} and Y^2_{sm}. If $|Y_{sm}| = 1$ then set $Y^1_{sm} = \emptyset$ if $s \leq m$ and set $Y^1_{sm} = Y_{sm}$ otherwise. If $|Y_{sm}| \neq 1$ then the partition classes have similar size, i.e. either both are empty or both are finite non empty or both are infinite. Let Z^1_{ms} and Z^2_{ms} be the induced partition of Z_{ms} i.e. $Z^r_{ms} = \varphi(Y^r_{sm})$ for $r = 1, 2$. Now we delete vertices of Y^1_{sm} and vertices of Z^2_{ms}.

We have now just to realize that no two vertices of the new $L_i \cup L_j$ have the same color, which gives us the injectivity of the coloring, and that from each l_{rl}, $r = i, j$; $l \in \omega$ we kept ω many vertices, which guarantees us that we still have an ω-forest after the operation. This proves Proposition 2.3. $\square$

Lemma 2.4 *For every natural number n there exists a natural number m such that for any matrix M with m rows and ω columns of natural numbers such that no two elements in the same row are the same, there exists an arithmetic progression $\mathcal{A}$ in $\{1, \ldots, m\}$ of length n and there exists a subset X of the columns whith $|X| = \omega$ such that for the submatrix $\tilde{M} := \mathcal{A} \times X$ (containing only the rows of $\mathcal{A}$ and the columns of X of the matrix M) holds that*

A) either no two numbers in the whole matrix $\tilde{M}$ are the same, or
B) $\exists f : X \to \mathbb{N}$ injective such that the element $\tilde{M}_{(i,j)} = f(j)$.

Proof: First we would like to find an infinite set $X \subseteq Y$ such that no element from a column is repeated in an other column. We show this by induction on $j \in \omega$.

For any column j, the numbers $M_{(i,j)}$, $i \in \{1, \ldots, m\}$, can be repeated only in a finite number of columns and therefore there exists $t = t(i, j) \in \omega$ such that for any $t \leq k \in \omega$ the number $M_{(i,j)}$ does not occur in the column k. Let R_j be the maximum of such $t(i, j)$ for all $i \in \{1, \ldots, m\}$. Let $s = min\ (\omega \setminus \{1, \ldots, R_j\})$ and we continue in the same way for the column $j = s$. Because we cannot stop after a finite numbers of steps (we shall always be able to find the required s), we get infinitely many such s and, therefore we get a set Y for which the following holds: in the submatrix $\{1, \ldots, m\} \times Y$, elements with the same number may occur only in the same column.

For every column i let $S_i \subseteq \mathcal{P}(\{1, \ldots, m\})$ be a such a partition of $\{1, \ldots, m\}$ that two elements are in the same cell of the partition iff they have the same number. The set S_i tells us what the pattern of the column i is; i.e. which rows are the same and which are not. For two columns i_1, i_2 whose have the same partition S of its rows, there exists a permutation π of the natural numbers such that $(j, i_1) = \pi((j, i_2))\ \forall j \in \{1, \ldots, m\}$; it is the two columns are the same up to a relabeling of their elements.

Having defined the sets S_i we define a coloring $\chi : Y \to 2^m$ by $\chi(i) = S_i$. The infinite Ramsey's theorem (for a formulation and proof see [NE–95]) guarantees us that there exists an infinite subset X of Y such that every column in X have a given structure S.

Now it is enough to use the canonical van der Waerden's theorem on the first column in X. We get an arithmetic progression $\mathcal{A}$ of length n such that the elements of the first column are either all different or all the same. As all columns in X have the same structure S, this holds for every column in X. Therefore we get a submatrix $\tilde{M} = \mathcal{A} \times X$ for which holds that

A) either no two element in $\tilde{M}$ are the same,

B) or there exists an injective mapping $f : X \to \mathbb{N}$ such that the elements $\tilde{M}_{(i,j)}$ have number $f(j)$, for $i \in \mathcal{A}$ and $j \in X$.

□

3 CANONICAL COLORING

We formulate now the main result of this paper. It is a canonical version of Brown's theorem 1.1: a generalization of the canonical van der Waerden's theorem.

Theorem 3.1 *For any coloring $\chi : \mathcal{P}_f(\mathbb{N}) \to \omega$ and any natural number n there exists an arithmetic copy $\mathcal{S}$ of an ω-forest $\mathcal{F}$ of height n, such that one of the following holds:*

A) χ restricted to $\mathcal{S}$ is injective.

B) χ restricted to S *is constant.*

C) $\chi(\phi(x)) = \chi(\phi(y)) \Leftrightarrow$ height of x = height of y.

D) $\chi(\phi(x)) = \chi(\phi(y)) \Leftrightarrow$ x and y are in the same tree.

E) $\chi(\phi(x)) = \chi(\phi(y)) \Leftrightarrow$ height of x = height of y and x and y are in the same tree.

Notation: Let $\binom{X}{k}$ denote the set of all k-element subsets of the set X and let $f \restriction A$ holds for the restriction of the mapping f to the set A.

Definition: (Constantly canonical) A coloring χ of a set system $\mathcal{S} \subseteq \mathcal{P}(X)$, where X is linearly ordered, is said to be *constantly canonical* iff there exists an index set V, such that for any natural number $r \leq |\mathcal{S}| \quad \chi \restriction \binom{\mathcal{S}}{r}$ is a V-canonical coloring.

Definition: (Injectively canonical) A coloring χ of a set system $\mathcal{S} \subseteq \mathcal{P}(X)$, where X is linearly ordered, is said to be *injectively canonical* iff there exists an injective mapping $g : \mathbb{N} \to \mathcal{P}_f(\mathbb{N}) \quad g : r \mapsto V_r$, such that $\chi \restriction \binom{\mathcal{S}}{r}$ is a V_r-canonical coloring.

By an ω-forest $\mathcal{F}_a$ in an ω-forest $\mathcal{F}_b$ corresponding to an arithmetic progression A we shall not understand any subgraph of $\mathcal{F}_b$, but an ω-forest defined as follows: For each ω-tree $\mathcal{T}_i \in \mathcal{F}_b$ we choose one vertex $v_i \in \mathcal{T}_i$ such that the height of v_i in $\mathcal{F}_b$ is the first element of the arithmetic progression A; these vertices will be the roots of $\mathcal{F}_a$. Having defined a vertex v in $\mathcal{F}_a$, let us reduce our attention to its successors in $\mathcal{F}_b$. These successors form an ω-forest. In each ω-tree of this forest choose one vertex which has the height of the next element in the arithmetic progression A in $\mathcal{F}_b$. We shall link these vertices with v by edges.

Proof of Theorem 3.1: By repeating the canonical Ramsey's theorem [RA–86] we get that for any natural number k there exists an infinite subset Y of $\mathbb{N}$ such that for any $r \le k \quad \chi \restriction \binom{Y}{r}$ is canonical. Now we shall construct an arithmetic copy $\tilde{S}$ of an ω-forest of height k such that $\tilde{S} \subseteq \mathcal{P}(Y)$. We use the following construction:

Construction:
Let us split Y into ω many sets Y_i of size ω. So we have: $Y = \bigcup_{i=1}^{\infty} Y_i$. The copy of the ith tree T_i will be a subset of $\mathcal{P}(Y_i)$ and the copy of every tree will be constructed in the same way as T_i. The root r_i of T_i will be mapped to $minY_i$. $\tilde{Y}_i = Y_i \setminus minY_i$ is infinite and can be split into ω many sets $\tilde{Y}_{ij}$ of size ω such that $\tilde{Y}_i = \cup_{j=1}^{\infty} \tilde{Y}_{ij}$. $T_i \setminus \{r_i\}$ is an ω-forest with ω-trees T_{ij} $j \in \omega$. The root r_{ij} of the tree T_{ij} will be mapped to the set $min\tilde{Y}_{ij} \cup minY_i$. Again $\bar{Y}_{ij} = \tilde{Y}_{ij} \setminus min\tilde{Y}_{ij}$ is infinite and $T_{ij} \setminus \{r_{ij}\}$ is an ω-forest. We continue in the same way until we get the required arithmetic copy.

For such an arithmetic copy $\tilde{S}$ the following holds:

$$\forall\, 1 \le r \le k\ \exists!\, j = j(r) \le r \quad \forall\, A, B \in \tilde{S},\ |A| = |B| = r$$

$$\chi(A) = \chi(B) \Leftrightarrow |A \cap B| \ge j$$

In words : for all natural number r smaller or equal to k there exists exactly one number j (depending on r and which is smaller or equal to r) such that for any two sets A, B of size r, A and B have the same iff only if the size of their intersection is at least j.

Let us define a function $f : \{0, \dots, k\} \to \{0, \dots, k\}$ such that for $r \ne 0$ $f(r) = j(r)$ for $r \ge 1$ and $f(0) = 0$. Now we can use Lemma 2.2 and we get an arithmetic progression $P \in \{1, \dots, k\}$ of length n' such that $g_{fP} \restriction P$,the stair mapping induced by f and P, is either (a) an identity or (b) a constant mapping.

We have to realize that $g_{fP}(r) = \bar{j}$, where $\bar{j}$ is the only element in P for which:$\forall A, B \in \bar{S}\ |A| = |B| = r \quad \chi(A) = \chi(B) \Leftrightarrow |A \cap B| \ge \bar{j}$. The stair mapping g_{fP} shows us how the coloring of any arithmetic copy $\bar{S} \subseteq \tilde{S}$ of an ω-forest $\mathcal{F}$, corresponding to P, is structured. Let us color ω-forest $\mathcal{F}$ so that the bijection between it and the arithmetic copy will preserve the coloring.

(a) If $g_{fP} \restriction P$ is an identity, then any two sets of the same size have different colors, i.e. any two vertices with same height in the ω-forest $\mathcal{F}$ have different colors. So using Proposition 2.3 we get A).

(b) $g_{fP} \restriction P$ is a constant mapping: We have to distinguish two cases:

1) If $g_{fP}(P) = 0$ then the coloring depends only on the size of the set. So for the coloring of the ω-forest $\mathcal{F}$, all the vertices with height i have the same color, say c_i. If we define a coloring $c : \{0, \ldots, n'\} \to \omega$ such that $c(i) = c_i$ we can apply the canonical van der Waerden's theorem on this coloring and we get an arithmetic progression P' of length n on which the coloring c is either constant or injective.

Let $\mathcal{F}'$ be an ω-forest in $\mathcal{F}$ corresponding to P'. Then $\mathcal{F}'$ is either monochromatic or each height has a different color. Its associated arithmetic copy implies cases B) or C).

2) If $g_{fP}(P) \neq 0$ then the coloring depends on two things: the size of the set and its first element. This implies that the coloring of a vertex of $\mathcal{F}$ depends on its height and on the tree in which it lies.

Let us define a coloring $c_0 : \omega \to \omega$ by $c_0(i) = b_i$, $b_i \in \omega$, if the root of the ith ω-tree has the color b_i. Using the canonical Ramsey's theorem we get a set B_0 of ω many ω-trees such that they have either roots of same color or each one has a root of a different color. Having defined a set B_j, $j \leq n'$, of ω many ω-trees we define a coloring $c_{j+1} : \omega \to \omega$ by $c_{j+1}(i) = d_i$, $d_i \in \omega$, if the vertices of the ith ω-tree at height $j+1$ have the color d_i. Again using the canonical Ramsey's theorem we get a set B_{j+1} of ω many ω-trees such that either all vertices at height $j+1$ of the ω-forest on B_{j+1} have the same color or two vertices at height $j+1$ of this ω-forest have the same color iff they are in the same ω-tree.

We get an ω-forest on $B_{n'}$ such that for every height i either the vertices have the same color or they have different colors from one ω-tree to another. Let us define a coloring $\tilde{c} : \{0, \ldots, n'\} \to \{0, 1\}$ by $\tilde{c}(i) = 0$ if the vertices of height i are monochromatic and $\tilde{c}(i) = 1$ if two vertices of height i have the same color iff they are in the same tree. If we use van der Waerden's theorem we get a monochromatic arithmetic progression $\tilde{P}$ of height $\tilde{n}$ in

$\{0, \dots, n'\}$. Let $\mathcal{F}'$ be the ω-forest in $\mathcal{F}$ corresponding to $\tilde{P}$.

2a) If $\tilde{P}$ has color 0 then we have an ω-forest in which each height is monochromatic. We can map the height to the color it has and using the canonical van der Waerden's theorem we get an arithmetic progression either monochromatic or with an injective coloring. This yields the cases B) or C).

2b) If $\tilde{P}$ has color 1 then we define an equivalence $\sim$ on the ω-forest $\mathcal{F}'$ such that $x \sim y$ iff x and y have the same height and belong to the same ω-tree. Then $\mathcal{F}'/\sim$ is isomorphic to a matrix $n \times \omega$ with elements of $\mathbb{N}$ such that no two numbers (=colors) in the same row are the same. Therefore we can use Lemma 2.4 which implies that for an arithmetic progression $\bar{P}$ of length n and for a set $X \in \omega$ we have that for the matrix $\bar{P} \times X$ holds: ♣ Either no two numbers in the whole matrix are the same or ♠ there exists an injective mapping $f : X \to \mathbb{N}$ such that all the elements of the jth column have number $f(j)$.

♣ The ω-forest corresponding to the arithmetic progression $\bar{P}$ restricted to the ω-tree of X has the property that two vertices have the same color iff they have the same height and belong to the same ω-tree. Let S be the image of this ω-forest in $\bar{S}$, then E) holds for S.

♠ The ω-forest corresponding to the arithmetic progression $\bar{P}$ restricted to the ω-tree of X has the property that two vertices have the same color iff they belong to the same ω-tree. Let S be the image of this ω-forest in $\bar{S}$, then D) holds for S. □

4 Canonical colorings are minimal

Next theorem shows us that Theorem 3.1 cannot be improved, i.e. we cannot take away any of the types of canonical coloring.

Proposition 4.1 *There exist colorings* $\chi_A, \chi_B, \chi_C, \chi_D, \chi_E : \mathcal{P}_f(\mathbb{N}) \to \omega$ *such that case J (which stands for one of the cases A),B),...,E) in theorem 3.1) occurs but none of the cases A),B),...,E) distinct of J can occur for coloring* χ_J.

Proof: Coloring χ_A: None of the cases B),C),D) and E) can occur for any injective coloring of $\mathcal{P}_f(\mathbb{N})$and any arithmetic copy of height n will imply case A).

Coloring χ_B: None of the cases A),C),D) and E) can occur for any constant coloring of $\mathcal{P}_f(\mathbb{N})$, and any arithmetic copy of height n will imply case B).

Coloring χ_C: Let us define the coloring χ_C by $\chi_C(A) = |A|$, where A is any finite subset of $\mathbb{N}$. It is clear that χ_C cannot be injective nor constant on any arithmetic copy. The colorings in cases D) and E) depend on the trees and therefore cannot happen as any two sets of the same size have the same color not depending on whether they intersect or not. Any arithmetic copy of height n implies case C).

Coloring χ_D: Let us define the coloring χ_D by $\chi_D(a) = minA$, where A is any finite set of $\mathbb{N}$. χ_D cannot be injective for any arithmetic copy of height n, as, if we consider a vertex x and vertices $y_1, y_2, y_3, \ldots$, its immediate successors, the following holds in any arithmetic copy:

$$\phi(x) = \bigcap_{j=1}^{\infty} \phi(y_j).$$

In the same time $min\, \phi(y_i) \neq min\, \phi(y_j)$ for only finitely many $i \neq j$, as there are only finitely many elements smaller or equal to $\phi(x)$. This implies that there will always be ω many y_i for which $\chi_D(\phi(x)) = \chi_D(\phi(y_j))$. For the same reason neither of the cases C) and E) can occur. χ_D cannot be constant on any arithmetic copy, as any two finite sets belonging to the image of different trees are disjoint and therefore cannot have the same minimal number. If we follow the construction on page 9 we get an arithmetic copy for which case D) holds.

Coloring χ_E: Let us define the coloring χ_E by $\chi_E = (minA, |A|)$, where A is again any finite subset of the set $\mathbb{N}$. Let us use the same notation as for the coloring χ_D. All $\phi(y_j)$ will have colors different from $\chi_E(\phi(x))$, which implies that cases B) and D) cannot occur. On the other hand, there exists a color c such that for ω many y_j holds that $\chi_E(\phi(y_j)) = c$. This implies that case A) cannot occur. Next, it is enough to realize that any two disjoint finite subsets of $\mathbb{N}$ have different colors to see that case C) cannot occur. Finally, if we follow the construction on page 9 we get an arithmetic progression of length n for which case E) holds. $\square$

5 OPEN PROBLEMS

It would be interesting to consider the combination of the polynomial van der Waerden's theorem with trees and lattices in the spirit of [Br–00], [BeLe–99] and [NeRo–84] and their canonical versions. For a proof of the polynomial extension of van der Waerden's theorem see [BeLe–96].

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