On Ground State Incongruence in Spin Glasses

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A construction supporting a conjecture that different ground state pairs exist in the 2-dimensional Edwards-Anderson Ising spin glass is presented.

The general Edwards-Anderson Ising spin glass model [1] on a graph $G = (V, E)$ is defined by coupling constants J_{ij} assigned to each edge $\{i, j\}$ of G. We will assume that J_{ij}'s are independently chosen from a mean zero Gaussian distribution. A physical state of the system is given by a *spin assignment* $\sigma : V \to \{\pm 1\}$ which has the corresponding energy

$$E(\sigma) = -\sum_{\{i,j\}\in E} J_{ij}\sigma_i\sigma_j.$$

A state is *groundstate* if its energy cannot be lowered by changing an arbitrary finite set of spins. Groundstates exist for the square lattice $\mathcal{S}$ and arbitrary coupling constants J_{ij} assigned to its edges by a compactness argument. Note that if we reverse all spins in a groundstate we again get a groundstate. Let us call these pairs *groundstate pairs*, or GSPs. As Newman and Stein write in [7], a fundamental problem in spin glass physics is to determine the multiplicity of infinite volume groundstates in finite-dimensional short-ranged systems, such as Edwards-Anderson Ising spin glass. In 1D, all edges may be satisfied, hence there is no frustration and only a single GSP exists. The mean-field model predicts many GSPs for the square lattice $\mathcal{S}$ ([2], [3]) while the droplet-scaling theories predict that only one GSP exists ([4], [5], [6]). The droplet-scaling picture has received support from seminal analytic work [7]. In this letter we present a combinatorial construction supporting the competing mean-field picture. Newman and Stein ([8], [7]) formalize the problem by means of *metastates.* A metastate has been introduced as a translation invariant measure constructed as follows: for each finite subsquare S_L of $\mathcal{S}$ with *periodic boundary conditions* consider the joint distribution of coupling constants and GSPs in S_L. When L goes to infinity, by compactness, there is a subsequence of L's so that the joint distributions converge to translation-invariant (since periodic boundary conditions are imposed) joint measure. The metastate induces a translation invariant measure on the sets of edges satisfied by exactly one of two GSPs (edge ij is *satisfied* by spin assignment σ if $J_{ij}\sigma_i\sigma_j > 0$). The connectivity components of such a set in the dual lattice are sometimes called 'domain walls'. The main question may be formulated in this setting as follows:

Is it possible that two randomly chosen GSPs (from the same metastate or from different metastates) are incongruent?

Two GSPs are called *incongruent* in [7] if the symmetric difference of their sets of satisfied edges has a positive density. Note an important fact (see Lemma 2 of [7]) which follows from the translation invariance: if two randomly chosen GSPs from a metastate are distinct, then with probability one they are incongruent. In theory different subsequences may lead to different metastates. Newman and Stein give support in [7] to the conjecture that no incongruent GSPs exist. They consider the case of two GSPs taken at random from the same metastate and show several properties. In particular, if the two GSPs are distinct, there is exactly one domain wall between them and it is a both-ways-infinite path. Newman and Stein consider this situation unlikely (such domain wall must look like a space-filing curve). Moreover Newman and Stein express belief that all the metastates are the same. In this letter we consider the finite sublattices with different than periodic boundary conditions: we fix the spins along the boundary of the finite sublattice so that maximum number of the edges of the boundary are satisfied. Note an important fact: a state of minimum energy with these boundary conditions neednot be a groundstate. We will call it a *c-groundstate*. We will also define incongruency in a different way: we say that two states are *finitely incongruent* if at least one of the domain walls between them contains an infinite both ways unbounded path. In this setting the main incongruency question may be formulated as follows:

Conjecture: For almost all couplings assignments J of $\mathcal{S}$ there is a nested sequence $S_i, i = 1, 2, ...$ of finite sublattices converging to $\mathcal{S}$ so that if e_j is a c-groundstate in S_{2j} and o_j is a c-groundstate in S_{2j-1} then both sequences (o_j), (e_j) converge and their limits o, e are weakly incongruent states.

Note that o, e are groundstates of the square lattice since the boundary conditions 'disappear to the infinity'. Also note that the boundary conditions mean that all or all but one boundary edges are satisfied depending on the parity of the number of edges with negative coupling constants. The aim of this letter is to prove a weaker statement (Theorem below) and argue that its proof may serve as a strategy for attacking the Conjecture.

Theorem. Let (S_i) be a nested sequence of finite sublattices monotonically converging to the square lattice and such that for each k there are sufficiently many lattices with width k. Let $\mathcal{J}_i$ be the distribution of the coupling constants in S_i. Then for almost all $(J_i)_{i\geq 1}$ from $(\mathcal{J}_i)_{i\geq 1}$ *there is a converging subsequence* (J_{m_j}) with the following properties: if e_j is a c-groundstate in S_{m_j} and o_j is a c-groundstate in S_{m_j-1} (with coupling constants given by J_{m_j}) then both sequences (o_j), (e_j) converge and their limits o, e are weakly incongruent states.

We start by considering the strip lattice $\mathcal{C}_k$: the vertical coordinates of its vertices are arbitrary integers, and its horisontal coordinates run through integers from $-k$ to k. As an introduction to our technics we show that the strip lattice satisfies the Conjecture. Let $C(n,k)$ be the finite induced subgraph of $\mathcal{C}_k$ with the vertices $(i,j) : |j| \leq n$. The basic building blocks of strip and square lattices are unit squares called *plaquettes*. A plaquette is *frustrated* if it has odd number of edges (out of 4) with negative coupling constants. Observe that a plaquette is frustrated if and only if arbitrary state satisfies an odd number of its edges. We will define graph $C(n,k)^*$ whose vertices are all the plaquettes of $C(n,k)$ and the edges are all edges e^* such that e is an edge of $C(n,k)$ not on the boundary; edge e^* connects two plaquettes p, q such that edge e lies on the boundary of p and q.If A is a set of edges of $C(n,k)$ then let A^* denote the set of 'dual' edges: $A^* = \{e^*; e \in A\}$.Let $n > m$, and consider the graph $D(n,m,k) = C(n,k) - C(m-1,k)$. Note that $D(n,m,k)$ has two connectivity components, each of them consists of $n-m$ levels of plaquettes. The two components of $D(n,m,k)$ are naturally called *upper* and *lower* and denoted by $DU(n,m,k)$ and $DL(n,m,k)$. We will use the following key observation:

Lemma 1. Let both boundaries of $C(n,k)$ and $C(m,k)$ have an even number of edges with negative coupling constants, and moreover assume that each of the two components of $D(n,m,k)$ has odd number of frustrated plaquettes. Let $c(i)$ be a c-groundstate of $C(i,k)$ and let $DIS(c(i))$ denote the set of edges dissatisfied by $c(i)$, $i = n, m$.Then the symmetric difference $DIS(c(n))^* \Delta DIS(c(m))^*$ contains a path from a plaquette of $DU(n,m,k)$ to a plaquette of $DL(n,m,k)$.

Proof. The subgraph formed by $DIS(c(i))^*$ induces odd degree in each frustrated plaquette and even degree in each happy plaquette of $C(i,k)$. Moreover for $i = m, n$, $DIS(c(i))$ contains no edge of the boundary of $C(i,k)$. Hence $DIS(c(n))^* \Delta DIS(c(m))^*$ induces odd degree in each frustrated plaquette of $D(n,m,k)$, and even degree in arbitrary other plaquette of $C(n,k)$. This easily implies the Lemma.

Now we are ready to show that the Conjecture holds for the strip lattice: Let $S_i = C(i,k)$. Clearly, for almost all coupling constants assignments J in the whole strip lattice there is a sequence (m_j) so that for each j, both S_{m_j} and S_{m_j-1} (with the coupling constants given by J) have an even number of boundary edges with the negative coupling constants and both $DU(m_j, m_j-1, k)$ and $DL(m_j, m_j-1, k)$ have an odd number of frustrated plaquettes. Let o_j be a c-groundstate in S_{m_j} and let e_j be a c-groundstate in S_{m_j-1}. Lemma 1 implies that for each j, $DIS(o_j)^*\Delta DIS(e_j)^*$ contains a path P_j of length at least $2j+1$.Now it is easy to see that

Claim 1. There is a subsequence P_{n_j} that converges to both ways infinite path P.

By compactness there is a subsequence (p_j) of (n_j) so that both sequences (e_{p_j}) and (o_{p_j}) converge. Let the respective limits be e and o. Then necessarily P is a subset of a domain wall between e and o and so e, o are weakly incongruent. Hence the Conjecture holds for the strip lattice.

If we want to apply the same proof method to the square lattice, we encounter difficulties with the Claim. It holds if we have an infinite subsequence (P'_i) of (P_i) such that each P'_i contains an edge in at most a fixed distance (say 100) from the origin. We are able to prove a 'pinning lemma' (see below) which simply implies the Theorem. We reformulate the lemma dually in Lemma 2 which we prove. Unfortunately our present proof gives only a very weak lower bound to the desired probability. We believe that proper understanding to Lemma 2 may prove the full Conjecture.

Pinning Lemma. There is a function c from positive integers to $(0,1)$ so that for each k and $n > k$, the probability of the following event $A(n,k)$ is at least $c(k)$.

$\mathbf{A(n,k)}$: Choose coupling constants at random in $C(n,k)$. We say that the instance belongs to $A(k,n)$ if both $C(n,k)$ and $C(n-1,k)$ have an even number of negative edges on the boundary, each of $DU(n,n-1,k)$ and $DL(n,n-1,k)$ has an odd number of frustrated plaquettes and there is a path P in $DIS(r)^*\Delta DIS(s)^*$ from a frustrated plaquette of $DU(n,n-1,k)$ to a frustrated plaquette of $DL(n,n-1,k)$ which contains an edge in distance at most 100 from the origin. Here r is a c-groundstate in $C(n-1,k)$ and s is a c-groundstate in $C(n,k)$.

Proof of the Theorem. Let $S_i = C(l_i, p_i), l_i > p_i$ and for each k there are sufficiently many lattices with $p_i = k$. The Theorem follows from Claim 2 below in the same way as the Conjecture for the strip lattices follows from Claim 1.

Claim 2. For almost all $(J_i)_{i\geq 1}$ from $(\mathcal{J}_i)_{i\geq 1}$ there is a subsequence $(C(n_i,k_i))$ of (S_i) so that both $C(n_i-1,k_i)$ and $C(n_i,k_i)$ (with the coupling constants given by J_{n_i}) have an even number of negative edges on the boundary, each of $DU(n_i,n_i-1,k_i)$ and $DL(n_i,n_i-1,k_i)$ has an odd number of frustrated plaquettes and there is a path P_i in $DIS(o_i)^*\Delta DIS(e_i)^*$ from a frustrated plaquette of $DU(n_i,n_i-1,k)$ to a frustrated plaquette of $DL(n_i,n_i-1,k)$ which contains an edge in distance at most 100 from the origin.Here o_i is a c-groundstate in $C(n_i-1,k_i)$ and e_i is a c-groundstate in $C(n_i,k_i)$.

Proof. Let k be an arbitrary positive integer. Since $p_i=k$ for sufficiently many i's, we know by the Pinning Lemma that the probability that one of $C(l_i,k)$ satisfies the properties of Claim 2 is very large. Hence the set of instances $(J_i)_{i\geq 1}$ from $(\mathcal{J}_i)_{i\geq 1}$ for which the propertiess of Claim 2 donot hold has measure zero. This finishes the proof of Claim 2 and the Theorem.

Claim 2 is a consequence of Pinning Lemma and *the fact that in each S_i we assign the coupling constants independently.* The remaining obstacle in proving the full Conjecture is that because of dependancies Pinning Lemma cannot be used independently in each S_i. We believe that understanding the Pinning Lemma may overcome this difficulty. It seems natural to formulate the Lemma as a property of the dual lattices. We start by listing some simple properties of lattices $C(n,k)$.

- $C(n,k)$ has an even number of negative coupling constants on the boundary if and only if it has an even number of frustrated plaquettes.
- A set R of edges of $C(n,k)$ not on the boundary is the set $DIS(r)$ of the dissatisfied edges of a state r (not necessarily a groundstate) if and only if R has an odd number of edges from each frustrated plaquette and an even number of edges from any other plaquette.
- A state r is a c-groundstate if and only if it satisfies the boundary conditions and $\sum_{(ij)\in DIS(r)}|J_{ij}|$ is as small as possible. Hence there is a natural bijection between the c-groundstate pairs of $C(n,k)$ and the sets A of edges not on the boundary and satisfying: a plaquette has an odd number of edges of A if and only if it is frustrated, and $\sum_{(ij)\in A}|J_{ij}|$ is as small as possible.

This means that regarding the Pinning Lemma we need only a subset of information given by the coupling constants: we need to know the value $|J_{ij}|$ for each edge (ij) not on the boundary, and we need to know which plaquette is frustrated. Each plaquette is equally likely to be frustrated or happy. If $\mathcal{J}$ is our distribution of coupling constants then we denote by $|\mathcal{J}|$

the distribution of their absolute values. We are interested only in those $C(n,k)$ that contain an even number of frustrated plaquettes. Hence instead of choosing the coupling constants from $\mathcal{J}$, we can choose them from $|\mathcal{J}|$ and choose uniformly at random an even set of plaquettes which we want to be frustrated. This means that the Pinning Lemma is about $C(n,k)^*$ rather than about $C(n,k)$. $C(n,k)^*$ is also a square grid, of width $2k$ and height $2n$. We need one more definition. Let $G=(V,E)$ be a graph and let T be a subset of an even number of vertices of G. We say that a set A of edges of a graph $G=(V,E)$ is *a T-join* if each vertex x of G is incident with an even number of edges of A if and only if $x \notin T$. Taking these considerations into account, note that the following implies the Pinning Lemma.

Lemma 2. There is a function g from positive integers to $(0,1)$ so that for each k and $n>k$, the probability of the following event $B(n,k)$ is at least $g(k)$.

$\mathbf{B(n,k)}$: Choose absolute values of coupling constants of $C(n,k)$ from $|\mathcal{J}|$. Choose an even subset T of vertices of $C(n-1,k) \subset C(n,k)$ uniformly at random. Choose a set X, $|X|$ odd, from $\{[i,n];|i| \leq k\}$, and a set Y, $|Y|$ odd, from $\{[i,-n];|i| \leq k\}$ uniformly at random. Choose a minimum T-join r in $C(n-1,k)$ and a minimum $(T \cup X \cup Y)$-join s in $C(n,k)$. Note that $r \Delta s$ contains a path P from X to Y. We say that the described instance belongs to $B(n,k)$ if P contains an edge in distance at most 100 from the origin.

Note that the statement of Lemma 2 is simply true for the cyllindrical lattices and for $g(k)=(2k+1)^{-1}$ by the translation invariance.Eventhough $g(k)$ may well be a constant, at present we are able to prove only a very bad inverse exponential lower bound for it. We believe that proper understanding to Lemma 2 may prove the full Conjecture.

Proof of Lemma 2. We will consider set $\mathcal{K}$ of *configurations* with joint distribution $\mathcal{U}_k$. A configuration is a quadruple (J,T,x,y) where J consists of the coupling constants, T is an even subset of vertices of $C(n-1,k)$ and $x=[i,n], y=[j,-n]$ is a particular choice of sets X,Y. We show that there is a function F from $\mathcal{K}$ to itself such that

- $\mathcal{U}_k(F(\mathcal{K})) \geq c_k > 0$, and
- Each $L \in F(\mathcal{K})$ is *positive*, i.e. there is a minimum T-join r in $C(n-1,k)$ and a minimum $T \cup \{x,y\}$-join s in $C(n,k)$ such that path P from x to y in $r \Delta s$ contains an edge in distance at most 100 from the origin.

Fix an arbitrary positive configuration K_0. Let $K=(J,T,x,y)$ be a configuration. If there is an edge e incident to a vertex $[i,j]$ with $|i| \leq 4$,

$|j| \le k$ and $|J_e| \ge 1$ then let $f(K) = K_0$, otherwise let P, r, s be as in the statement of Lemma 2. If P passes in distance at most 100 from the origin then let $f(K) = K$. Otherwise let m be the smallest positive integer such that P contains a vertical edge with the x-coordinate of its vertices equal to m or $-m$ and with the absolute value of both y-coordinates at most 4. Note that P has no vertex $[i,j]$ with $|i| < m$ and $|j| < 4$. Let Z be the graph induced on the vertices $[i,j]; |i| \le m, |j| \le 4$. We let $K = K_1$, $P = P_1$, $r = r_1$, $s = s_1$, $Z = Z_1$, $T = T_1$, $m = m_1$, let n_1 be the number of vertical edges of $P \cap Z$ and let p_1 be the number of (all) edges of $P \cap Z$.

Next we describe a procedure with input $I_i = (K_i = (J_i, T_i, x, y), P_i, r_i, s_i, Z_i, m_i, p_i)$ which produces $F(K)$ or I_{i+1}.

The Procedure. If $r_i \Delta s_i$ contains a cycle then let $F(K) = K_0$. Otherwise let H_i be the segment of $P_i \cap Z_i$ defined as follows: If P_i contains a vertical edge $e = \{[z,a],[z,b]\}$ so that $|z| = m_i$ and $|a| < |b| < 4$ then let H_i consist of e. If P_i contains no such vertical edge but it does contain a horisontal edge $e = \{[a,z],[b,z]\}$ so that $|z| = 4$ and $|a| < |b| < m_i$ then again let H_i consist of e. Finally let there be only 'corner' edges in $P_i \cap Z_i$. Let e be such vertical edge (it exists by the choice of m_i), $e = \{[z,a],[z,b]\}$, $|z| = m_i$ and without loss of generality $z = -m_i, a = 3, b = 4$. Then we let H_i consist of e if $\{[z,4],[z-1,4]\} \notin P_i$, and e together with $\{[z,4],[z-1,4]\}$ otherwise. Let W_i be the set of edges of a path in Z_i between the end-points of H_i such that it contains some edges in distance at most 100 from the origin, no vertical edge of the boundary of Z_i, and as few horisontal edges of the boundary of Z_i as possible (i.e. at most two). Let M_i be an integer upper bound of the coupling constants of the edges incident with a vertex of Z_i. For instance $M_1 = 1$. For each edge e of Z_i such that $e \notin W_i \cup (P_i - H_i)$ we let $(J_{i+1})_e = (J_i)_e + 100kM_i$, and we let $(J_{i+1})_e = (J_i)_e$ otherwise.

T_{i+1} is defined as folows: let r'_i be obtained from r_i by deleting all the edges of $r_i \cap s_i$ which belong to Z_i. Analogously define s'_i. Let U_i^b (U_i^0 respectively) be the set of vertices of $Z_i - T_i$ ($Z_i \cap T_i$ respectively) such that we deleted an odd number of edges of r_i incident with them. We let $T'_{i+1} = (T_i - U_i^o) \cup U_i^b$. Observe that T'_{i+1} has no vertices in the interior of Z_i and r'_i is a T'_{i+1}-join and s'_i is a $T'_{i+1} \cup \{x,y\}$-join. If $H_i \subset r'_i$ or $H_i \subset s'_i$ then let $T_{i+1} = T'_{i+1}$ else necessarily H_i contains two edges incident to a 'corner vertex' of Z_i and without loss of generality assume that the vertical edge of H_i belongs to r'_i. In this case T_{i+1} is obtained from T'_{i+1} by changing the status of both vertices of the horisontal edge of H_i. We also modify r'_i and s'_i so that we delete the horisontal edge of H_i from s'_i and add it to r'_i. Observe that T_{i+1} has no vertices in the interior of

Z_i and r'_i is a T_{i+1}-join and s'_i is a $T_{i+1} \cup \{x,y\}$-join. Moreover $H_i \subset r'_i$ or $H_i \subset s'_i$, $r'_i \Delta s'_i = P_i$ and $r'_i \cap s'_i$ has no edges in Z_i. Without loss of generality assume $H_i \subset r'_i$. Let r''_i be obtained from r'_i by exchanging H_i for W_i. Clearly r''_i is a T_{i+1}-join, $r''_i \Delta s'_i$ is a path P'_i obtained from P_i by exchanging H_i for W_i and $E(s'_i) \leq E(s_i)$ and $E(r''_i) \leq E(r_i) + 32M_i k$. Let $K_{i+1} = (J_{i+1}, T_{i+1}, x, y)$, r_{i+1} be a minimum T_{i+1}-join, s_{i+1} be a minimum $T_{i+1} \cup \{x,y\}$-join, and let P_{i+1} be the x,y-path in the symmetric difference of r_{i+1}, s_{i+1}. If P_{i+1} contains an edge in distance at most 100 from the origin then let $F(K) = K_{i+1}$ otherwise we output vector $(K_{i+1} = (J_{i+1}, T_{i+1}, x, y), P_{i+1}, r_{i+1}, s_{i+1}, S_{i+1}, m_{i+1}, n_{i+1}, p_{i+1})$. This finishes the describtion of the Procedure.

Now observe that $R = r_{i+1} \Delta (r_i \Delta r'_i)$ is a T_i-join such that R and r_{i+1} differ only on the edges of Z_i. Hence r_{i+1} cannot contain an edge of $Z_i - (W_i \cup (P_i - H_i))$ since otherwise $E(r_{i+1}) \geq E(R) - 32kM_i + 100kM_i \geq E(R) + 60kM_i \geq E(r_i) + 60kM_i > E(r''_i)$. Hence r_{i+1} contains all the edges of W_i of the interior of Z_i or none of them, and its edges from the boundary of Z_i form a subset of $Z_i \cap [(P_i - H_i) \cup D]$, where D is non-empty only if H_i has a vertical 'corner' edge and then D consists of one or two horisontal edges (by the definition of W_i). The same holds for s_{i+1}. If H_i has at least one vertical edge of Z_i then we have that $m_i < m_{i+1}$ or $m_i = m_{i+1}, n_i > n_{i+1}$. If H_i has no vertical edge then it consists of exactly one horisontal edge and all the edges of W_i belong to the interior of Z_i. Hence r_{i+1} contains all the edges of W_i or none of them and the edges of r_{i+1} from the boundary of Z_i belong of $P_i - H_i$. The same holds for s_{i+1}. Hence we have $m_i < m_{i+1}$ or $m_i = m_{i+1}, n_i = n_{i+1}, p_i > p_{i+1}$. Summarising, $m_i < m_{i+1}$ or $m_i = m_{i+1}, n_i > n_{i+1}$, or $m_i = m_{i+1}, n_i = n_{i+1}, p_i > p_{i+1}$.

Hence after at most $(16k)^2$ repetitions of the Procedure we have $F(K)$ defined. Moreover $F(K)$ is defined only if the path with desired properties exists. Finally $c_k > 0$ exists since the set $\mathcal{Z}$ of configurations with $|J_{ij}| < 1$ for $|i| \leq 4, |j| \leq k$ clearly satisfies $\mathcal{U}_k(\mathcal{Z}) \geq c'_k > 0$ and with probability one $L = F(K)$ from $\mathcal{Z}$ may be viewed as $K + \alpha_K$, where each component of α_K is a bounded integer and the number of possible α_K's is bounded from above by the number of paths on vertices $[i,j], |i| \leq k, |j| \leq 4$ (which is a modest function of k). This finishes the proof.

We believe that the construction described in the letter shows a possibility how multiple groundstates may exist in spin glasses.

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[1] S. Edwards, P.W. Anderson, J. Phys. F 5, 965 (1975).
[2] M. Mezard, G. Parisi, M.A. Virasoro, Spin Glass Theory and Beyond, World Scientific, Singapore, 1987.
[3] K. Binder, A.P. Young, Rev. Mod. Phys. 58, 801 (1986).
[4] W.L. McMillan, J. Phys. C 17, 3179 (1984).
[5] A.J. Bray, M.A. Moore, Phys. Rev. Lett. 58, 57 (1987).
[6] D.S. Fisher, D.A. Huse, Phys. Rev. Lett. 56, 1601 (1986).
[7] C.M.Newman, D.L.Stein, Nature of Ground State Incongruence in Two-Dimensional Spin Glasses, Physical Review Letters 84, 17, 3966-3969 (2000).
[8] C.M.Newman, D.L.Stein, Spatial Inhomogeneity and Thermodynamic Chaos, Physical Review Letters 76, 25, 4821-4824 (1996).