The computational complexity of the role assignment problem

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Abstract

A graph G is R-role assignable if there is a locally surjective homomorphism from G to R, i.e. a vertex mapping $r : V_G \to V_R$, such that the neighborhood relation is preserved: $r(N_G(u)) = N_R(r(u))$. Kristiansen and Telle conjectured that the decision problem whether such a mapping exists is an NP-complete problem for any connected graph R on at least three vertices. In this paper we prove the conjecture and show further corollaries for disconnected graphs and related problems.

Keywords: computational complexity, graph homomorphism, role assignment

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1 Introduction

Given two graphs, say G and R, an *R-role assignment* for G is a vertex mapping $r : V_G \to V_R$, such that the neighborhood relation is maintained, i.e. all roles of the image of a vertex appear on the vertex's neighborhood. Such a condition can be formally expressed as

$$\text{for all } u \in V_G : r(N_G(u)) = N_R(r(u)),$$

where $N(u)$ denotes the set of neighbors of u in the corresponding graph.

Such assignments have been introduced by Everett and Borgatti [6], who called them role colorings. They originated in the theory of social behavior. The graph R, i.e. the *role graph*, models roles and their relationships, and for a given society we can ask whether its individuals can be assigned roles such that the relationships are preserved: Each person playing a particular role has among its neighbors exactly all necessary roles as they are prescribed by the model.

From the computational complexity point of view it is interesting to know whether it is possible to decide quickly (i.e. in polynomial time) whether such assignment exists. This problem was considered by Roberts and Sheng [15], who focus on a more generalized problem called the 2-role assignment problem. If both graphs G and R are part of the input, the problem is NP-complete already for $R = K_3$ [12].

In order to make a more precise study we consider a class of R-role assignment problems, $\mathrm{RA}(R)$, parameterized by the role graph R. Here the instance is formed only by the graph G, and we ask whether an R-role assignment of G exists.

The complexity study of this class of problems is closely related to a similar approach for locally constrained graph homomorphism problems [9]. A graph homomorphism from G to H is a vertex mapping $f : V_G \to V_H$ satisfying the property that whenever an edge (u, v) appears in E_G, then $(f(u), f(v))$ belongs to E_H as well.

The adjective "locally constrained" expresses the condition that the mapping f restricted to the neighborhood of any vertex u must satisfy further properties. (See [14, 7] for a general model of such conditions.)

It may be required to be locally

- *bijective*, then the mapping is called a *full cover* of H, and the corresponding decision problem is called H-Cover [1, 13],

- *injective*, then it is called a *partial cover* of H, and the problem H-PCover [8, 9],
- *surjective*, then we get a *locally surjective cover* of H, and decision problem H-ColorDomination [14].

All these problems are parameterized by a fixed graph H, and the instance is formed only by a graph G. The question is whether an appropriate graph homomorphism from G to H exists. Observe that the definition of a locally surjective cover is equivalent with the definition of an R-role assignment for $R = H$.

Full covers have important applications, for example in distributed computing [5], in recognizing graphs by networks of processors [2, 3], or in constructing highly transitive regular graphs [4]. Similarly partial covers are used in distance constrained labelings of graphs [10].

Even if the first attempt to get some results on the computational complexity for the class of H-Cover problems was made a decade ago in [1], it is not fully classified yet neither for H-PCover nor for H-ColorDomination (RA(H)) problems. However, several partial results are known. For example, if the H-Cover problem is NP-complete, then the corresponding H-PCover [9] and H-ColorDomination problems [14] are NP-complete as well. Moreover, the H-Cover problem is known to be NP-complete for all k-regular graphs H of valency $k \geq 3$ [9], and the NP-hardness hence propagates for partial and locally surjective covers of such graphs as well.

The H-ColorDomination problem was proven to be NP-complete for paths, cycles and stars in [14]. It was conjectured there that for simple connected graphs the H-ColorDominationproblem is NP-complete if and only if H has at least three vertices. Our main theorem proves this conjecture.

The paper is organized as follows. The next section provides necessary definitions and basic observations. The third section provides technical lemmas used in the main construction which is showed in the fourth section. The fifth section describes the complexity of the role assignment problem for disconnected role graphs. We finally apply the main theorem to prove NP-completeness for a generalized k-role assignment problem [15] in the sixth section.

2 Preliminaries

Through the paper we use terminology stemming from the role assignment problems.

We consider simple graphs, denoted by $G = (V_G, E_G)$, where V_G is a finite vertex set of vertices and E_G is a set of unordered pairs of vertices, called edges. For a vertex $u \in V_G$ we denote its neighborhood, i.e. the set of adjacent vertices, by $N_G(u) = \{v \mid (u, v) \in E_G\}$.

The *degree* $\deg_G(u)$ of a vertex u is the number of edges incident with it, or equivalently the size of its neighborhood. The symbol $\delta(G)$ is the minimum degree among all vertices of G.

A graph G is called *connected* if for every pair of distinct vertices u and v, there exists a *path* connecting u and v, i.e. a sequence of distinct vertices starting by u and ending by v where each pair of consecutive vertices forms an edge of G. The *length* of the path is the number of its edges.

A graph that is not connected is called *disconnected*. Each maximal connected subgraph of a graph is called a *component*. A vertex whose removal causes a component of a graph to become disconnected is called a *cutvertex*. We say that a cutvertex u *separates* vertex v from w in G if v, w belong to different components of $G \setminus u$.

Two graphs G and $\tilde{G}$ are called *isomorphic*, denoted by $G \simeq \tilde{G}$, if there exists a one-to-one mapping f of vertices of G onto vertices of $\tilde{G}$ such that $(u, v) \in E_G$ if and only if $(f(u), f(v)) \in E_{\tilde{G}}$.

In the sequel the symbol G denotes the instance graph and R the so-called *role graph*.

Definition *We say that G is* R-role assignable *if a mapping $r : V_G \to V_R$ exists satisfying:*

$$\text{for all } u \in V_G : r(N_G(u)) = N_R(r(u)),$$

where we use the notation $r(S) = \bigcup_{u \in S} r(u)$ for a set of vertices $S \subseteq V_G$. The function r is called an R-role assignment *of G.*

The goal of this paper is a full characterization of the computational complexity for the following class of problems:

R-Role Assignment (RA(R))
Instance: A graph G.
Question: Does the graph G allow an R-role assignment?

We continue with some observations that we use later in the paper. We note first that role assignments are closed under composition:

Observation 2.1 *If G is S-role assignable and S is R-role assignable, then G is R-role assignable.*

Proof: Let $s : V_G \to V_S$ be an S-role assignment for G and $r : V_S \to V_R$ be an R-role assignment for S. Then $t : V_G \to V_R$ defined by $t(u) = r(s(u))$ for all $u \in V_G$ is an R-role assignment for G. $\square$

Observation 2.2 *If G is R-role assignable, then $\deg_G(u) \geq \deg_R(r(u))$ for all vertices $u \in V_G$.*

Proof: $\deg_G(u) = |N_G(u)| \geq |r(N_G(u))| = |N_R(r(u))| = \deg_R(r(u))$. $\square$

From this we easily derive that $\delta(G) \geq \delta(R)$, and moreover:

Lemma 2.3 *If G is R-role assignable and u is a vertex of G with $\deg_G(u) = \delta(R)$, then $\deg_R(r(u)) = \delta(R)$ and r restricted to $N_G(u)$ is an isomorphism between $N_G(u)$ and $N_R(r(u))$.*

Lemma 2.4 *Let G be R-role assignable and x, y be vertices of R connected by a path P_R. Then for each u with $r(u) = x$ a vertex $v \in V_G$ and a path P_G connecting u and v exist, such that r restricted to P_G is an isomorphism between P_G and P_R.*

Proof: We prove the statement by induction on the length of the path P_R. If x and y are adjacent, then the vertex u has a neighbor v mapping onto y, by the definition of the R-role assignment r.

Now assume that the path P_R is of length $k \geq 2$, and that the hypothesis is valid for all paths of length at most $k-1$. Denote by y' the predecessor of y in P_R and by P'_R the truncation of P_R by the last edge, i.e. the path of length $k-1$ connecting x and y'. By the induction hypothesis G contains a vertex v' and a path P'_G such that $P'_G \simeq P'_R$ under r. Then it is easy to find a neighbor v of v' satisfying $r(v) = y$ and tack it to P'_G to get the desired path P_G. $\square$

We get immediately the following claims:

Observation 2.5 *If G is R-role assignable and R is connected, then each vertex $v \in V_R$ appears as a role for some vertex $u \in V_G$.*

If $G \simeq R$ then G is R-role assignable, because every isomorphism satisfies the condition of the role assignment. Due to the previous observation we have:

Observation 2.6 *Let R be a connected role graph. If G is R-role assignable and $|V_G| = |V_R|$, then $G \simeq R$.*

Lemma 2.7 *Let G be R-role assignable, $u \in V_G$ be a vertex of role x, and $z, y \in V_R$ be some other roles. If in G each path connecting u to a vertex of role y contains a vertex of role z, then the vertex z is a cutvertex in R.*

Proof: Since vertices of roles x and y are connected by a path in G, there exists a path in R connecting x to y. Moreover if z were not a cutvertex, then we can find such a path avoiding the role z. But then by Lemma 2.4 we can find a path in G from u to some vertex of role y avoiding any vertex of role z. □

Lemma 2.8 *Let u be a cutvertex of G and C be a component of $G \setminus u$. Then the roles of V_C form a connected subgraph of R.*

Proof: By the definition, whenever $(u, v) \in E_G$ then $(r(u), r(v)) \in E_R$, in other words, every role assignment is also a graph homomorphism. In any homomorphism the image of a connected graph is connected. □

Lemma 2.9 *Let G be R-role assignable, u, u' be vertices of G such that $N_G(u) \subseteq N_G(u')$, and $\deg_G(u) = \delta(R)$. If all vertices of minimum degree in R are cutvertices then $r(u) = r(u')$.*

Proof: We denote $z = r(u)$. Since $\deg_R(z) \le \deg_G(u) = \delta(R)$ we get that z is a vertex of minimum degree, and by our assumptions it is also a cutvertex in R. Let x, y be two of its neighbors that are separated by z and let $v, w \in N_G(u)$ be their preimages. (Their uniqueness is even guaranteed by Lemma 2.3.) The image of the path v, u', w is connected, hence it contains the vertex z as the role of u'. □

3 Gadgets

3.1 Graphs with two role assignments

For the garbage collection in our NP-completeness proof we need to construct a graph that allows two different role assignments:

Lemma 3.1 *Let R be a role graph. Then a graph H exists that has two R-role assignments r_1 and r_2, such that for any two roles v and w, a vertex u exists in H with $r_1(u) = v$, and $r_2(u) = w$. Moreover, H can be constructed in time being polynomial with respect to the size of R.*

Proof: Take H as the Cartesian product $R \times R$, defined by the vertex set $V_H = V_R \times V_R$, and edges $((a,b),(c,d)) \in E_H$ if and only if $(a,c),(b,d) \in E_R$.

The projections $r_1 : (a,b) \to a$ and $r_2 : (a,b) \to b$ are valid R-role assignments, and the vertex $u = (v,w)$ satisfies the statement of the Lemma. □

Note that for our purposes, it is possible for any two roles v, w to construct a connected H with two role assignments — it is enough to select the component of $R \times R$ containing the vertex $u = (v,w)$.

3.2 Glued subgraphs

Definition *We say that a graph $\tilde{R}$ is* glued *in a graph G by a vertex $\tilde{v}$, if G can be obtained from $\tilde{R}$ and some other graph G' by identifying a vertex $x \in V_{G'}$ with the vertex $\tilde{v}$.*

See Fig. 1 for a more intuitive picture of such a glued graph.

As a convention we use letters x, y, z to denote roles, while u is reserved for vertices of the instance. The symbols v, w stand for roles, while $\tilde{v}$ or w' are vertices of the instance graph isomorphic to v, w.

Lemma 3.2 *Let R be a connected role graph. Let G be an R-role assignable graph and $\tilde{R}$ be glued in G by a vertex $\tilde{v}$, where $\tilde{R}$ is isomorphic to R and v, the isomorphic copy of $\tilde{v}$ in $\tilde{R}$, is not a cutvertex of R. Then $r(V_{\tilde{R}}) = V_R$.*

Proof: We use induction on the number of vertices.

If R consists of only one vertex, then G has no edges. Similarly if R consists of only two vertices, then G must be bipartite without isolated vertices and the statement is valid directly.

For graphs R on at least three vertices we prove the statement by contradiction. Suppose a role $z \in V_R \setminus r(V_{\tilde{R}})$ exists.

Claim 1: The role v' of the vertex $\tilde{v}$ is a cutvertex of R and does not appear as a role of any other vertex of $\tilde{R}$.

Select any vertex u of $\tilde{R}$ other from $\tilde{v}$. An arbitrary path from u to a vertex with the role z uses v', hence v' is a cutvertex by Lemma 2.7. If $r(u) = v'$ then by Lemma 2.4 it would be possible to select a path using distinct nodes. This contradicts the fact that v' is used on the path at least twice.

Based on this claim we can explore the structure of R. Denote by S the subgraph of R induced by $r(V_{\tilde{R}})$. Since v is not a cutvertex, the graph $\tilde{R} \setminus \tilde{v}$ is connected. Lemma 2.8 shows that $S \setminus v'$ is connected as well. Hence the vertex v' is not a cutvertex in S, but it is a cutvertex of R.
Claim 2: The mapping $s : V_{\tilde{R}} \to V_S$ defined by $s(u) = r(u)$ for all $u \in \tilde{V_R}$ is an S-role assignment for $\tilde{R}$.

For all $u \in V_{\tilde{R}} \setminus \tilde{v}$ we get

$$s(N_{\tilde{R}}(u)) = r(N_G(u)) = N_R(r(u)) = N_S(s(u)).$$

We have to show the same statement also for $\tilde{v}$. Firstly we get that $s(N_{\tilde{R}}(\tilde{v})) = r(N_{\tilde{R}}(\tilde{v})) \subseteq N_R(v') \cap V_S = N_S(v') = N_S(s(\tilde{v}))$. The inclusion "$\subseteq$" is in fact equality "$=$". Assume that some role $z \in N_R(v') \cap V_S$ does not appear among roles of $N_{\tilde{R}}(\tilde{v})$ under the assignment s. Since $z \in V_S$, a vertex u of $\tilde{R}$ exists such that $s(u) = z$. Then $u \neq \tilde{v}$ and it has a neighbor $u^* \neq \tilde{v}$ with role v', a contradiction with Claim 1.

Let $\tilde{S}$ be the subgraph of $\tilde{R}$ corresponding to S. Since the role graph S has less vertices than R, and the mapping s is a proper S-role assignment, we get by the induction hypothesis that

$$r(V_{\tilde{S}}) = s(V_{\tilde{S}}) = V_S.$$

We focus our attention on other properties of cutvertices:
Claim 3: If u is a cutvertex of $\tilde{R}$ then $r(u)$ is a cutvertex of R.

To see this consider in $\tilde{R}$ a vertex u^* separated from $\tilde{v}$ by u. According to Claim 1 the role v' appears only once on $\tilde{R}$, namely as the image of the vertex $\tilde{v}$, Then every path in G from u^* to a vertex of role v' uses the vertex u. By Lemma 2.7 the role $r(u)$ is a cutvertex in R.

We formulate a similar statement for cutvertices in $\tilde{S}$.
Claim 4: If u is a cutvertex in $\tilde{S}$ then $r(u)$ is a cutvertex in S.

Because the graph S is separated from the rest of the graph R by the vertex v', it follows that for each cutvertex of S, its isomorphic copy is not only a cutvertex in $\tilde{S}$ but also in $\tilde{R}$. Then, according to Claim 3, the cutvertices of $\tilde{S}$ map onto cutvertices of R, which are not equal to v' due to Claim 1. Hence they must be mapped on cutvertices of S.

We are ready to conclude our proof. Since $r(V_{\tilde{S}}) = V_S$ and $\tilde{S}$ is isomorphic to S, the role assignment r acts as a bijection between $V_{\tilde{S}}$ and V_S.

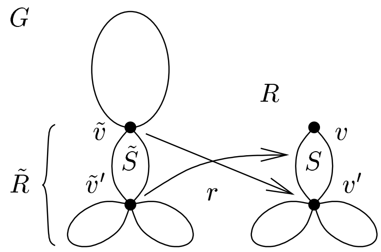

Figure 1: A role assignment of a glued subgraph

This fact together with Claim 4 implies that r maps all non-cutvertices of $\tilde{S}$ on non-cutvertices of S.

The vertex $\tilde{v}'$, i.e the isomorphic copy of v', maps on a cutvertex in S, because it is a cutvertex in $\tilde{R}$. However, it is not a cutvertex in $\tilde{S}$ (see Fig. 1).

$\square$

Lemma 3.3 *Let R be a connected role graph. Let G be an R-role assignable graph and $\tilde{R}$ be glued in G by a vertex $\tilde{v}$, where $\tilde{R}$ is isomorphic to R and v, the isomorphic copy of $\tilde{v}$ in $\tilde{R}$, is not a cutvertex of R. Then an R-role assignment r exists such that $r(\tilde{w}) = w$ for every $w \in V_R$.*

Proof: By Lemma 3.2 every R-role assignment $s : V_G \to V_R$ acts as a bijection when restricted to $\tilde{R}$. This mapping defines an isomorphism $\pi : V_R \to V_R$ by

$$\pi(x) = w \quad \Leftrightarrow \quad s(\tilde{w}) = x$$

which is also an R-role assignment of R. We have already observed that the composition of two role assignments is a role assignment. Then $r = \pi \circ s$ is an R-role assignment of G satisfying $r(\tilde{w}) = \pi(s(\tilde{w})) = w$. $\square$

4 The main result

Now we are ready to prove the conjecture of Kristiansen and Telle [14]:

Theorem 1 *Let R be a connected role graph. Then the R-role assignment problem is polynomially solvable if $|V_R| \leq 2$ and it is* NP*-complete if $|V_R| \geq 3$.*

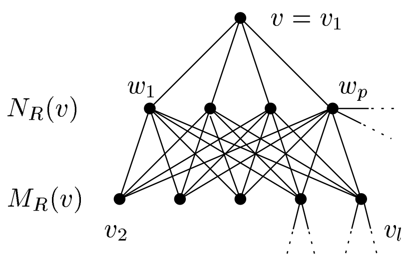

Figure 2: Neighborhood of a vertex v in R.

Proof: First we show that $\mathrm{RA}(R)$ is polynomially solvable for $|V_R| \leq 2$.

- $|V_R| = 1$. Clearly, a graph G is R-role assignable if and only if G contains only isolated vertices.
- $|V_R| = 2$. Clearly, a graph G is R-role assignable if and only if G is a bipartite graph that does not contain any isolated vertices.

Now let $|V_R| \geq 3$. Since we can guess a mapping $r : V_G \to V_R$ and check in polynomial time if r is an R-role assignment, the problem $\mathrm{RA}(R)$ is a member of NP. We prove NP-completeness by reduction from hypergraph 2-colorability. This is a well-known NP-complete problem (cf. [11]).

HYPERGRAPH 2-COLORABILITY (H2C)
Instance: A set $Q = \{q_1, \ldots, q_m\}$ and a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ with $S_j \subseteq X$ for $1 \leq j \leq n$.
Question: Is there a 2-coloring of $(Q, \mathcal{S})$, i.e., a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$?

With such a hypergraph we associate its incidence graph I, which is a bipartite graph on $Q \cup \mathcal{S}$, where (q, S) forms an edge if and only if $q \in S$.

To prove the theorem we choose a vertex $v \in V_R$ of minimum degree. Because we cannot apply Lemma 3.3 if v is a cutvertex, we have to distinguish between the case, in which all vertices of minimum degree are cutvertices, and the case, in which a non-cutvertex of minimum degree exists.

Assume first that the vertex v is a vertex of minimum degree that *is not* a cutvertex. Denote the neighbors of v by $N_R(v) = \{w_1, \ldots, w_p\}$ and also the second common neighborhood as $M_R(v) = \bigcap_{u \in N_R(v)} N_R(u) = \{v, v_2, \ldots, v_l\}$. See Fig. 2 for a drawing of a possible situation.

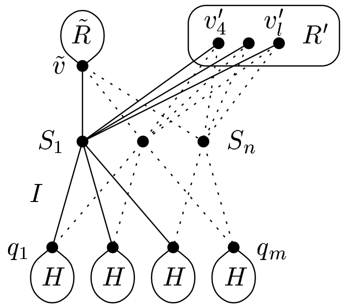

Figure 3: Construction of the graph G in Case 2.

We distinguish four cases according to possible values of p and l:
Case 1: $p = 1, l = 1$. Then $R = K_2$ and we have already discussed this case above.
Case 2: $p = 1, l \geq 3$. We extend the incidence graph I as follows: According to Lemma 3.1 we construct a graph H for which two role assignments exist mapping a particular vertex u to v_2 and v_3. We form an instance G as the union of the graph I and m disjoint copies of the graph H, where the vertex u of the i-th copy is identified with the vertex q_i of I. Finally we insert into G two extra copies $\tilde{R}, R'$ of the role graph R and add the following edges (cf. Fig 3):

- $(\tilde{v}, S_j)$ for all $S_j \in \mathcal{S}$,
- (v_k', S_j) for all $S_j \in \mathcal{S}$ and all $4 \leq k \leq l$ (this set may be empty).

We show that the graph G formed in this way allows an R-role assignment if and only if $(Q, \mathcal{S})$ is 2-colorable.

Assume first that G is R-role assignable. Then according to Lemma 3.3 we assume that the vertex $\tilde{v}$ is assigned role v and all vertices S_j are mapped to role w_1. Since their neighborhoods are saturated by common $l-3$ roles on $v_4', \ldots, v_l'$, at least two distinct roles $v_a, v_b \in M_R(v) \setminus r(\{v_4', \ldots, v_l'\})$ exist that are used on some neighbors of each S_j in the set $\mathcal{S}$.

The partition $Q_1 = \{q_i \,|\, r(q_i) = v_a\}$ and $Q_2 = Q \setminus Q_1 \supseteq \{q_i \,|\, r(q_i) = v_b\}$ is the desired 2-coloring of $(Q, \mathcal{S})$.

In the opposite direction, any 2-coloring Q_1, Q_2 can be transformed into an R-role assignment r of G by letting $r(q_i) = v_a$ if $q_i \in Q_a$ for $a = 1, 2$ and

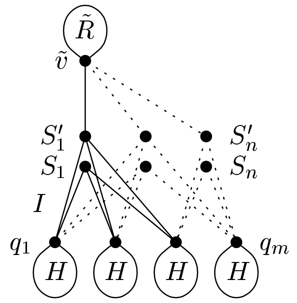

Figure 4: Construction of the graph G in Case 3.

by further extension according to the two projections of the graph H and graph isomorphisms $\tilde{R} \to R$, $R' \to R$.

Case 3: $p = 1, l = 2$. The case when R is isomorphic to the path P_4 was already shown to be NP-complete in [14]. If R is not isomorphic to a path on four vertices but v_2 is incident with a vertex v^* of degree one, then we can reduce this case to the previous case ($p = 1, l \geq 3$) by selecting v^* as the non-cutvertex of minimum degree. So without loss of generality we may assume that v_2 is not incident with a vertex of degree one.

We construct G from I as follows. First we insert n new vertices $S_1', \dots, S_n'$ and a copy $\tilde{R}$ of the role graph R. We identify each q_i with the vertex u of an extra copy of the graph H as in the previous case, but here H is constructed such that u can be assigned v or v_2.

These parts are linked as follows (cf. Fig. 4):

- $(\tilde{v}, S_j') \in E_G$ for all $j \in \{1, \dots, n\}$,
- $(q_i, S_j') \in E_G$ if and only if $(q_i, S_j) \in E_I$.

If G is R-role assignable, then without loss of generality we may assume that $\tilde{v}$ has role v. Then all S_j' have role w_1 since w_1 is the only neighbor of v. The roles of all q_i hence belong to $N_R(w_1) = \{v, v_2\}$. Each S_j' requires the role v_2 to be present among its neighbors in Q. Moreover, if all neighbors of some S_j' in Q are assigned the role v_2, we get that S_j must be mapped to a neighbor of v_2 that is a leaf, but this is in contradiction with our assumptions. We conclude that each S_j is mapped to w_1. Hence both roles v, v_2 appear on its neighborhood and the partition $Q_1 = \{q_i \mid r(q_i) = v\}$ and $Q_2 = \{q_i \mid r(q_i) = v_2\}$ is a 2-coloring of $(Q, \mathcal{S})$.

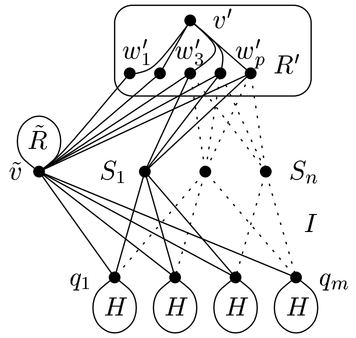

Figure 5: Construction of the graph G in Case 4.

In the opposite direction, an R-role assignment of G can be constructed from a 2-coloring of $(Q, \mathcal{S})$ in a straightforward way as in the previous case.
Case 4: $p \geq 2$. As above we first build the graph H which allows two R-role assignments mapping a vertex u either to w_1 or to w_2.

The graph G consists of the graph I, where each q_i is unified with the vertex u of an extra copy of H. We further include two copies of R denoted by $\tilde{R}$ and R'. Finally we extend the set of edges by (cf. Fig. 5):

- $(\tilde{v}, q_i)$ for all $q_i \in Q$,
- $(\tilde{v}, w'_k)$ for all $1 \leq k \leq p$,
- (S_j, w'_k) for all $3 \leq k \leq p$ (this set may be empty).

If an R-role assignment exists, then we assume that $r(\tilde{v}) = v$. For each S_j we have $N_G(S_j) \subseteq N_G(\tilde{v})$. So we know that S_j is assigned some role v_i for which $N_R(v_i) = N_R(v)$. However only $p-2$ roles appear on vertices $w'_3, \ldots, w'_p$, so two distinct roles w_a and w_b are used on none of $w'_3, \ldots, w'_p$. Then we define a 2-coloring of $(Q, \mathcal{S})$ by selecting $Q_1 = \{q_i \,|\, r(q_i) = w_a\}$ and $Q_2 = Q \setminus Q_1 \supseteq \{q_i \,|\, r(q_i) = w_b\}$.

An R-role assignment can be derived from a 2-coloring of $(Q, \mathcal{S})$ as in the previous cases.

Finally, we return to the situation when all vertices of minimum degree in R are cutvertices. (Observe, that $\delta(R) \geq 2$ since vertices of degree one are not cutvertices.)

We construct the graph G as in Case 4 above (cf. Fig. 5). The argumentation goes in the same manner: Since $N_G(v') \subseteq N_G(\tilde{v})$, we get by Lemma 2.9 that $\tilde{v}$ is mapped to a role of minimum degree. For each S_j we have $N_G(S_j) \subseteq N_G(\tilde{v})$. So we know that S_j is assigned a role that has the same neighbors in R as role $r(\tilde{v})$. Each S_j then lacks two roles w_a, w_b that do not appear on $w'_3, \ldots, w'_p$. Hence we can define a valid 2-coloring of $(Q, \mathcal{S})$ according to the appearance of roles w_a and w_b on the set Q. $\square$

Observe that all graphs G involved in our constructions were connected, even if the incidence graph I was not connected.

5 Disconnected Role Graphs

Up to now we have only considered role graphs that were connected. Due to this property we could easily derive that all roles appear as the image of the vertex in the instance graph (cf. Observation 2.5). We now focus our attention to the case of disconnected role graphs. Suppose R is a role graph with set of components $\mathcal{C} = \{C_1, \ldots C_m\}$. We order the components such that the latter have a higher number of vertices. (Formally, for all $i \leq j : |V_{C_i}| \leq |V_{C_j}|$.)

Note that the identity mapping $\pi : V_{C_1} \to V_R$ preserves the local constraint for role assignment, but Observation 2.5 is no longer valid here (take $G \simeq C_1$). Our argument guarantees that a locally surjective cover is globally surjective only for connected role graphs. Within some social network models it is natural to demand that all roles appear on the vertices of the instance graph. We show below that the computational complexity of the role assignment problem for disconnected role graphs depends whether such a property $r(V_G) = V_R$ is required or not.

We call an R-role assignment $r : V_G \to V_R$ a *globally R-role assignment* for G if r is an R-role assignment and $r(V_G) = V_R$ holds. Our generalized role assignment problem can now be formulated as

Global R-Role Assignment (GRA(R))
Instance: A graph G.
Question: Is G globally R-role assignable?

With respect to the computational complexity we obtain the following result.

Theorem 2 *Let R be a disconnected role graph. Then the* GRA(R) *problem is polynomially solvable if all components have at most two vertices and it is* NP*-complete otherwise.*

Proof: Clearly the GRA(R) problem belongs to NP. For connected role graphs the statement immediately follows from Theorem 1.

Suppose R has $m \geq 2$ components ordered as shown above. If all components consist of only one vertex, then a graph G is R-role assignable if and only if G is a collection of at least m isolated vertices. Suppose R consists of k isolated vertices and $m-k$ isolated edges. Then a graph G is R-role assignable if and only if G contains at least k isolated vertices and at least $m-k$ bipartite components, each on at least two vertices.

Now suppose $|V_{C_m}| \geq 3$. We prove NP-completeness by reduction from RA(C_m). Without loss of generality we assume that the instance graph G for the RA(C_m) problem is connected. Let G' be the graph with components $G, \tilde{C}_1, \ldots, \tilde{C}_{m-1}$, where $\tilde{C}_i$ is isomorphic to C_i for $1 \leq i \leq m-1$. It is straightforward to see that G' is R-role assignable if G is C_m-role assignable.

Now assume that G' is R-role assignable. Observe that both G' and R have the same number of components, so each component of R provides roles for exactly one component of G'. It is impossible to make a role assignment from $\tilde{C}_i$ to C_j when $|V_{\tilde{C}_i}| < |V_{C_j}|$. Hence the component G can only be assigned roles of one of the components of maximum size.

If the roles of the component G belong to C_m, then we are finished. Suppose the roles of the component G are in C_i with $i \neq m$. Then $\tilde{C}_i$ maps to some other component C_j such that $|V_{\tilde{C}_i}| = |V_{C_j}|$ and by Observation 2.6 we get $\tilde{C}_i \simeq C_j$. Then G allows also a C_j-role assignment. If $j \neq m$ we repeat the argument, and after at most m iterations we find a desired C_m-role assignment of G. $\square$

Now we show that without the condition of global surjectivity "$r(V_G) = V_R$", some polynomially solvable RA(R) problems exist for role graphs R with large components.

Take any role graph R with bipartite components (of arbitrary size) but assure that at least one of these components is isomorphic to K_2 (i.e. to a graph consisting of two vertices forming an edge). For simplicity assume that R has no isolated vertices. We claim that G is R-role assignable if and only if G is bipartite without isolated vertices. The necessity of such condition follows from the fact that non-bipartite graphs have no homomorphism to bipartite graphs. In the opposite direction, any homomorphism from G to K_2 can be viewed as an R-role assignment of G.

This observation leads us to propose the following conjecture:

Conjecture *For a simple role graph R, the* RA(R) *problem is solvable in polynomial time if and only if either all components of R have at most two vertices, or R is bipartite and at least one component is isomorphic to K_2. In all other cases the* RA(R) *problem is* NP*-complete.*

Although we have shown above a proof of the polynomial part of the statement, we do not see a direct way for a possible NP-hardness construction.

6 k-Role Assignability

In this section we study a more general version of the role assignment problem. We call a graph G *k-role assignable* if there exists a role graph R on k vertices, such that G is globally R-role assignable.

k-Role Assignment (k-RA)
Instance: A graph G.
Question: Is G k-role assignable?

This problem was studied by [15] and is of interest in social network theory where networks are modeled in which individuals of the same social role relate to other individuals in the same way. The networks of individuals are represented by simple graphs. Contrary to our previous results, in this new model two individuals that are related to each other may have the same role. Hence role graphs *that contain loops* are allowed.

Again our aim is to fully characterize the computational complexity of the k-RA problem. Clearly the 1-RA problem is solvable in linear time, since it is sufficient to check whether G has no edges ($R = K_1$) or whether all vertices in G have degree at least one (R consists of one vertex with a loop). The 2-RA problem is proven to be NP-complete in [15]. We generalize this result as follows:

Corollary 3 *The k-*RA *problem is polynomially solvable for $k = 1$ and it is* NP*-complete for all $k \geq 2$.*

Proof: We show that k-RA is NP-complete for $k \geq 3$. First assume that $k \geq 3$ but $k \neq 4$. We prove NP-completeness by reduction from RA(P_k), where P_k is a path on k vertices.

Let G be an instance of RA(P_k) constructed in the proof of Theorem 1. Without loss of generality we may assume that G is connected and that a graph $\tilde{P}_k \simeq P_k$ is glued in G by vertex $\tilde{v}$, the isomorphic copy of one of the two leaves of P_k. Let G' be the graph obtained after linking a path P' on $2k-2$ vertices to G via an edge from $\tilde{v}$ to one of the leaves of P'. Our claim is that G is P_k-role assignable if and only if G' is k-role assignable.

Clearly, if G is P_k-role assignable, then G' is k-role assignable.

In the opposite direction, consider any k-role assignment of G with a connected role graph R on k vertices. Denote vertices of $\tilde{P}_k$ by $\{\tilde{v} = u_1, u_2, \ldots, u_k\}$. Since u_k is a leaf, it must be mapped to a leaf, and then by downward induction each $u_i : 2 \leq i < k$ has neighbors of two distinct roles. Otherwise R cannot be connected and hence cannot be used for a global R-role assignment of G.

From the above we conclude that R must be isomorphic to P_k or otherwise to a path on k vertices with a loop in one of its end points. However, the latter case leads to a contradiction if we try to assign roles to P'. Hence G' can only be P_k-role assignable if it is k-role assignable. Clearly, this implies that G is P_k-role assignable as well.

When $k = 4$ we can either adjust the NP-completeness proof for $\mathrm{RA}(P_4)$ from [14] or reduce the P_3-role assignment problem by extending a graph G' obtained from a connected instance G of $\mathrm{RA}(P_3)$ by an extra isolated vertex u. Then, by the same arguments we get that the original G has to have an P_3-role assignment. This is because exactly one isolated vertex must be reserved as a role for u in a possible R. (In fact R is isomorphic to a disjoint union of P_3 and K_1.) $\square$

7 Conclusion

We have fully characterized the computational complexity for the R-role assignment problem for simple connected role graphs R. We have provided further arguments for disconnected role graphs too.

The computational complexity of the role assignment problem can be studied also for role graphs that contain some loops. If all components of R either consist of exactly one vertex or are isomorphic to K_2, the $\mathrm{RA}(R)$ problem is polynomially solvable. The conjecture is that in all other cases the problem is NP-complete, even if instances are restricted to simple graphs.

We expect that our constructions would work in a similar way. Instead of a graph isomorphic to the role graph an other appropriate graph should be glued in the instance graph to obtain a reduction from the H2C problem as we have used in the proof of Theorem 1.

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