# Cuts and Bounds


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**Abstract**

We consider the colouring (or homomorphism) order $\mathcal{C}$ induced by all finite graphs and the existence of a homomorphism between them. This ordering may be seen as a lattice which is however far from being complete. In this paper we study bounds and suprema and maximal elements in $\mathcal{C}$ of some frequently studied classes of graphs (such as bounded degree, degenerated and classes determined by a finite set of forbidden subgraphs). We relate these extrema to cuts of subclasses $\mathcal{K}$ of $\mathcal{C}$ (cuts are finite sets which are comparable to every element of the class $\mathcal{K}$). We determine all cuts for classes of degenerated graphs. For classes of bounded degree graphs this seems to be a very difficult problem which is also mirrored by the fact that these classes fail to have a supremum. We note a striking difference between undirected and oriented graphs. This is based on the recent work of C. Tardif and J. Nešetřil. Also minor closed classes are considered and we survey recent results obtained by authors. A bit surprisingly this order setting captures Hadwiger conjecture and suggests some new problems.



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# 1 Introduction

Graph theory receives its mathematical motivation mostly from two areas of mathematics: algebra and geometry (topology) and it is fair to say that graphical notions stood at the birth of algebraic topology (in the beginning called combinatorial topology). Consequently, various operations and relations for graphs stress either its algebraic aspects (as exhibited e.g. by colourings and by various products and spaces associated with graphs) or its geometrical aspects (represented e.g. drawings, contractions, embeddings). It is only natural that the key place in modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by various modifications of the notion of graph minor. However from the algebraic point of view perhaps the most natural notion which captures comparison of two graphs is that of a homomorphism.

A *homomorphism* $G \to H$ is a mapping $f : V(G) \to V(H)$ which satisfies $f(u)f(v) \in E(H)$ for any edge $uv \in E(G)$. (We shall consider both directed and undirected graphs. This will be always clearly specified. Section 5 is devoted entirely to oriented graphs.)

The central notion of this paper is the quasiorder (and partial order) induced by the existence of a homomorphism:

Given graphs $G, H$ we denote by $G \leq H$ the existence of a homomorphism $G \to H$. Clearly $\leq$ is a quasiorder. If we consider isomorphism types of minimal retracts (or *cores*, see [13]) then we obtain a partial order. This quasiorder (and partial order) is called *colouring order* (or *homomorphism order*, [13]) and it is denoted by $\mathcal{C}$. We denote by $G \sim H$ the equivalence given by $G \leq H \leq G$; in this case the graphs $G$ and $H$ are said to be *homomorphismequivalent*. We also denote by $<$ the strict version of $\leq$ (thus $G < H$ iff $G \leq H$ and $G \not\sim H$). For a graph $H$ we denote by $\mathcal{C}_H$ the principal ideal determined by $H$: $\mathcal{C}_H = \{G; G \leq H\}$. $\mathcal{C}_H$ is also called a *colour class*. This name is justified by interpreting homomorphisms as generalized colourings: Indeed, a homomorphism $G \to K_k$ is a just a (proper) $k$-colouring of graph $G$ and, more generally, a homomorphism $G \to H$ is called a $H$–colouring. Thus $\mathcal{C}_H$ is the class of all $H$-colourable graphs; hence the name colour class. It follows that the question whether $G \leq H$ is difficult to decide (and it is NP-complete in a very strong sense). We refer to [12, 13, 5] as a background information, our graph–theory terminology is standard.

It is perhaps surprising how many fine combinatorial questions are captured by *order − theoretic* properties of the colouring order $\mathcal{C}$. In this paper we concentrate on extremal elements of this order: greatest and maximal elements, suprema and (upper) bounds in general. It appears that these extremal graphs are related to various problems which are as remote as *duality theorem* ([19]) and celebrated Hadwiger conjecture. These interpretations lead also to some, hopefully interesting, problems.

Given a class $\mathcal{K}$ of graphs it is usually a difficult question to find a graph $H$ which is maximal (or greatest, or supremum) of $\mathcal{K}$ in $\mathcal{C}$ as such a result yields maximal chromatic number of a graph in $\mathcal{K}$. As these concepts are the subject of this paper we recall the corresponding definitions in the setting of colouring order $\mathcal{C}$:

A graph $H$ is said to be an (*upper*)*bound* of $\mathcal{K}$ if every graph $G \in \mathcal{K}$ satisfies $H \leq G$. If in addition $H \in \mathcal{K}$ then $H$ is said to be *greatest* graph in $\mathcal{K}$.

A graph $H$ is said to be *maximal* of $\mathcal{K}$ if $H \in \mathcal{K}$ and no graph $G \in \mathcal{K}$ satisfies $G < H$.

A graph $H$ is said to be *supremum* of $\mathcal{K}$ if $G \leq H$ for every $G \in \mathcal{K}$ and if for every graph $H' < H$ there exists a graph $G \in \mathcal{K}$ such that $G \not\leq H'$.

For example, in this setting, the 4-colour theorem says that $K_4$ is the greatest graph in the class of all planar graphs. This obviously cannot be improved. On the other hand, Grötzsch's theorem says that $K_3$ is an upper bound of the class of all planar $K_3$–free graphs. However, as we will see, this may be improved as $K_3$ fails to be a supremum of this class.

Let us add one more, less standard, order theoretic notion: Given a class $\mathcal{K}$ of graphs a finite subset $C$ of $\mathcal{K}$ is said to be a *cut* if every graph $G \in \mathcal{K}$ is comparable with a graph $H \in C$: i.e. it is either $G \leq H$ or $H \leq G$. Obviously graphs $K_1$ and $K_2$ are the cuts of every class which contains them. We call these two cuts *trivial cuts*. The existence of other cuts is a nontrivial question which is studied in this paper. This is related to suprema and greatest elements of classes of graphs and we investigate in this context some of the frequently studied classes of graphs (compare [5]). These include classes $\mathrm{Forb}(\mathcal{F})$ where $\mathcal{F}$ is a finite set of connected graphs: We denote by $\mathrm{Forb}(\mathcal{F})$ the class of all finite graphs $G$ which satisfy $F \not\leq G$ for every $F \in \mathcal{F}$. Alternatively, $\mathrm{Forb}(\mathcal{F})$ is the class of all graphs which do not contain a homomorphic image of a graph from $\mathcal{F}$. In yet another way we can say that $\mathrm{Forb}(\mathcal{F})$ is the class of all graphs defined by forbidden homomorphisms from a finite set of graphs $\mathcal{F}$. In our context this is a natural class of graphs. A bit surprisingly all related questions can be solved for classes $\mathrm{Forb}(\mathcal{F})$. For undirected graphs this is much easier than for directed graphs where we rely on strong results obtained jointly with C.

Tardif [19, 20].

As an approximation to the minor closed classes we also consider extrema relativized by classes of bounded degree graphs (in Section 3) and classes of $d$-degenerated graphs (in Section 4). While for degenerated graphs we have a full discussion of cuts and extremal properties for bounded degrees this seems to be a very difficult problem.

This paper surveys some of the results proved in a companion paper [14].

While in the Sections 2, 3, 4 we consider undirected graphs in Section 5 we survey oriented graphs and in Section 6 we conclude with some remarks and more open problems.

## 2 Classes induced by forbidden homomorphisms

In this section we consider undirected graphs. As a warm up we consider the simplest case of classes $\mathrm{Forb}(\mathcal{F})$ for a finite set $\{F_1, F_2, \ldots, F_t\}$ of connected graphs. Note that in our setting we can assume that all graphs $F_i$ are non–bipartite. We shall see that in this case we can determine suprema and cuts. For example we have the following:

**Theorem 2.1** *For any finite set $\mathcal{F}$ of non bipartite graphs, the class* $\mathrm{Forb}(\mathcal{F})$ *is not bounded. Moreover the class* $\mathrm{Forb}(\mathcal{F})$ *does not have any non-trivial cut.*

**Proof.** The class $\mathrm{Forb}(\mathcal{F})$ contains graphs which have any given chromatic number with girth $\geq$ maximal cycle in a graph $F_i$. As the homomorphic image of an odd cycle contains an odd cycle we get that the class $\mathrm{Forb}(\mathcal{F})$ has no bound in $\mathcal{C}$. Similarly we prove that there is no finite non-trivial cut in the class $\mathrm{Forb}(\mathcal{F})$: Let $C = \{H_1, \ldots, H_r\}, r \geq 1$ be a cut in $\mathrm{Forb}(\mathcal{F})$. Assume that $C$ is non-trivial. This amounts to say that none of the graphs $H_i$ is bipartite. Let $k$ denotes the maximal chromatic number of a graph $H_i$ and let $l$ denote the maximal length of the shortest odd cycle in $H_i$ (i.e. the $odd--girth$ of $H_i$). Let $H$ be a graph with $\chi(G) > k$ and with the odd-girth of $H > l$. Then $H \not\leq H_i$ by chromatic number and $H_i \not\leq H$ by the odd girth monotonicity. ∎

In this context one should also mention the following result for countable graph proved recently in [18]:

**Theorem 2.2** $K_1, K_2$ *and the infinite complete graph* $K_\omega$ *are the only minimal* $1$*–cuts for the class of all countable graphs.*

As opposed to the finite case countable graphs allow non-trivial finite cuts of any size. And contrary to the 1–cuts (which are characterized by Theorem 2.1), the minimal cuts of size $t > 1$ are abundant:

**Theorem 2.3** *For every positive integer $t, t > 1$ the class of all (undirected) countable graphs has infinitely many minimal cuts of size $t$.*

**Proof.** Let $t > 1$ be fixed. Let $F_1, F_2, \ldots, F_{t-1}$ be finite connected graphs which are pairwise incomparable in $\mathcal{C}$. We can use Theorem 2.1, a random $(t-1)$–tuple of graphs will do as well. Now we can apply a result of [1] which gives the existence of a countable graph $H$ which is universal for the class $\mathrm{Forb}(F_1, F_2, \ldots, F_{t-1})$ (when considered as the class of all countable graphs). Explicitly: $H$ is a graph such that $F_i \not\leq H$ for every $i = 1, \ldots, t-1$ and if $G$ is a countable graph satisfying $F_i \not\leq G$ for every $i = 1, \ldots, t-1$ then $G$ is an induced subgraph of $H$. However then the set $C = \{F_1, F_2, \ldots, F_{t-1}, H\}$ is obviously a cut in the class of all countable graphs. It is also easy to check that $C$ is a minimal cut. ∎

This proof is perhaps more interesting than the statement of Theorem 2.3: presently there are no other known minimal cuts for infinite graphs. This perhaps suggest the following problem (which is also related to some results for oriented graphs, see Section 4):

**Problem 2.4** *Is it true that any minimal cut of size at least 2 for the class of all countable graphs contains always a finite graph?*

One can refine Theorem 2.1 to subclasses of $\mathrm{Forb}(\mathcal{F})$ which are $H$-colourable; or in the other words to the classes $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$.

**Theorem 2.5** *For any finite set $\mathcal{F}$ of non bipartite graphs and for any graph $H$, the class $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ has supremum $H$. Moreover, if $F \in \mathrm{Forb}(\mathcal{F})$ then the singleton set $\{H\}$ is its only non-trivial cut of the class $\mathrm{Forb}(\mathcal{F})$.*

**Proof.** We only note that both of these statements are non-trivial and both are related to the following statement isolated in [17, 24, 13].

*SparseIncomparabilityLemma* :

For every pair of graphs $G, H, G < H$ and for every positive integer $l$ there exists a graph $G'$ with girth $l$ with the following properties: $G' \leq H$ and $G$ and $G'$ are incomparable graphs (in $\mathcal{C}$).

Now, Sparse Incomparability Lemma implies that, while $H$ is an obvious bound of the class $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$, no strictly smaller graph $G < H$ is a bound of $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ (as $G' \not\leq G$, it is $G' \in \mathrm{Forb}(\mathcal{F})$ as $G'$ has a high girth). By the same token either $H \in \mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$, in which case $H$ is both greatest element and cut of $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$, or all elements of $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ are strictly smaller than $H$. In the later case, for any finite subset $G_1, G_2, \ldots, G_r$ of non-bipartite graphs in $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$, there exists a graph $G \in \mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ such that $G$ is incomparable with any of the graphs $G_i$. The existence of $G$ follows again from Sparse Incomparable Lemma. Hence, $\{G_1, G_2, \ldots, G_r\}$ may not be a cut of $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$. ∎

Thus for undirected graphs the principal ideals even when restricted by a finite set of forbidden homomorphisms have only trivial cuts. Bellow we shall strengthen this result in various directions. Despite of its simplicity this result is a prototype of a statement we want to prove (in a more complex situations).

# 3 Bounded degrees

Denote by $\Delta_d$ the class of all graphs $G$ with maximal degree $\Delta(G) \leq d$. For a finite set $\mathcal{F} = \{F_1, \ldots, F_t\}$ of connected graphs denote also $\Delta_d(\mathcal{F})$ the class of all graphs $G \in \Delta_d$ with $F_i \not\to G$ for $i = 1, \ldots, t$. Thus $\Delta_d(\mathcal{F}) = \Delta_d \cap \mathrm{Forb}(\mathcal{F})$.

Celebrated Brook's theorem states that while $K_{d+1}$ is a bound (and indeed greatest element) of the class $\Delta_d$ by forbidding $K_{d+1}$ this may be improved to a better bound $K_d$. It follows that $K_{d+1}$ fails to be supremum of the class $\Delta_d(\{K_{d+1}\})$. This is not an accident and a similar statement holds in general thus yielding a whole hierarchy of Brook's type bounds. We say that a graph $H$ is a *proper supremum* of a class $\mathcal{K}$ if $H$ is supremum of $\mathcal{K}$ and $H \notin \mathcal{K}$(as we are working with equivalence classes the later condition of course means that $H \not\sim G$ for every graph $G \in \mathcal{K}$).

**Theorem 3.1** *The class $\Delta_d(\mathcal{F})$ has no proper suprema for any $d \geq 3$ and any finite set $\mathcal{F}$ of connected graphs.*

Remark that the assumption "$d \geq 3$" may not be dropped.

We see that this property of bounded degree graphs is in a sharp contrast with properties of all graphs and classes $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ (which are disscussed in Section 2). Bellow we shall see that also the classes of degenerated graphs behave very differently (see Section 4).

Theorem 3.1 will be proved in the following more technical form (cf. [14]):

**Theorem 3.2** *Let $\mathcal{F}$ be a finite set of connected graphs, $d \geq 3$. Let $H$ be a bound for the class $\Delta_d(\mathcal{F})$, $H \notin \Delta_d(\mathcal{F})$. Then there exists a bound $H'$ for $\Delta_d(\mathcal{F})$ with $H' < H$.*

First, we show that Theorem 3.2 implies Theorem 3.1:

**Proof.**

In the situation of Theorem 3.1 assume for a contradiction that a graph $H$ is a proper supremum of $\Delta_d(\mathcal{F})$. If $H$ is connected then consider the set $\mathcal{F}' = \mathcal{F} \cup \{H\}$ and apply Theorem 3.2 to get a bound $H'$ for the class $\Delta_d(\mathcal{F}')$, $H' < H$. However note that $\Delta_d(\mathcal{F}) = \Delta_d(\mathcal{F}')$ (as for no graph $G \in \Delta_d(\mathcal{F})$ we have $H \leq G$. This shows that $H$ fails to be a supremum of $\Delta_d(\mathcal{F})$. It remains to be proved that (hypothetical) proper supremum $H$ is a connected graph. Suppose contrary, let $H$ be a disconnected core and let $K$ be a component of $H$ that does not belong to $\Delta_d(\mathcal{F})$ (it exists as $H \notin \Delta_d(\mathcal{F})$ and as $\mathcal{F}$ is a set of connected graphs). We have $H-K < H$ and thus there exists a connected graph $G \in \Delta_d(\mathcal{F})$ such that $G \not\leq H - K$. It follows that any connected graph $G'$ containing $G$ satisfies also $G' \not\leq H - K$ As all graphs $G' \in \Delta_d(\mathcal{F})$ satisfy $G' < H$ we get that any connected graph $G'$ from $\Delta_d(\mathcal{F})$ satisfies $G' \leq K$. Thus $H$ fails to be a supremum of $\Delta_d(\mathcal{F})$. (Note that the last part of the proof holds generally for all classes $\mathrm{Forb}(\mathcal{F})$ for a finite set $\mathcal{F}$ of connected graphs.) ∎

**Proof.(Theorem 3.2)**

Let $d \geq 3$, $\mathcal{F}$ be a finite set of connected graphs. Let $H$ be a bound for the class $\Delta_d(\mathcal{F})$, $H \notin \Delta_d(\mathcal{F})$. We may assume that $H$ is connected (see the above proof). Put $\mathcal{F}' = \mathcal{F} \cup \{H\}$. It suffices now to prove that the class $\Delta_d(\mathcal{F}')$ is bounded in the class $\mathrm{Forb}(\mathcal{F}') \cap \mathcal{C}_H$ (Recall: $\mathcal{C}_H$ is the class of all $H$-colourable graphs). Explicitly, we want to prove that there exists a graph $H', H' \in \mathrm{Forb}(\mathcal{F}'), H' < H$ such that $G \leq H'$ for every graph $G \in \Delta_d(\mathcal{F}')$. The existence of such a graph is proved in [2] (as a refinement of [6]). We only sketch this for a completeness: Let $a$ be the maximal number of vertices of a graph in $\mathcal{F}$. Put $b = 1 + d^a$, put $X = \{1, 2, \ldots, b\}$. Put also $H = (V, E)$. The vertices of graph $H'$ will be all triples $(A, x, v)$ where $A$ is any connected graph that belongs to $\Delta_d(\mathcal{F})$ with vertex set $V(A) \subset X$, $x \in V(A)$, and $v \in V$. Two vertices $(A, x, v)$ and $(A', x', v')$ form an edge if $vv' \in E, xx' \in E(A) \cap E(A')$ and $E(A)$, $E(A')$ coincide both on the set $V(A) \cap V(A')$ and finally if for any vertex $y \in X$ holds: $d_A(x,y) \leq a iff d_{A'}(x,y) \leq a$ and, also, $d_A(x',y) \leq a iff d_{A'}(x',y) \leq a$ (where $d_A(x,y)$ denotes the distance of $x$ and $y$ in $A$).

One can prove that the graph $H'$ has all the desired properties. ∎

It follows that any class of form $\Delta_d \cap \mathrm{Forb}(\mathcal{F})$ has a supremum only if there exists $H \in \Delta_d \cap \mathrm{Forb}(\mathcal{F})$ such that $\Delta_d \cap \mathrm{Forb}(\mathcal{F}) \subset \mathcal{C}_H$; this then means that $H$ is the greatest element of $\Delta_d(\mathcal{F})$.

However the order structure of classes $\Delta_d \cap \mathcal{C}_H$ is far from obvious. For example the following two problems (analogous to the results in Section 2) have been isolated:

**Problem 3.3** *Let $d \geq 3$. Is it true that for every graph $G \in \Delta_d$, $G < K_d$ there exists a graph $G' \in \Delta_d$ such that neither $G \leq G'$ nor $G' \leq G$ (i.e. graphs $G$ and $G'$ are incomparable graphs in $\Delta_d$.*

A positive solution of this Problem would imply that the the non–trivial cuts of classes $\Delta_d$ are $K_d$ and $K_{d+1}$ only. But this seems to be a hard question. This problem is related to the complexity of $H$-colourings of bounded degree graphs which have been studied e.g. in [4, 9].

**Problem 3.4 (Pentagon problem)** *Does there exists an $l$ such that any cubic (i.e. 3-regular) graph $G$ with girth $l$ is homomorphic to $C_5$ (i.e. is $C_5$-colourable)?*

Partial results related to this problem were obtained in [23, 8, 9]. One should perhaps note that for $C_{2k+1}, k > 3$, (instead of $C_5$) the answer is negative.

# 4 Degenerated classes

Recall that an (undirected) graph $G = (V, E)$ is said to be *$d$-degenerated* if there exists a linear ordering $v_1 < v_2 < \ldots < v_n$ of vertices of $G$ satisfying

$$|\{v_i; j < i, v_j v_i \in E\}| \leq d$$

for every $j, 1 \leq j \leq n$. Alternatively, a $d$-degenerated graph can be defined by the condition

$$\delta(G') \leq d$$

for every subgraph $G'$ of $G$ ($\delta(G)$ denotes the minimal degree of a vertex of $G$). (Yet another way is to define $d$-degenerated graphs by the hereditary edge-density.)

The class of all $d$-degenerated graphs will be denoted by $DEG_d$. The class $DEG_1$ is just the class of all forests. For $d > 1$ these classes are more interesting. Similarly as in the previous section we denote by $DEG_d(\mathcal{F})$ the class of all $d$-degenerated graphs which belong to the class Forb$(\mathcal{F})$. While these definitions are formally similar the extremal properties of these classes are strikingly different.

**Theorem 4.1** *Let $d \geq 2$. Then the following holds:*
*i. $K_{d+1}$ is the greatest graph in $DEG_d$.*
*ii. For every finite set $\mathcal{F}$ of non-bipartite graphs the class $DEG_d(\mathcal{F})$ has supremum $K_{d+1}$.*

Note that for a forbidden set $\mathcal{F}$ which contain a bipartite graph the situation is much simpler and different - $K_1$ is a bound.

We do not prove this result here (see [14]). Instead we prove the following related result about cuts:

**Theorem 4.2** *For any finite set $\mathcal{F}$ of non-bipartite graphs and $d \geq 2$ the only non–trivial cut of the class $DEG_d(\mathcal{F})$ is $\{K_{d+1}\}$.*

**Proof.** For contradiction assume that $\{H_1, \ldots, H_r\}$ be a cut. Let $d \geq 2$ and $\mathcal{F}$ be as assumed. Let $\mathcal{F}'$ denotes the set of all non–bipartite blocks of graphs belonging to $\mathcal{F}$. As any graph $F \in \mathcal{F}$ contains a non–bipartite block it follows that the class $DEG_d(\mathcal{F}')$ is a subclass of the class $DEG_d(\mathcal{F})$. Put $l$ the maximal number of vertices of a graph belonging to $\mathcal{F}'$. Put $k = \max|V(H_i)|$. We shall construct a graph $H$ with the following properties:

1. $H$ has girth $> k + l$ and thus particularly $F \not\leq H$ for any $F \in \mathcal{F}'$ and consequently also $F \not\leq G$ for any $F \in \mathcal{F}$.

2. $H$ is $d$-degenerated;

3. Any homomorphic image of $H$ with at most $k$ vertices contains $K_{d+1}$.

1. and 2. imply that $H \in DEG_d(\mathcal{F})$ and that $H_i \not\leq H, i = 1, \ldots, r$. It follows from 3. that $H \not\leq H_i, i = 1, \ldots, r$ and thus $\{H_1, \ldots, H_r\}$ fails to be a cut of $DEG_d(\mathcal{F})$.

The graph $H$ will be constructed by means of Descartes–Tutte–type of construction as follows (compare [8]):

We shall construct graphs $G_1, G_2, \ldots, G_{d+1}$; $G_{d+1}$ will be the desired graph $H$. Put $G_1 = K_1$ and $G_2 = K_2$. In the induction step assume that $G_i$ is constructed. Put $|V(G_i)| = p_i$ and let $(X_{i+1}, \mathcal{M}_{i+1})$ be $p_i$-uniform hypergraph without cycles of length $\leq l$ and with chromatic number $> k$ (this exits by [3, 10]). For every $M \in \mathcal{M}_{i+1}$ take an isomorphic copy $G_i^M$ of $G_i$ and assume $V(G_i^M) \cap X_{i+1} = \emptyset, V(G_i^M) \cap V(G_i^{M'}) = \emptyset$, for all $M \neq M' \in \mathcal{M}_{i+1}$. Finally for every $M \in \mathcal{M}_{i+1}$ fix a bijection $\pi_{i+1}^M : V(G_i^M) \to M$. Define the graph $G_{i+1} = (V_{i+1}, E_{i+1})$ as follows:

$$V_{i+1} = X_{i+1} \cup \bigcup (V(G_i^M); M \in \mathcal{M}_{i+1})$$

$$E_{i+1} = \bigcup (E(G_i^M); M \in \mathcal{M}_{i+1}) \cup \{v\pi_{i+1}^M(v); v \in V(G_i^M), M \in \mathcal{M}_{i+1}\}.$$

$G_{i+1}$ does not contain cycles of length $\leq l$ (in fact, by our choice of $G_1$ and $G_2$ it does not contain cycles of length $\leq 3l$; we do not optimize here). We also prove by induction for $i = 1, 2, \dots d+1$ that $G_i$ is an $(i-1)$–degenerated graph. In the induction step assume that $G_i$ has an $(i-1)$–degenerated ordering. For $V(G_{i+1})$ choose such an ordering which satisfies $x < v$ for all $x \in X_{i+1}$ and $v \in V(G_i^M)$ and coincides on any set $V(G_i^M)$ with $(i-1)$–degenerated ordering (of $G_i^M$). Clearly this is an $i$–degenerated ordering of $G_{i+1}$.

Finally, let $f : V(G_{d+1}) \to H$ be a homomorphism, $|V(H)| \leq k$. By the downward induction for $j = d+1, d, \dots, 1$ we prove that for every $j$ there exists $M_j \in \mathcal{M}_j$ such that $f$ restricted to the set $M_j$ is a constant. However this is nearly obvious as the building blocks of our construction –the hypergraphs $(X_i, \mathcal{M}_j)$– have all chromatic number $> k$. As every $M_j$ is joined by an edge to all vertices of $V(G_{j-1}^{M_j})$ we get that the homomorphic image of $G$ under $f$ contains $K_{d+1}$, which is a contradiction.

∎

# 5 Minor closed classes

A class of graphs $\mathcal{K}$ is said to be *minor closed* if contains all minors of any of its member. We say that $\mathcal{K}$ is *proper* if it does not contain all graphs. Note that all graphs in a proper minor closed class of graphs are $d$-degenerated for a $d$ (by Mader's Theorem [11]). Consequently any minor closed class of graphs is bounded (in $\mathcal{C}$). However extremal graphs are much more difficult for minor closed classes then for bounded degree and $d$-degenerated classes. One of the few general results was obtained recently [16]:

**Theorem 5.1** *Let $\mathcal{K}$ be any proper minor closed class of graphs, $k$ a positive integer. Then the class of all $K_k$-free graphs from $\mathcal{K}$ is bounded by a $K_k$-free graph.*

*Explicitly: there exists a graph $H = H(\mathcal{K}, k)$ with the following properties:*

*i.* $K_k \not\leq H$;

*ii.* $G \leq H$ *for any* $G \in \mathcal{K}$;

*Additionally we may assume that the chromatic number of* $H$ *is equal to the maximal chromatic number of a graph in* $\mathcal{K}$.

We shall not prove this here. Let us just remark that Theorem 5.1 implies that the celebrated Grötzsch's theorem (which asserts that $K_3$ is a bound for all triangle–free planar graphs) does not yield the best bound: By virtue of Theorem 5.1 there exists a bound $H$ satisfying $H < K_3$. The bounds for minor closed classes are related to the *Hadwiger conjecture* which we state in two following ways: *i.* is the usual formulation and *ii.* is a formulation in the spirit of this paper. We also add a localized version *iii.* (A class of graphs is said to be *principal ideal* in the minor order if the class consists from minors of a graph.)

**Conjecture 5.2** *(***Hadwiger***)*

*i. For every graph* $G$ *holds* $\chi(G) \leq h(G)$ *where* $h(G)$ *is the maximal complete graph which is a minor of* $G$.

*ii. Any proper minor closed class* $\mathcal{K}$ *of graphs has the greatest element which is a complete graph.*

*iii. Any principal ideal of the minor quasiorder has greatest graph in the homomorphism order and this graph is a complete graph.*

Let us prove that these three forms of Hadwiger conjecture are indeed equivalent: *i.* $\Rightarrow$ *ii.* holds as for any graph $G$ we can apply *i.* to the class $\mathcal{K}$ formed by all $G$ together with all its minors. If $H$ is the greatest element of $\mathcal{K}$ then $\chi(G) \leq \chi(H)$ and thus *i.* implies $\chi(G) \leq h(G)$.

Conversely, assume *ii.* and let $\mathcal{K}$ be a proper minor closed class. Let $H$ be a graph in $\mathcal{K}$ with the maximal chromatic number, put $k = \chi(H)$. Then $\mathcal{K}$ is bounded by $K_k$ and by *ii.* applied to the graph $H$ we know that $K_k \in \mathcal{K}$. The equivalence of *i.* and *iii.* follows similarly.

In view of results of the previous two sections perhaps one should consider the following weaker:

**Conjecture 5.3** *Any proper minor closed class of graphs has a greatest element in the homomorphism order* $\mathcal{C}$.

It is not even clear whether any minor closed class of graphs has a supremum. However this weaker Conjecture 5.3 seems to be more accessible as illustrated also by the following:

**Proposition 5.4** *Let $\mathcal{K}$ be a minor closed class which is defined by a set of connected forbidden minors. Then $\mathcal{K}$ has greatest element iff every ascending chain of graphs from $\mathcal{K}$ is bounded in $\mathcal{K}$.*

Let us state the *ascending chain* condition explicitly: If $G_1, G_2, \ldots$ are graphs from $\mathcal{K}$ such that $G_1 < G_2 < \ldots < G_n < \ldots$ then there exists $H \in \mathcal{K}$ such that $G_i \leq H$ for every $i = 1, 2, \ldots$.

**Proof.** One direction is clear. So let us assume that $\mathcal{K}$ satisfies ascending chain condition and that it is defined by a set of connected forbidden minors. It is then clear that for every $G \in \mathcal{K}$ there exists a maximal element $H \in \mathcal{K}$ (in the homomorphism order $\mathcal{C}$) such that $G \leq H$. All maximal elements of $\mathcal{K}$ are mutually incomparable (in the homomorphism order). Let there be two incomparable maximal elements $H$ and $H'$. Then their disjoint union $H + H'$ satisfies $H + H' \in \mathcal{K}$ $H < H + H', H' < H + H'$ which is a contradiction. Thus $\mathcal{K}$ has the greatest element. ∎

We have the following corollary which verifies Conjecture 5.3 for principal ideals.

**Corollary 5.5** *For every connected graph $G$ the class of all its minors has the greatest element.*

**Proof.** Denote by $\mathcal{K}$ the class of all graphs $H$ for which every component $H_i$ of $H$ is a minor of $G$. The class $\mathcal{K}$ is minor closed a it is defined by a set of connected forbidden minors (namely by the set of all connected forbidden minors of $G$; this set may be taken finite as the minor relation is WQO, [22]. Moreover $\mathcal{K}$ satisfies ascending chain condition (as there are only finitely many choices for components). Thus we can apply Proposition 5.4 ∎

Let us also note that by Theorem 5.1 any proper minor closed class of graphs is bounded by a graph $H$ with clique number $\omega(H) = h(\mathcal{K})$ where $h(\mathcal{K})$ is the largest clique contained in $\mathcal{K}$.

Some minor closed classes seem to have more restricted cuts. For example we have that $\{K_1\}, \{K_2\}, \{K_3\}, \{K_4\}$ are all cuts for the class of all planar graphs. Are there any other? This is equivalent to the following problem:

**Problem 5.6** *Let $G_1, G_2, \ldots, G_t$ be a set of incomparable planar graphs, $t > 1$. Does there exists a planar graph $G$ which is incomparable with all graph $G_i, i = 1, \ldots, t$?*

For general minor closed classes a similar statement fails to be true. For example if we consider any pair of two incomparable graphs (for example $K_3$ and the Grötzsch's graph) and we consider all minors of these graphs then this class has obviously a non–trivial cut. Also, considering, say, series parallel graphs we see that not every 1-cut is a complete graph (in the case of series parallel graphs we get all odd cycles). On the other hand 1-cuts have certain special structure:

**Proposition 5.7** *Let $\mathcal{K}$ be any class of graphs. Then* 1*-cuts of $\mathcal{K}$ form a chain.*

By virtue of Corollary 5.5 for every non–bipartite graph $G$ the class of all its minors has a non-trivial cut.

# 6 Directed graphs – Suprema and Dualities

Until now we considered undirected graphs only. It is a special feature of this area that there is a big gap between directed and undirected graphs. We briefly review some recent results which are relevant to the context of this paper.

First we consider classes $\mathrm{Forb}(\mathcal{F})$ (of all directed graphs $G$ which do not contain any $F \in \mathrm{Forb}(\mathcal{F})$ with $F \leq G$). While for the undirected graphs these classes are bounded in trivial instances only for directed graphs we have a much richer an interesting spectrum of results. Recall that an oriented graph $G$ is said to be *balanced* iff every cycle in $G$ has the same number of forwarding and backwarding arcs. In terms of homomorphisms this is the same as to say that there exists a homomorphism $G \to \vec{P}_n$ where $\vec{P}_n$ is the directed path of length $n$ (i.e. with $n+1$ vertices). For a balanced graph $G$ we also put $al(G) = \min\{n; G \to \vec{P}_n\}$ (*algebraic length* of $G$).

We start with the following:

**Theorem 6.1** *For a finite set $\mathcal{F}$ of directed graphs, the following statements are equivalent:*
*$i$. The class* $\mathrm{Forb}(\mathcal{F})$ *is bounded;*
*$ii$. At lest one of the graphs $F \in \mathcal{F}$ is balanced.*

**Proof.** This is yet another version of sparse high chromatic graphs. $ii$. implies $i$. as the chromatic number of graphs in $\mathrm{Forb}(\mathcal{F})$ is bounded by $al(F)$ for a balanced $F \in \mathcal{F}$. Conversely, suppose that no $F \in \mathcal{F}$ is balanced.

Alternatively we know that any homomorphic image of any $F \in \mathcal{F}$ contains a cycle. It suffices to take any orientation of a high chromatic graph without short cycles. These graphs all belong to $\mathrm{Forb}(\mathcal{F})$ and thus there is no bound for this class. ∎

The characterization of classes of form $\mathrm{Forb}(\mathcal{F})$ with a greatest element is a more difficult result:

**Theorem 6.2** *For a finite set $\mathcal{F}$ of directed graphs the following statements are equivalent:*

*i. The class* $\mathrm{Forb}(\mathcal{F})$ *has greatest element;*

*ii. $F \in \mathcal{F}$ is a set of (orientations of) trees.*

Theorem 6.2 is proved in [19] in a different context (compare [13]).

Let us remark that Theorem 6.2 may be seen as characterization of all Gallai–Roy (and Hasse) – type theorems which correspond to the case $\mathcal{F} = \{\vec{P}_n\}$. In this case the dual graph is the transitive tournament with $n$ vertices. This point of view is taken in [20].

Let us finally discuss the existence of suprema for the classes $\mathrm{Forb}(\mathcal{F})$. Here we have also a full solution which is perhaps surprising (this result is taken from [14]):

**Theorem 6.3** *For a finite set $\mathcal{F}$ of connected graphs the following statements are equivalent:*

*i. The class* $\mathrm{Forb}(\mathcal{F})$ *has supremum;*

*ii. At least one of the graphs $F \in \mathcal{F}$ is balanced.*

*In the other words every bounded class* $\mathrm{Forb}(\mathcal{F})$ *of oriented graphs has a supremum.*

Let us add a few remarks:

1. Connectedness assumption in Theorem 6.3 (similarly as in Theorem 3.1 is a necessary assumption.

2. Note that many of these suprema in Theorem 6.3 are proper. An example is the class $\mathrm{Forb}(\{F_1, F_2, \ldots, F_t, H\})$ with supremum $H$.

3. Finite oriented graphs contain an abundance of minimal cuts of every size (as opposed to undirected graphs). An example is provided by any finite set $\{F_1, F_2, \ldots, F_t\}$ of trees together with the corresponding dual graph $H_{\{F_1, F_2, \ldots, F_t\}}$. It follows that for any $t \geq 1$ there are many cuts of size $t+1$ and their structure is complicated. On the other side minimal cuts of size 1 (i.e. 1–cuts) are simple to describe. These are just graphs $K_1, \vec{P}_1, \vec{P}_2$ full details of this will appear elsewhere).

The following recent result of C. Tardif and J. Nešetřil [20] characterizes all minimal cuts of size 2:

**Theorem 6.4** *A pair $\{A, B\}$ of oriented graphs is a minimal cut if and only if $A$ is (homomorphism equivalent to) a tree and $A$ is not any of the graphs $K_1, \vec{P}_1, \vec{P}_2$ and $B$ is (homomorphism equivalent to) its dual graph.*

*(Recall that two graphs are homomorphism equivalent if each has an homomorphism to the other).*

For cuts of size 3 a similar result is presently open.

Above we noted that all the cuts for the class of all undirected graphs (and many other classes of graphs) are just trivial cuts. Here the difference to oriented graphs could not be bigger: cuts in oriented graphs form universal poset. Let us be more precise:

Denote by $CUT$ the class of all cuts (for the homomorphism order of all oriented graphs). For class $CUT$ define the order $\leq$ by putting $C \leq C'$ if for every graph $G \in C$ there exists a graph $G' \in C'$ such that $G \leq G'$. Denote also $CUT_t$ the class of all $t$-cuts. We know that $CUT_1$ is a chain. It is easy to determine this chain. It consists for $\{P_0\}$, $\{P_1\}$, $\{P_2\}$, where $\{P_i\}$ is the monotone path of length $i$ (see [20]). However the structure of $CUT_2$ and thus the structure of $CUT$ is as rich as possible:

**Theorem 6.5** *The order $CUT_2$ (when restricted to the core graphs) is universal partial order.*

*Explicitly, every countable partially ordered set is is an (induced) subposet of $CUT_2$.*

**Proof.** Recall that for any oriented path $T$ there exists a dual $D_T$ with the property: For every graph $G$ we have

$$G \leq D_T \Leftrightarrow T \not\leq G$$

.

Let $T_1, T_2$ be any two paths each with a monotone path of length 3. We have the following:

$T_1 \leq T_2 \Leftrightarrow T_2 \not\leq D_{T_1}$ iff $D_{T_1} \leq D_{T_2}$.

We shall prove that

$$\{T_1, D_{T_1}\} \leq \{T_2, D_{T_2}\} \Leftrightarrow T_1 \leq T_2$$

.

One direction is clear. Thus assume $T_1 \not\leq T_2$ and $\{T_1, D_{T_1}\} \leq \{T_2, D_{T_2}\}$. Thus $T_1 \leq D_{T_2}$. Further we have $D_{T_1} \not\leq D_{T_2}$ (as $T_1 \not\leq T_2$) and thus necessarily $D_{T_1} \leq T_2$. However the last statement is impossible if the paths contain a monotone path of length $\geq 3$ (as the dual graph has chromatic number $\geq 3$).

Now we can apply (non–trivial) result of [7] which asserts that the homomorphism order when restricted to oriented path is universal. This concludes the proof. ∎

# 7 Summary and Concluding Remarks

The purpose of this paper is to initiate the study of graph bounds in a homomorphism and partial order setting. From this point of view greatest elements and suprema present tight bounds (which cannot be "improved"). We have proved (Theorem 6.3) that classes which are defined by forbidden homomorphisms from a finite set of connected graphs have suprema if and only if they are bounded. On the other hand the same classes when relativized by bounded degrees do not have suprema at all (with a few isolated cases; see Theorem 3.1). This is in a sharp contrast with the situation for degenerated graphs where suprema are easy to describe (and they form a chain). This perhaps sheds some light on questions like Hadwiger conjecture which can be expressed in the same vain.

On the other side some (perhaps many) questions, theorems and even proofs considered in this paper can be carried over to more general situations: coloured graphs and even relational structures (finite models). An example of this is given in [19]. This provides a connection with universal algebra and model theory. We hope to return to this in near future.

What we propose here is a global approach to extremal–theory estimates (such as bounds for chromatic number) by means of colouring (homomorphism) order. We studied some local properties of the colouring order (such as suprema and greatest elements). To present a good bound (i.e. supremum) for a class of graphs is equivalent to finding a smallest finite homomorphism universal graph. Whether this hom–universal graph can have the same local properties as the class itself is one of the central questions of this paper. We gave instances with both positive and negative answer. A satisfactory solution we could provide for classes which are defined by finitely many homomorphism obstructions. We relativized these results by bounded degree–, degeneracy– and minor closed–restrictions. This leads to some seemingly difficult problems but it also shows how these questions are relevant and that global structure of colourings can capture some of the key combinatorial conjectures.

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