A General View of Approximation

B. Banaschewski and A. Pultr*

Introduction

To motivate the subject studied in this paper, we consider the following way of describing the approximation of positive real numbers by rational numbers which is specifically rooted in actual calculation. The rational numbers used in this context are given by the decimal expansions

$$\alpha = b_m b_{m-1} \cdots b_0.a_1 a_2 \cdots a_n$$

where n will be called the order $\mathrm{ord}\alpha$ of α. Such an α will be considered as an approximation to a real number ξ if ξ belongs to the open interval $(\alpha - 10^{-\mathrm{ord}\alpha}, \alpha + 10^{-\mathrm{ord}\alpha})$, the precision of the approximation then evidently measured by $\mathrm{ord}\alpha$. Further, in accordance with this view, we put

$$\alpha \leq \beta \quad \text{iff} \quad (\alpha - 10^{-\mathrm{ord}\alpha}, \alpha + 10^{-\mathrm{ord}\alpha}) \subseteq (\beta - 10^{-\mathrm{ord}\beta}, \beta + 10^{-\mathrm{ord}\beta}),$$

expressing the notion that α *is a better approximation than* β for any ξ it approximates. As a suggestive example we note that $\alpha \leq \beta$ whenever $\mathrm{ord}\alpha \geq \mathrm{ord}\beta$ and α is an extension of β, say

$$b_m b_{m-1} \cdots b_0.a_1 a_2 \cdots a_{n+1} \leq b_m b_{m-1} \cdots b_0.a_1 a_2 \cdots a_n,$$

but of course there are other instances, such as $0.12 \leq 0.2$.

It is clear that this defines a partial order but it should be pointed out that, while this certainly captures the essence of the way approximations are calculated, it does not coincide with the notion of *metric* approximation based on the absolute value. Thus 1.412 is metrically closer to $\sqrt{2}$ than

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1.4, but while the latter is an approximation of $\sqrt{2}$ in the present sense the former is not. Similarly, it is important to note that 1.11 and 1.1100 are different in this setting: $1.1100 \leq 1.11$ but not the reverse, in accordance with the fact that 1.11, signifying the interval between 1.10 and 1.12, contains strictly less information than 1.1100.

If D is the partially ordered set thus described it is clear that its subsets $C_n = \{\alpha \in D \mid \mathrm{ord}\alpha = n\}$ represent *levels of precision* of approximations, and D together with this collection of subsets may be viewed as expressing the notion of computational approximation of the positive reals.

Furthermore, the C_n turn out to be subsets of the partially ordered set D of a special kind called *covers*, and the collection of these covers has certain particular properties in view of which it defines what we shall call a *nearness* on D. Finally, the positive reals themselves appear in this setting as what will be called its *points*, technically its regular Cauchy filters.

Now, abstracting from this example, we are led to consider arbitrary partially ordered sets, assumed to have a top for convenience, equipped with a suitable collection of covers. Here, the elements are viewed as representing approximations to a certain kind of entity, $x \leq y$ meaning that x is a better approximation than y, and the given covers are taken to describe levels of precision of the approximations. We call these entities *approximation systems*.

In addition, we define morphisms between approximation systems as binary relations between their underlying tests which, in particular, preserve compatibility of approximations and are uniform in some natural sense. As a result we then have the category **ApprSyst** of approximation systems.

Our first step after this will be to relate this category to an appropriate category of *information systems*. The crucial idea here is that the notion of implication between bits of information, as encoded by information systems (in the original sense of Scott [10]) can usefully be augmented by a suitable concept of *accuracy of information*, again given by appropriate sets of covers, this time on the preordered set of the consistent finite subsets of the system. Furthermore, it is then natural to consider only those maps of the underlying information systems which are uniform relative to the latter. This will then define the category **StratInfSyst** of stratified approximation systems, and it will be shown that this is equivalent to **ApprSyst** up to the partial order between the maps of either category given by inclusion.

Next, motivated by the fundamental dual equivalence between the category **InfSyst** of (mere) information systems and the category **ESLat** of semilattices and their extended homomorphisms (Banaschewski-Pultr [6]) we consider semilattices equipped with the same type of sets of covers considered in the two previous contexts together with those extended semilattice homomorphisms which are uniform in the appropriate sense. For the resulting category **ENearS** it will then be shown that the original dual equivalence induces a dual equivalence between it and **StrInfSyst**. In addition, as a specific feature of the enriched situation which has no counterpart for the original category **ESLat**, we establish the equivalence, again up to the partial order between maps, of **ENearS** with the corresponding category of *complete nearness frames*.

Further, we introduce the space PtX of points of an approximation system X as the space of regular Cauchy filters of X, with its topology and nearness structure naturally arising from X, and establish that this is isomorphic to the spectrum of the complete nearness frame FX which arises from X by the successive application of the equivalences described above.

Finally, we (1) provide the proof that the situation considered above in our motivating example indeed defines an information system and then describe how this is related to the more familiar frame of non-negative reals, and (2) discuss an alternative functor between nearness frames and approximation systems.

0 Background

For the relevant details concerning the notions involved in this paper we refer to the following:

Johnstone [8] or Vickers [11] for frames in general,

Banaschewski-Pultr [5] , Banaschewski [2], and Banaschewski-Hong-Pultr [4] specifically for nearness frames, the original ancestors of the different kinds of structures introduced here as well as the entities to which these structures will ultimately be related.

Banaschewski-Pultr [6] for the connection between information systems and semilattices, and

Mac Lane [9] for the minimal number of category theoretical concepts employed here.

For convenience, we specifically describe the material from [6] which is needed here.

Recall that an *information system* (Scott [10]) is an entity $S = (E, Con, \vdash)$ where

E is a set,
Con is a collection of finite subsets $u, v, w, \ldots$ of E, and
$\vdash$ is a binary relation between Con and E such that

(IS1) If $u \subseteq v$ and $v \in Con$ then $u \in Con$.
(IS2) $\{x\} \in Con$ for all $x \in E$.
(IS3) If $u \vdash x$ then $u \cup \{x\} \in Con$.
(IS4) If $u \in Con$ and $x \in u$ then $u \vdash x$.
(IS5) If $u \vdash x$ for all $x \in v$ and $v \vdash y$ then $u \vdash y$.

We use the following notation for information systems: if $S = (E, Con, \vdash)$ we put

$$|S| = E,\ ConS = Con,\ \vdash_S = \vdash$$

and allow omission of the index in the latter whenever suitable.

Next, a *map* $f : S \to T$ between information systems (called approximable mapping in [10]) is a binary relation $f \subseteq ConS \times ConT$ such that

(M1) $\emptyset f \emptyset$.
(M2) If $u' \subseteq u$, ufv, and $v \subseteq v'$ then $u'fv'$.
(M3) If ufv and ufw then $uf(v \cup w)$.

Further, these notions form a category, the category **InfSyst** of information systems, in which composition is given by the usual relational composite, that is, for $f : S \to T$ and $g : T \to U$, $gf : S \to U$ is $f \circ g$, and the identity map of S is the extension to $ConS$ of the relation $\vdash_S$ such that $u \vdash_S v$ iff $u \vdash_S x$ for all $x \in v$.

In addition, one considers the category **ESLat** of *semilattices and their extended homomorphisms*, where semilattices $A, B, \ldots$ are always taken as bounded meet semilattices and an extended homomorphism from A to B, denoted $\varphi : A \multimap B$, is a bounded semilattice homomorphism $\varphi : A \to \mathfrak{D}B$ where $\mathfrak{D}B$ is the lattice of all (non-void) downsets of B ($0 \in U$ and $x \leq y$, $y \in U$, implies $x \in U$). Here, the composite of $\varphi : A \multimap B$ and $\psi : B \multimap C$ is $\psi \bullet \varphi : A \multimap C$, defined by

$$\psi \bullet \varphi(a) = \bigcup \{\psi(x) \mid x \in \varphi(a)\},$$

and the identities for this composition are the homomorphisms $\eta_A : A \to \mathfrak{D}A$ such that $\eta_A(a) = {\downarrow} a\ = \{x \in A \mid x \leq a\}$.

Finally, we have a *dual equivalence* $\mathfrak{L}$: **InfSyst** $\to$ **ESLat** between these categories, with the following effects on objects S and maps $f : S \to T$:

$$\mathfrak{L}S = (ConS/ \dashv\vdash_S) \cup \{\bot\}$$

where $\dashv\vdash_S$ is the equivalence relation $u \vdash_S v \vdash_S u$ on $ConS$ which determines the *partially ordered* set associated with $ConS$ preordered by $\vdash_S$, and $\perp$ is an added bottom.

$\mathfrak{L}f : \mathfrak{L}T \multimap \mathfrak{L}S$ is given by

$$\mathfrak{L}f([v]) = \{[u] \in ConS/\dashv\vdash_S \mid ufv\} \cup \{\perp\},\ \mathfrak{L}f(\perp) = \perp$$

where $[\cdot]$ stands for the block of the equivalence relations $\dashv\vdash_T$ and $\dashv\vdash_S$, respectively.

Concerning the notion of completeness of strong nearness frames which plays a central rôle in Section 4, the reader is reminded that

(1) a *dense surjection* $h : M \to L$ of strong nearness frames is a frame homomorphism which is dense ($a = 0$ whenever $h(a) = 0$), onto on elements, and onto on uniform covers,

(2) L is called *complete* if any dense surjection $M \to L$ is an isomorphism, and

(3) A *completion of* L is a dense surjection $M \to L$ with complete M.

We note in addition that any L has completion, and the corresponding dense surjection is the coreflection map to L from complete strong nearness frames.

1 Nearness

In the following we shall be dealing with several kinds of preordered or partially ordered sets, always taken to have a unit (= top) e, including (bounded) meet semilattices. Our first object will be to define a concept of nearness on any of these, modelled after the familiar notion in frames (Banaschewski [2]) which embodies the pointfree form of regular nearness (and specifically uniform) spaces.

In order to do this two separate things are needed:

(1) a notion of *cover*, as the obvious basic ingredient, and

(2) a notion of *non-disjointness* which is needed for the definition of the admissibility of a set of covers.

We use the following general terminology and notation for a preordered set X. For any $x \in X$, $\downarrow x = \{z \in X \mid z \leq x\}$, and for any $A \subseteq X$, $\downarrow A = \bigcup\{\downarrow x \mid x \in A\}$. Further, for any $A, B \subseteq X$, $A \leq B$ (A refines B) means $A \subseteq \downarrow B$, that is, each element of A is below some element of B.

Finally, for any $x \in X$ and $A \subseteq X$, $x = \bigvee A$ (x is the join of A) means $A \subseteq \downarrow x$ and $x \leq z$ whenever $A \subseteq \downarrow z$, and a subset B of X is called *join-dense* if $x = \bigvee B \cap \downarrow x$ for each $x \in X$.

Now, a *cover* of a preordered set X is a subset C of X such that

$$x = \bigvee \downarrow C \cap \downarrow x$$

for each $x \in X$, that is, $\downarrow C$ is join-dense. Note that in a frame L, $C \subseteq L$ is a cover iff $\bigvee C = e$, the unit of L, saying that C is a cover in the usual sense. Similarly, for any partially ordered subset X of a frame L, any subset of X which is a cover of L is also a cover of X. Further, in a semilattice A, the cover condition obviously simplifies to

$$a = \bigvee \{a \wedge s \mid s \in C\}$$

for all $a \in A$. We note in passing that, for semilattices, this is strictly stronger than just $e = \bigvee C$.

Concerning covers in a preordered set we have two obvious rules:

(1) if C is a cover then any subset D such that $C \leq D$ is a cover, and
(2) if C and D are covers then $\downarrow C \cap \downarrow D$ is a cover, denoted $C \wedge D$.

Note that in (2), if $E \leq C$ and $E \leq D$ for any covers then $E \subseteq C \wedge D$. Of course, in a meet semilattice $C \wedge D$ is equivalent, in the sense provided by the relation $\leq$ between sets, to $\{s \wedge t \mid s \in C,\ t \in D\}$.

Next, regarding non-disjointness, we distinguish between preordered sets and pre-ordered sets with *specified bottom* 0, referring to both cases jointly as general preordered sets. A finite subset u will be called a *cluster* in the first case of u has lower bounds and in the second case of u has lower bounds not less than or equal to the specified 0. For $u = \{x_1, \ldots, x_n\}$ and $\ell b(u) = \downarrow x_1 \cap \cdots \cap \downarrow x_n$ this says $\ell b(u) \neq \emptyset$ and $\ell b(u) \neq [0]$, the set of all $x \leq 0$, respectively. In either case, if $u = \{x_1, \ldots, x_n\}$ we may express this formally by $x_1 \wedge \cdots \wedge x_n \neq 0$, with $x_1 \wedge \cdots \wedge x_n = 0$ meaning that u is not a cluster.

Now, for any cover C of a general preordered set X and any $x \in X$, let

$$C_x = \{s \in C \mid x \wedge s \neq 0\}$$

and put $x \triangleleft_C z$ if $C_x \subseteq \downarrow z$. Note that the latter implies $x \leq z$: it makes z an upper bound of $\downarrow x \cap \downarrow C$ because z is trivially an upper bound of the $\downarrow x \cap \downarrow s$ for $s \notin C_x$ in C. The relation $\triangleleft_C$ gives rise to a new relation between covers: we put $C \leq^* D$ (C star-refines D) provided for each $x \in C$ there exist $t \in D$ such that $s \triangleleft_C t$.

Further, for any set $\mathfrak{A}$ of covers of X, $x \triangleleft_{\mathfrak{A}} z$ shall mean that $x \triangleleft_C z$ for some $C \in \mathfrak{A}$, and we call $\mathfrak{A}$ *admissible* whenever

$$z = \bigvee \{x \in X \mid x \triangleleft_{\mathfrak{A}} z\}$$

for each $z \in X$. Finally, a *nearness* on X is an admissible set $\mathfrak{N}$ of covers of X such that $C \in \mathfrak{N}$ and $C \leq D$ implies $D \in \mathfrak{N}$ and $C \wedge D \in \mathfrak{N}$ for any $C, D \in \mathfrak{N}$.

Note that, for any set $\mathfrak{A}$ of covers which is admissible and down-directed in the sense of $\leq$, the set of covers refined by some member of $\mathfrak{A}$ is a nearness, said to be *generated by* $\mathfrak{A}$.

Regarding special kinds of nearnesses, a nearness $\mathfrak{N}$ is called *strong* if the cover

$$\check{C} = \{x \in X \mid x \triangleleft_{\mathfrak{N}} z \text{ for some } z \in C\}$$

belongs to $\mathfrak{N}$ for each $C \in \mathfrak{N}$, and a *uniformity* if, for each $C \in \mathfrak{N}$ there exist $D \leq^* C$ in $\mathfrak{N}$. Note that $D \leq^* C$ implies $D \subseteq \check{C}$ and hence any uniformity is a strong nearness (but certainly not conversely.

It is perfectly obvious that, in the case of a frame, all these notions amount to the familiar ones referred to by the same terminology, and hence the present definitions provide a considerable extension of the scope of these concepts.

Regarding the existence of nearnesses on a given X, note that if X has a nearness then the set $CovX$ of all its covers is a nearness; clearly, this is the case iff $CovX$ is admissible, and whenever this holds the nearness $CovX$ is trivially strong. Further, if $x \prec z$ is defined to mean that

$$\{z\} \cup \{y \in X \mid x \wedge y = 0\}$$

is a cover then $CovX$ is admissible iff $z = \bigvee\{x \in X \mid x \prec z\}$, extending the familiar fact that a frame has a nearness iff it is regular. Note that, as an obvious consequence of this, no totally ordered set with more than two elements has a nearness: in this case, $x \prec z$ only holds if $x = 0$ or $z = e$.

We close with a simple observation concerning the two basic processes involving preordered sets, the passage to partially ordered sets by taking the familiar quotient modulo the equivalence "$x \leq y$ and $y \leq x$" and the enlargement $X \subseteq X_\perp$ of a partially ordered set without specified bottom by the addition of a specified bottom.

If $\nu : X \to \tilde{X}$ is the quotient map in the first case then $C \subseteq X$ is a cover iff $\nu[C]$ is a cover; $C \leq D$ for covers in X iff $\nu[C] \leq \nu[D]$; for any cover C of X and $x, z \in X$, $x \triangleleft_C z$ iff $\nu(x) \triangleleft_{\nu[C]} \nu(z)$; and ν preserves and reflects all existing joins. Consequently, the nearnesses on X and on $\tilde{X}$ correspond to each other by the map ν, and the same holds, for analogous reasons, for the nearnesses on X and $X_\perp$ in relation to the embedding $X \to X_\perp$. Moreover, it is clear that strong nearnesses and uniformities correspond to each other, respectively, in both cases.

2 Approximation systems and stratified information systems

An *approximation system* is a partially ordered set with a top together with a strong nearness on it. We denote such systems by $X, Y, \ldots$, the corresponding nearness by $\mathfrak{N}X, \mathfrak{N}Y, \ldots$, and allow notational confusion between X and its underlying set or partially ordered set; the partial orders will be $\leq_X, \leq_Y, \ldots$, or without the index whenever convenient. Further, an approximation system will be called *uniform* if $\mathfrak{N}X$ is a uniformity.

As indicated in the Introduction, we view the elements of the underlying partially ordered set of an information system as approximations to some entity or other where the partial order indicates "is a better approximation than," the covers belonging to $\mathfrak{N}X$ signify levels of precision, and $\mathfrak{N}X$ as a whole provides a scale of such levels, with refinement indicating increased precision. We leave it to the reader to spell out the meaning of the formal properties of covers and the admissibility of $\mathfrak{N}X$ in terms of this view.

Examples 2.1 (1) We now consider D as the partially ordered set of terminating decimal expansions discussed in the Introduction augmented by a formal top element. Then, as shown in the appendix, the sets $C_n = \{\alpha \in D \mid \mathrm{ord}\alpha = n\}$ form an admissible set of covers such that $C_{n+1} \leq^* C_n$. Hence we have a uniform approximation system, to be called the *decimal approximation system*, again denoted by D.

(2) For any strong nearness frame, the partially ordered set of all its nonzero elements, together with the uniform covers not containing 0 obviously forms an approximation system.

(3) The singleton partially ordered set $\{0\}$ clearly has $\mathfrak{N} = \{\{0\}\}$ as its unique nearness, and we let **1** be the resulting approximation system. Note this results from the two-element strong nearness frame by the process

described in (2).

Below we use the following notation for any binary relation $f \subseteq X \times Y$ between sets X and Y: for $x \in X$, $y \in Y$, $S \subseteq X$, and $T \subseteq Y$,

$$\begin{aligned} f(x) = \{y \in Y \mid xfy\} &\quad , \quad f^{-1}(y) = \{x \in X \mid xfy\} \\ f[S] = \bigcup\{f(x) \mid x \in S\} &= \{y \in Y \mid xfy \text{ for some } x \in S\} \\ f^{-1}[T] = \bigcup\{f^{-1}(y) \mid y \in T\} &= \{x \in X \mid xfy \text{ for some } y \in T\} \end{aligned}$$

Now, for any approximation systems X and Y, an *approximation map* (*map* for short) $f : X \to Y$ is a binary relation $f \subseteq X \times Y$ such that

(AM1) efe.
(AM2) If $x' \leq x$, xfy, and $y \leq y'$ then $x'fy'$.
(AM3) If $x_1 \wedge \cdots \wedge x_n \neq 0$ in X and $x_i f y_i$ for $i = 1, \ldots, n$ then $y_1 \wedge \cdots \wedge y_n \neq 0$.
(AM4) For any $C \in \mathfrak{N}Y$, $f^{-1}[C] \in \mathfrak{N}X$.

We consider the first three conditions to define cluster preserving morphisms between partially ordered sets and view the last one as an added uniformity requirement.

Note that, for any approximation system X, its partial order is a map $X \to X$, the first three conditions being quite obvious while (AM4) holds because $(\leq)^{-1}(x) = \downarrow x$ and $C \leq \bigcup\{\downarrow s \mid s \in C\}$. As a consequence, we may view the $f : X \to Y$ as externally provided modes of comparing the quality of approximations in X with that of approximations in Y which is suitably compatible with the specified levels of precision.

Lemma 2.1 (1) *For any approximation maps $f : X \to Y$ and $g : Y \to Z$, $f \circ g$ is approximation map $X \to Z$.*
(2) *For any approximation map $f : X \to Y$, $(\leq_X) \circ f = f = f \circ (\leq_Y)$.*

Proof. (1) (AM1) and (AM3) are quite straightforward to check, and the other two conditions result from the general observation that

$$(f \circ g)[S] = g[f[S]] \text{ and } (f \circ g)^{-1}[T] = f^{-1}[g^{-1}[T]]$$

for any $S \subseteq X$ and $T \subseteq Z$.

(2) $x((\leq_X) \circ f)y$ means $x \leq z$ and zfy for some z and hence xfy by (AM1); on the other hand, xfy implies $x \leq x$ and xfy so that $x((\leq_X) \circ f)y$, proving the reverse inclusion. The second case is obtained in the same way. $\square$

As a result, if we take $f \circ g : X \to Z$ as the composite gf of $f : X \to Y$ and $g : Y \to Z$ the associativity of relational composition together with (2) of the lemma shows that this defines a category which will be denoted **ApprSyst**.

Examples 2.2 (1) For any approximation system X there is a unique map $f : X \to \mathbf{1}$, obviously such that $xf0$ for all $x \in X$. Hence $\mathbf{1}$ is the terminal object of **ApprSyst**.

(2) Defining the operation "multiplication by 10" on the underlying set of the decimal approximation system D in the obvious way we obtain corresponding maps $f, g : D \to D$ such that $\alpha f \beta$ iff $10\alpha \le \beta$ and $\alpha g \beta$ iff $\alpha \le 10\beta$ for which $gf = \mathrm{id}_C = fg$, expressing multiplication and division by 10 in terms of the category **ApprSyst**.

(3) For any uniform homomorphism $h : M \to L$ between strong nearness frames, if X and Y are the corresponding approximation systems determined by $L - \{0\}$ and $M - \{0\}$ respectively then h determines an approximation map $f : X \to Y$ such that xfy iff $x \le h(y)$. The first three conditions for maps hold quite obviously, and regarding the fourth one we have, for any $C \in \mathfrak{N}Y$, that is: $C \in \mathfrak{N}M$ and $0 \notin C$,

$$f^{-1}[C] = \{x \in X \mid x \le h(y) \text{ for some } y \in C\} = (\downarrow h[C]) - \{0\}$$

which belongs to $\mathfrak{N}X$ since $h[C] - \{0\} \in \mathfrak{N}L$.

For the following we refer to the background on information systems given in Section 0.

A *stratified information system* is an entity $S = (E, Con, \vdash, \mathfrak{N})$ where $(E, Con, \vdash)$ is an information system and $\mathfrak{N}$ a strong nearness on $ConS$ taken with its usual preorder $\vdash$. We let $|S| = E$, $ConS = Con$, and $\mathfrak{N}S = \mathfrak{N}$.

Further, for stratified information system S and T, a map $f : S \to T$ is a map of the underlying information systems which is *uniform* in the sense that $f^{-1}[C] \in \mathfrak{N}S$ for each $C \in \mathfrak{N}T$.

Lemma 2.2 *For stratified information systems, their identity maps and the composites of any uniform maps are uniform.*

Proof. For any S, $\mathrm{id}_S = \vdash$; hence if $C \in \mathfrak{N}S$ then

$$(\vdash)^{-1}[C] = \{u \in ConS \mid u \vdash v \text{ for some } v \in C\}$$

which contains C and consequently belongs to $\mathfrak{N}S$. Further, for uniform $f : S \to T$ and $g : T \to U$, if $C \in \mathfrak{N}U$ then

$$(gf)^{-1}[C] = (f \circ g)^{-1}[C] = f^{-1}[g^{-1}[U]]$$

which belongs to $\mathfrak{N}S$ by the given uniformity of f and g. $\square$

As a result, the stratified information systems together with the uniform maps between them form a category **StrInfSyst** with units and composition as in the category of (mere) information systems. It is our aim now to relate this category to our earlier category **ApprSyst**.

As a first step, we note that any approximation system X determines a stratified information system $\mathbf{S}X$ as follows.

$|\mathbf{S}X|$ is the underlying set of X,

$Con(\mathbf{S}X)$ is the set $CluX$ of clusters of X, that is (to recall) the finite $u \subseteq X$ which have a lower bound in X,

$u \vdash x$ for any cluster u of X and $x \in X$ iff $\ell b(u) \subseteq {\downarrow}x$,

$\mathfrak{N}(\mathbf{S}X)$ is generated by the sets $C_* = \{\{s\} \mid s \in C\}$, $C \in \mathfrak{N}X$.

Regarding the axioms of information systems, we have:

(IS1) if $u \subseteq v$ where $v \in CluX$ then $\ell b(v) \subseteq \ell b(u)$ and $\ell b(v) \neq \emptyset$ so that $\ell b(u) \neq \emptyset$ and hence $u \in CluX$.

(IS2) Any $\{x\}$ is trivially a cluster.

(IS3) If $u \vdash x$ then $\ell b(u) \subseteq {\downarrow}x$ trivially, hence $\ell b(u \cup \{x\}) = \ell b(u) \cap {\downarrow}x = \ell b(u) \neq \emptyset$ and therefore $u \cup \{x\} \in CluX$.

(IS4) For any cluster u and $x \in u$, $\ell b(u) \subseteq {\downarrow}\, x$ trivially and hence $u \vdash x$.

(IS5) For any clusters u and v, if $\ell b(u) \subseteq {\downarrow}x$ for all $x \in v$ then $\ell b(u) \subseteq \bigcap\{{\downarrow}x \mid x \in v\} = \ell b(v)$ and consequently $\ell b(v) \subseteq {\downarrow}y$ implies $\ell b(u) \subseteq {\downarrow}y$.

Note further for the extension of $\vdash$ to $Con(\mathbf{S}X)$ that $u \vdash v$ iff $\ell b(u) \subseteq \ell b(v)$.

Concerning $\mathfrak{N}(\mathbf{S}X)$ it has to be checked that the C_* are indeed covers of $Con(\mathbf{S}X) = CluX$ and that the set of these covers has all the required properties.

To see any C_* is a cover, consider $u, v \in CluX$ such that

$$u \cup \{s\} \vdash v \text{ for each } s \in C \text{ where } u \cup \{s\} \text{ is a cluster.}$$

Then $\ell b(u) \cap {\downarrow}\, s = \emptyset$ and $\ell b(u) \cap {\downarrow}\, s \subseteq \ell b(v)$. Hence for $x \in \ell b(u)$ and $z \in v$, ${\downarrow}x \cap {\downarrow}s \subseteq {\downarrow}z$ and because C is a cover of X this shows $x \leq z$. As a result, $\ell b(u) \subseteq {\downarrow}\, z$ for all $z \in v$, saying that $\ell b(u) \subseteq \ell b(v)$ and therefore $u \vdash v$. This proves the claim since ${\downarrow}\, u \cap {\downarrow}\{s\} \neq \emptyset$ iff $u \ \cup \{s\}$ is a cluster, and for each $x \in \ell b(u)$, ${\downarrow}x \cap {\downarrow}s \neq \emptyset$ obviously implies $u \cup \{s\}$ is a cluster.

Next, if $C \leq D$ in $\mathfrak{N}X$ then $C_* \leq D_*$ because $x \leq z$ in X implies $\{x\} \vdash \{z\}$ in $Con(\mathbf{S}X)$, and hence the C_*, $C \in \mathfrak{N}X$, form a filter basis of covers of $Con(\mathbf{S}X)$.

Regarding strong inclusion, note that $x \triangleleft_C z$ in X iff $\{x\} \triangleleft_{C_*} \{y\}$ in $Con(\mathbf{S}X)$ because

$$\begin{aligned} (C_*)_{\{x\}} &= \{\{s\} \in C_* \mid u \vdash \{x\} \text{ and } u \vdash \{s\} \text{ for some } u \in CluX\} \\ &= \{\{s\} \in C_* \mid \{x, s\} \in CluX\} = \{\{s\} \in C_* \mid s \in C_*\} \end{aligned}$$

and $s \leq z$ iff $\{s\} \vdash \{z\}$.

Before we turn to admissibility, note the following important property of the $\{x\}$ in $CluX$: If $\{x\} \vdash u$ implies $\{x\} \vdash v$ for all $x \in X$ then $u \vdash v$ since $\{x\} \vdash w$ iff $x \in \ell b(w)$ so that $\ell b(u) \subseteq \ell b(v)$ which means $u \vdash v$. As a result $u = \bigvee\{\{x\} \mid \{x\} \vdash u\}$ in $CluX$, saying that $\{\{x\} \mid x \in X\}$ is join-dense in $CluX$.

Now, for each $x \in X$, $\{x\} = \bigvee\{\{z\} \mid \{z\} \triangleleft \{x\}\}$. To see this, consider any $u \in CluX$ such that $\{z\} \vdash u$ for all $\{z\} \triangleleft \{x\}$. Then $\{z\} \vdash u$ for any $z \triangleleft u$ by our earlier observation and consequently $z \leq y$ for each $y \in u$ whenever $z \triangleleft x$. Hence $x \leq y$ by the admissibility of $\mathfrak{N}X$, and in all this implies $x \in \ell b(u)$, that is $\{x\} \vdash u$. Finally, given that the $\{x\}$, $x \in X$ are join-dense in $CluX$ this proves the admissibility.

As a result, the C_*, $C \in \mathfrak{N}X$, generate a nearness on $Con(\mathbf{S}X)$, and since $x \triangleleft s$ iff $\{x\} \triangleleft \{s\}$ we have $(\check{C})_* \subseteq (C_*)\check{}$, showing this nearness is strong. Similarly, $C \leq^* D$ in $\mathfrak{N}X$ implies $C_* \leq^* D_*$ and hence this nearness is a uniformity whenever this holds for $\mathfrak{N}X$.

Our next aim is to extend the object correspondence $X \mapsto \mathbf{S}X$ from **ApprSyst** to **StrInfSyst** to an appropriate correspondence of maps. Ideally one would like this to be a functor but that does not quite work out: what we obtain here is a functor only modulo a certain equivalence on the class of maps in either category. The crucial point here is that, in both cases, the set of maps between any pair of objects is partially ordered, namely by set inclusion, with composition of maps preserving this partial order, and one can then consider the equivalence relations generated by these partial orders. We refer to this as "modulo order" for short. Similarly, by a lax functor between the categories involved here we mean a correspondence of objects to objects and maps to maps which takes identity maps to identity maps but for which the equality of composition preservation is replaced by the partial order – which certainly implies equality modulo order.

For any map $f : X \to Y$ of approximation systems consider $\mathbf{S}f \subseteq Con(\mathbf{S}X) \times Con(\mathbf{S}Y) = CluX \times CluY$ such that

$\emptyset \mathbf{S}f \emptyset$ and

$u\mathbf{S}fv$ iff there exist $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ such that

$$u \vdash \{x_1, \ldots, x_n\},\ x_i f y_i \text{ for all } i,\ \{y_1, \ldots, y_n\} \vdash v.$$

Note here that $\{x_1, \ldots, x_n\}$ is a cluster by the first condition and so is $\{y_1, \ldots, y_n\}$ by the properties of f.

Now, given $u\mathbf{S}fv$ and $u\mathbf{S}fw$, that is

$$u \vdash \{x_1, \ldots, x_n\},\ x_i f y_i,\ \{y_1, \ldots, y_n\} \vdash v$$

and

$$u \vdash \{x'_1, \ldots, x'_m\},\ x'_i f y'_i,\ \{y'_1, \ldots, y'_m\} \vdash w,$$

it follows that $u \vdash \{x_1, \ldots, x'_m\}$ and $\{y_1, \ldots, y'_m\} \vdash v \cup w$ and hence $u\mathbf{S}f(v \cup w)$. Since the remaining conditions are obvious this shows $\mathbf{S}f$ is a map of the underlying information systems. Furthermore, it is uniform: for any $C \in \mathfrak{N}Y$,

$$(\mathbf{S}f)^{-1}[C_*] = \{u \in CluX \mid u \vdash \{x_1, \ldots, x_n\},\ x_i f y_i,\ \{y_1, \ldots, y_n\} \vdash s,\ s \in C\}$$
$$\supseteq \{\{x\} \mid xfs,\ s \in C\} = (f^{-1}[C])_*$$

and the latter belongs to $\mathfrak{N}(\mathbf{S}X)$ since $f^{-1}[C] \in \mathfrak{N}X$.

Next, the correspondence $f \mapsto \mathbf{S}f$ takes identities to identities: $u\mathbf{S}(\leq_X)v$ means that $u \vdash \{x_1, \ldots, x_n\}$ where $x_i \leq z_i$ and $\{z_1, \ldots, z_n\} \vdash v$, and the middle condition implies $\{x_1, \ldots, x_n\} \vdash \{z_1, \ldots, z_n\}$ so that $u \vdash v$. Conversely, if $u \vdash v$ then also $u \vdash u \cup v \vdash v$ and hence $u\mathbf{S}(\leq_X)v$.

Regarding composition, if $f : X \to Y$ and $g : Y \to Z$ then $u\mathbf{S}(f \circ g)w$ says that

$$u \vdash \{x_1, \ldots, x_n\},\ x_i(f \circ g)z_i,\ \{z_1, \ldots, z_n\} \vdash w$$

and hence by the middle condition $x_i f y_i$ and $y_i g z_i$ for suitable $y_i \in Y$; thus $u\mathbf{S}fv$ and $v\mathbf{S}gw$ with $v = \{y_1, \ldots, y_n\}$ so that $u(\mathbf{S}f) \circ (\mathbf{S}g)w$, showing $\mathbf{S}(gf) \leq (\mathbf{S}g)(\mathbf{S}f)$. Thus $\mathbf{S} : \mathbf{ApprSyst} \to \mathbf{StrInfSyst}$ is a lax functor in the sense explained earlier.

Proposition 2.1 *Modulo order,* $\mathbf{S}$ *is an equivalence between* **ApprSyst** *and* **StrInfSyst**.

Proof. This will be given in the following steps: (1) $\mathbf{S}$ is onto on objects, up to isomorphism. (2) For any $g : \mathbf{S}X \to \mathbf{S}Y$ there exists $\hat{g} : X \to Y$ such that $\mathbf{S}\hat{g} \leq g$, showing $f \mapsto \mathbf{S}f$ is full modulo order. (3) For any $f, g : X \to y$, $f \sim g$ iff $\mathbf{S}f \sim \mathbf{S}g$ for the equivalence relation $\sim$ generated by $\leq$.

(1) Given any stratified information system S, let X be the partially ordered set $ConS/\dashv\vdash$ of $\dashv\vdash$-blocks $[u],[v],\ldots$ as discussed in Section 0 together with the strong nearness determined by $\mathfrak{N}S$, that is, consisting of the covers $C_{\#} = \{[s] \mid s \in C\}$ of $ConS/\dashv\vdash$ for $C \in \mathfrak{N}S$. Then $\mathbf{S}X$ is given by the following specifications:

$|\mathbf{S}X| = ConS/\dashv\vdash$,

$Con(\mathbf{S}X) = Clu(ConS/\dashv\vdash)$, consisting of the $\{[u_1],\ldots,[u_n]\}$ such that $\bot \neq [u_1]\wedge\cdots\wedge[u_n]$ in terms of the semilattice $(ConS/\dashv\vdash)\cup\{\bot\}$ and hence of the $\{[u_1],\ldots,[u_n]\}$ for which $u_1\cup\cdots\cup u_n \in ConS$,

$\{[u_1],\ldots,[u_n]\} \vdash [u]$ iff $[u_1]\wedge\cdots\wedge[u_n] \leq [u]$ iff $u_1\cup\cdots\cup u_n \vdash u$, and

$\mathfrak{N}(\mathbf{S}X)$ is generated by the covers $(C_{\#})_* = \{\{[s]\} \mid s \in C\}$.

Now consider the relations $f \subseteq Con(\mathbf{S}X) \times ConS$ and $g \subseteq ConS \times Con(\mathbf{S}X)$ such that

$$\{[u_1],\ldots,[u_n]\} f v \quad \text{iff} \quad u_1\cup\cdots\cup u_n \vdash v,\ \emptyset f \emptyset$$

and

$$u g \{[v_1],\ldots,[v_n]|] \quad \text{iff} \quad u \vdash v_1\cup\cdots\cup v_n,\ \emptyset g \emptyset.$$

It is easily checked that these are maps of the information systems involved, based on the observation that

$$\{[w_1],\ldots,[w_m]\} \vdash \{[u_1],\ldots,[u_n]\} \quad \text{iff} \quad w_1\cup\cdots\cup w_m \vdash u_1\cup\cdots\cup u_n$$

for any clusters $\{w_1,\ldots,w_m\}$ and $\{u_1,\ldots,u_n\}$ in $ConS$. In the same way, one readily verifies that $f\circ g = \vdash_{\mathbf{S}X}$ and $g \circ f = \vdash_S$, and hence f and g are isomorphism, inverse to each other, of the underlying information systems.

Regarding the uniformity conditions we have, for any $C \in \mathfrak{N}S$,

$$\begin{aligned} f^{-1}[C] &= \{\{[u_1],\ldots,[u_n]\} \in Con(\mathbf{S}X) \mid u_1\cup\cdots\cup u_n \vdash s, s \in C\} \\ &\supseteq \{\{[s]\} \mid s \in C\} = (C_{\#})_*, \end{aligned}$$

and since $\mathfrak{N}(\mathbf{S}X)$ is generated by the $(C_{\#})_*$ this shows $f^{-1}[C] \in \mathfrak{N}(\mathbf{S}X)$. Similarly,

$$\begin{aligned} g^{-1}[(C_{\#})_*] &= \{u \in ConS \mid ug\{[s]\},\ s \in C\} \\ &= \{u \in ConS \mid u \vdash s,\ s \in C\} \supseteq C \end{aligned}$$

which proves the condition for g, again since the $(C_{\#})_*$ generate $\mathfrak{N}(\mathbf{S}X)$.

(2) For any $g : \mathbf{S}X \to \mathbf{S}Y$, let $\hat{g} \subseteq X \times Y$ be defined by $x\hat{g}y$ iff $\{x\}g\{y\}$. We claim this is a map $\hat{g} : X \to Y$.

$e\hat{g}e$ because $\emptyset g \emptyset$ and $\{e\} \vdash\!\dashv \emptyset$.

If $x' \leq x$, $\{x\}g\{y\}$, and $y \leq y'$ then $\{x'\} \vdash \{x\}$ and $\{y\} \vdash \{y'\}$ and hence $\{x'\}g\{y'\}$.

For any cluster $\{x_1, \ldots, x_n\}$, if $\{x_i\}g\{y_i\}$ for $y_1, \ldots, y_n \in Y$ then $\{x_1, \ldots, x_n\} \vdash \{x_i\}$ and therefore $\{x_1, \ldots, x_n\}g\{y_i\}$ for each i which implies $\{x_1, \ldots, x_n\}g\{y_1, \ldots, y_n\}$, showing $\{y_1, \ldots, y_n\}$ is a cluster.

For any $C \in \mathfrak{N}Y$, $g^{-1}[C_*] \in \mathfrak{N}(\mathbf{S}X)$ since $C_* \in \mathfrak{N}(\mathbf{S}Y)$, and hence $B_* \leq g^{-1}[C_*]$ for some $B \in \mathfrak{N}X$ by the definition of $\mathfrak{N}(\mathbf{S}X)$. Now, the latter means for each $z \in B$ there exist $u \in CluX$ for which

$$\{z\} \vdash u \quad \text{and} \quad ug\{s\} \text{ for some } s \in C;$$

consequently $\{z\}g\{s\}$, showing that $z \in \hat{g}^{-1}[C]$, hence $B \subseteq \hat{g}^{-1}[C]$ and therefore $\hat{g}^{-1}[C] \in \mathfrak{N}X$.

This proves the claim. Further, for any $u \in CluX$ and $v \in CluY$, if $u\mathbf{S}\hat{g}v$ then

$$u \vdash \{x_1, \ldots, x_n\},\ \{x_i\}g\{y_i\}, \quad \text{and} \quad \{y_1, \ldots, y_n\} \vdash v$$

for suitable $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$, hence $ug\{y_i\}$ for each i so that $ug\{y_1, \ldots, y_n\}$ and consequently ugv. Thus $\mathbf{S}\hat{g} \leq g$, as desired.

(3) For any $f : X \to Y$, xfy trivially implies $\{x\}\mathbf{S}f\{y\}$ and therefore $f \leq (\mathbf{S}f)\hat{}$. Hence, for any $f, g : X \to Y$, if $\mathbf{S}f \sim \mathbf{S}g$ then $(\mathbf{S}f)\hat{} \sim (\mathbf{S}g)\hat{}$ as $()\hat{}$ clearly preserves inclusion, and consequently $f \sim g$ because $f \leq (\mathbf{S}f)\hat{} \sim (\mathbf{S}g)\hat{} \geq g$. The converse being trivial since $\mathbf{S}$ also preserves inclusion, this shows $f \sim g$ iff $\mathbf{S}f \sim \mathbf{S}g$ for any $f, g : X \to Y$. □

Remark 2.1 It is clear from the construction of $\mathbf{S}X$ and from part (1) of the above proof that $\mathbf{S}$ induces an equivalence modulo order on the subcategories of *uniform* entities in either category.

3 Nearness semilattices and stratified information systems

A *nearness semilattice* is a meet semilattice, always understood to be bounded, together with a strong nearness on it. We denote these by $A, B, \ldots$, the corresponding nearness by $\mathfrak{N}A, \mathfrak{N}B, \ldots$, and again permit notational confusion between A and its underlying set. For notation and basic results concerning mere semilattices we refer to Section 0.

For nearness semilattices A and B, an extended homomorphism $\varphi : A \multimap\!\!\bullet\, B$ of the underlying semilattices is called *uniform* if

$$\varphi[C] = \bigcup\{\varphi(s) \mid s \in C\}$$

belongs to $\mathfrak{N}B$ for each $C \in \mathfrak{N}A$.

Lemma 3.1. *For any nearness semilattices, the identity maps and composites of uniform maps are uniform.*

Proof. Recall that the identity map $A \multimap\!\!\bullet\, A$ is $\eta_A : A \to \mathfrak{D}A$ taking a to $\downarrow a$. Hence, for any $E \in \mathfrak{N}A$,

$$\eta_A[E] = \bigcup\{\downarrow s \mid s \in E\}$$

which contains E and hence belongs to $\mathfrak{N}A$.

Further, for any uniform $\varphi : A \multimap\!\!\bullet\, B$ and $\psi : B \multimap\!\!\bullet\, C$ and any $E \in \mathfrak{N}A$,

$$\begin{aligned} \psi \bullet \varphi[E] &= \bigcup\{\psi \bullet \varphi(s) \mid s \in E\} = \bigcup\{\bigcup\{\psi(b) \mid b \in \varphi(s)\} \mid s \in E\} \\ &= \bigcup\{\psi(b) \mid b \in \bigcup\{\varphi(s) \mid s \in E\} = \psi[\varphi[E]] \end{aligned}$$

which belongs to $\mathfrak{N}C$ since φ and ψ are uniform. □

As a result, we have the category of nearness semilattice and their uniform extended homomorphisms which will be denoted **ENearS**.

For any stratified information system S, the semilattice $(ConS/ \vdash\!\dashv) \cup \{\bot\}$ associated with S inherits from S the nearness which is derived from the nearness $\mathfrak{N}S$ on $ConS$ via the processes discussed at the end of Section 1. Hence, in the present setting, we shall consider $\mathfrak{L}S$ as the *nearness semilattice* thus defined.

Lemma 3.2. *For any stratified information systems S and T, a map of information systems $f : S \to T$ is uniform iff $\mathfrak{L}f : \mathfrak{L}T \to \mathfrak{L}S$ is uniform.*

Proof. If $\nu : ConS \to \mathfrak{L}S$ is the quotient map taking u to $[u]$ then, for any $C \in \mathfrak{N}T$,

$$\mathfrak{L}f[\nu[C]] = \{[u] \in \mathfrak{L}S \mid ufs \text{ for some } s \in C\} = \nu[f^{-1}[C]],$$

and this belongs to $\mathfrak{N}(\mathfrak{L}S)$ iff $f^{-1}[C] \in \mathfrak{N}S$ by the relation between nearnesses on $ConS$ and $(ConS/ \vdash\!\dashv) \cup \{\bot\}$. □

It follows that we have a contravariant functor, again to be denoted by $\mathfrak{L}$, from **StrInfSyst** to **ENearS** given by the correspondences $S \mapsto \mathfrak{L}S$, $f \mapsto \mathfrak{L}f$. Moreover, any nearness semilattice A is isomorphic to $\mathfrak{L}S$ for the stratified information system S given as follows:

$|S| = |A| - \{0\}$,

$ConS$ consists of all finite $u \subseteq |S|$ such that $\bigwedge u \neq 0$,

$u \vdash x$ for $u \in ConS$ and $x \in X$ iff $\bigwedge u \leq x$, and

$\mathfrak{N}S$ is generated by the $C_* = \{\{s\} \mid s \in C\}$. That the last specification indeed determines a nearness on $ConS$ follows immediately from the proof of the corresponding fact for $\mathbf{S}X$ in the last section, and since the familiar algebraic isomorphism $A \to \mathfrak{L}S$ taking a to $[\{a\}]$ for $a \neq 0$ and 0 to $\perp$ is also a nearness isomorphism it is clear that $A \simeq \mathfrak{L}S$ in **ENearS**.

In all, we therefore have

Proposition 3.1. *$\mathfrak{L}$ is a dual equivalence between* **StrInfSyst** *and* **ENearS**.

Corollary 3.1. *The composite* $\mathfrak{L}\mathbf{S}$: **ApprSyst** $\to$ **ENearS** *is a dual equivalence modulo order.*

It may be of interest to describe the effect of this composite directly, at least for the objects. For any approximation system X, the underlying semilattice of $\mathfrak{L}\mathbf{S}X$, apart from the added bottom $\perp$, consists of the clusters of X modulo the equivalence relation

$$\downarrow x_1 \cap \cdots \cap \downarrow x_n = \downarrow y_1 \cap \cdots \cap \downarrow y_m,$$

with partial order derived from the preorder $\vdash$ defined by

$$\{x_1, \ldots, x_n\} \vdash \{y_1, \ldots, y_m\} \quad \text{iff} \quad \downarrow x_1 \cap \cdots \cap \downarrow x_n \subseteq \downarrow y_1 \cap \cdots \cap \downarrow y_m.$$

Hence this may just as well be taken as the set of all non-void $\downarrow x_1 \cap \cdots \cap \downarrow x_n$, partially ordered by inclusion. Also, there is the embedding $x \mapsto \downarrow x$ of X into this whose image is a join-dense generating set for $\cap$, and the underlying semilattice of $\mathfrak{L}\mathbf{S}X$ is abstractly characterized by this. Furthermore, the nearness determined by $\mathfrak{N}X$ on this semilattice is generated by the covers $\{\downarrow x \mid x \in C\}$, $C \in \mathfrak{N}X$; hence if we regard X as a subset of $\mathfrak{L}\mathbf{S}X$ $\mathfrak{N}X$ itself generates the nearness.

Remark 3.1. It is obvious from the closing remarks of Section 1 that the above dual equivalence induces a dual equivalence between the subcategories of *uniform* entities in either category, and the corresponding result holds for the situation covered by the corollary.

4 Nearness semilattices and nearness frames

The purpose of this section is to relate our category of nearness semilattices to the more familiar category of *complete strong nearness frames* which will be denoted **CSNearF**. For details regarding the latter we refer to Section 0.

We begin with a special notion which will play a central rôle in this context.

In the following, a semilattice homomorphism $\varphi : A \to L$ between a (bounded meet) semilattice A and a frame L is always understood to be a *meet* homomorphism preserving zero and unit. Furthermore, for nearness semilattices and nearness frames, φ is called *uniform* if $\varphi[C] \in \mathfrak{N}L$ for all $C \in \mathfrak{N}A$. Finally, a uniform $\varphi : A \to L$ is called *regular* if

$$\varphi(a) = \bigvee\{\varphi(x) \mid x \triangleleft a\}$$

for all $a \in A$.

Lemma 4.1. *For any uniform semilattice homomorphism $\varphi : A \to L$, $\varphi^o : A \to L$ defined by*

$$\varphi^o(a) = \bigvee\{\varphi(x) \mid x \triangleleft a\}$$

is a regular uniform semilattice homomorphism, and if $\psi \le \varphi$ is any such homomorphism then $\psi = \varphi^o$.

Proof. φ^o is a homomorphism: it preserves zero and unit since $0 \triangleleft 0$ and $e \triangleleft e$, and

$$\varphi^o(a) \wedge \varphi^o(b) = \bigvee\{\varphi(x \wedge y) \mid x \triangleleft a,\ y \triangleleft b\} \le \varphi^o(a \wedge b)$$

because $x \triangleleft a$ and $y \triangleleft b$ implies $x \wedge y \triangleleft a \wedge b$, and as φ^o is clearly order preserving it follows that it preserves $\wedge$.

Further, φ^o is uniform: for any $C \in \mathfrak{N}A$,

$$\varphi^o[C] = \left\{\bigvee\{\varphi(x) \mid x \triangleleft s\} \mid s \in C\right\} \ge \varphi[\check{C}]$$

and the latter belongs to $\mathfrak{N}L$ since $\mathfrak{N}A$ is strong and φ is uniform.

Next, for any $C \in \mathfrak{N}A$, if $x \triangleleft_C a$ so that $C_x \subseteq {\downarrow}a$ then

$$(*) \qquad C \le \{a\} \cup \{s \in C \mid s \wedge x = 0\},$$

showing the set on the right belongs to $\mathfrak{N}A$, and since φ^{oo} is uniform it follows that

$$\varphi^{oo}(a) \vee \bigvee \{\varphi^{oo}(s) \mid s \in C,\ s \wedge x = 0\} = e.$$

Further, $\varphi^{oo}(s) \leq \varphi(s)$ and $\varphi(s) \wedge \varphi(x) = 0$, and this implies $\varphi(x) \leq \varphi^{oo}(a)$, hence $\varphi^{o}(a) \leq \varphi^{oo}(a)$ and consequently $\varphi^{oo}(a) = \varphi^{o}(a)$, showing φ^{o} is regular.

Finally, if $\psi : A \to L$ is any regular uniform semilattice homomorphism such that $\psi(a) \leq \varphi(a)$ for all $a \in A$ then by $(*)$ we also have

$$\psi(a) \vee \bigvee \{\psi(s) \mid s \in C,\ s \wedge x = 0\} = e$$

whenever $x \triangleleft_C a$, and since $\psi(s) \wedge \varphi(x) \leq \varphi(s \wedge x) = 0$ it follows that $\varphi(x) \leq \psi(a)$, showing $\varphi^{o}(a) \leq \psi(a)$. On the other hand, $\psi \leq \varphi$ implies $\psi = \psi^{o} \leq \varphi^{o}$, and in all $\psi = \varphi^{o}$, as claimed. □

Next, we construct a nearness frame for any given nearness semilattice as follows. For the details concerning *prenuclei* see Banaschewski [1].

In the frame $\mathfrak{D}A$ of all downsets of A, that is, the $U \subseteq A$ such that $0 \in U$ and $x \in U$ whenever $x \leq y$ and $y \in U$, the closure system of all $U \in \mathfrak{D}A$ for which

$$(C) \qquad \begin{array}{ll} \{a\} \wedge C \subseteq \text{ implies } a \in U, & \text{for all } a \in A \text{ and } C \in \mathfrak{N}A, \\ k(a) = \{x \in A \mid x \triangleleft a\} \subseteq U & \text{implies } a \in U, \text{ for all } a \in A \end{array}$$

is a quotient frame of $\mathfrak{D}A$: the corresponding order preserving and expansive operator ℓ_0 on $\mathfrak{D}A$ such that

$$\ell_0(U) = \{a \in A \mid \{a\} \wedge C \subseteq U \text{ for some } C \in \mathfrak{N}A \text{ or } k(a) \subseteq U\}$$

is a prenucleus, that is, $\ell_0(U) \cap V \subseteq \ell_0(U \cap V)$ because

$$\{a\} \wedge C \subseteq U \text{ and } a \in V \quad \text{ implies } \{a\} \wedge C \subseteq U \cap V$$

and

$$k(a) \subseteq U \text{ and } a \in V \quad \text{ implies } k(a) \subseteq U \cap V.$$

We let ℓ be the associated nucleus, recalling that the present closure system is $\mathrm{Fix}(\ell) = \mathrm{Fix}(\ell_0)$.

Note that any $\downarrow a$ satisfies (C), the first part by the properties of covers and the second by admissibility. Further, the sets $\tilde{C} = \{\downarrow s \mid s \in C\}$

for $C \in \mathfrak{N}A$ are covers of $\mathrm{Fix}(\ell)$ because non-void join in this is union followed by the action of ℓ and $C \subseteq U = \ell(U)$ implies $e \in U$. Also, for any $C, D \in \mathfrak{N}A$,

$$\tilde{C} \wedge \tilde{D} = \{\downarrow s \cap \downarrow t \mid s \in C,\ t \in D\} = \{\downarrow (s \wedge t) \mid s \in C,\ t \in D\} = (C \wedge D)\tilde{}$$

showing the $\tilde{C}$ form a filter basis of covers $\mathrm{Fix}(\ell)$. We claim this is admissible.

To begin with, note that $\downarrow x \triangleleft_{\tilde{C}} \downarrow a$ iff $x \triangleleft_C a$ for any $x, a \in A$ and $C \in \mathfrak{N}A$ because the map $z \mapsto \downarrow z$ is a semilattice embedding. Hence, for any $a \in A$,

$$\downarrow a = \bigvee \{\downarrow x \mid x \triangleleft a\} = \bigvee \{\downarrow x \mid \downarrow x \triangleleft \downarrow a\}$$

in $\mathrm{Fix}(\ell)$, the first step by (C) and the second $\triangleleft$ referring to the $\tilde{C}$, $C \in \mathfrak{N}A$, and since any $U \in \mathrm{Fix}(\ell)$ is the join of the $\downarrow a$, $a \in U$, this proves the claim.

We let $\mathfrak{F}A$ be the resulting nearness frame, noting that it is indeed strong because $(\tilde{C})\check{}$ contains $(\check{C})\tilde{}$; moreover, it will be uniform iff A is uniform because $C \leq^* D$ implies $\tilde{C} \leq^* \tilde{D}$ and conversely. Further, $\varepsilon_A : A \to \mathfrak{F}A$ will be the map $a \mapsto \downarrow a$. This is a semilattice embedding, as already noted, uniform because $\varepsilon_A[C] = \tilde{C}$, and regular since $\varepsilon_A(a) = \bigvee\{\varepsilon_A(x) \mid x \triangleleft a\}$, again as observed already.

Proposition 4.1. *$\varepsilon_A : A \to \mathfrak{F}A$ is the universal regular uniform semilattice homomorphism from A to nearness frames.*

Proof. Given any $\varphi : A \to L$ as stated, the obvious choice for the desired $\tilde{\varphi} : \mathfrak{F}A \to L$ is $\tilde{\varphi}(U) = \bigvee \varphi[U]$. Indeed, for arbitrary $U \in \mathfrak{D}A$ this defines the standard frame homomorphism $\mathfrak{D}A \to L$ by which $a \mapsto \downarrow a$ is shows to be the universal semilattice homomorphism to mere frames, and the point to check is that this factors through the quotient homomorphism $\ell : \mathfrak{D}A \to \mathfrak{F}A$ defining $\mathfrak{F}A$. This amounts to saying that $\tilde{\varphi}\ell = \tilde{\varphi}$ which will follow, by general principles (Banaschewski [1]) if we show $\tilde{\varphi}\ell_0 = \tilde{\varphi}$. Now, for any $a \in A$, $C \in \mathfrak{N}A$, and $U \in \mathfrak{D}A$, if $\{a\} \wedge C \subseteq U$ then $\{\varphi(a)\} \wedge \{\varphi[C]\} \subseteq \varphi[U]$, hence $\{\varphi(a)\} \wedge \varphi[C] \subseteq \downarrow \tilde{\varphi}(U)$, and since $\varphi[C]$ is a cover of L this shows $\varphi(a) \leq \tilde{\varphi}(U)$. Similarly, if $k(a) \subseteq U$ then $\varphi[k(a)] \subseteq \downarrow \tilde{\varphi}(U)$, and since $\varphi(a) = \bigvee \varphi[k(a)]$ by regularity we again obtain $\varphi(a) \leq \tilde{\varphi}(U)$. In all, then, $\tilde{\varphi}(\ell_0(U)) \leq \tilde{\varphi}(U)$, the non-trivial part of the desired equality. Further, $\tilde{\varphi}\varepsilon_A = \varphi$ immediately be the definition of $\tilde{\varphi}$, and this condition clearly characterizes $\tilde{\varphi}$ since $\mathrm{Im}(\varepsilon_A)$ is join-dense in $\mathfrak{F}A$. Finally, $\tilde{\varphi}$ is uniform because $\mathfrak{N}(\mathfrak{F}A)$ is generated by the covers $\varepsilon_A[C]$, $C \in \mathfrak{N}A$. □

Corollary 4.1. *For any uniform semilattice homomorphism $\varphi : A \to L$ from a nearness semilattice to a nearness frame there exists a unique uniform frame homomorphism $h : \mathfrak{F}A \to L$ such that $h\varepsilon_A \leq \varphi$.*

Proof. By Lemma 4.1 we have the regular $\varphi^o : A \to L$ and hence the proposition supplies a uniform frame homomorphism $h : \mathfrak{F}A \to L$, unique such that $h\varepsilon_A = \varphi^o$. Now, the latter trivially implies $h\varepsilon_A \leq \varphi$, but this condition equally characterizes h. For any uniform homomorphism $k : \mathfrak{F}A \to L$ such that $k\varepsilon_A \leq \varphi^o$, $k\varepsilon_A$ is a regular since ε_A is and k is a frame homomorphism, hence $k\varepsilon_A = \varphi^o$ by Lemma 4.1 and therefore $k\varepsilon_A = h\varepsilon_A$, which proves $k = h$ because $Im(\varepsilon_A)$ generates $\mathfrak{F}A$. □

Corollary 4.2. *For any nearness semilattice A, $\mathfrak{F}A$ is complete.*

Proof. Let $k : K \to \mathfrak{F}A$ be the completion of $\mathfrak{F}A$. Then the right adjoint $k_* : \mathfrak{F}A \to K$ is a uniform semilattice homomorphism (Banaschewski [2]), and Corollary 4.1 provides a frame homomorphism $h : \mathfrak{F}A \to K$ such that $k\varepsilon_A \leq k_*\varepsilon_A$. It follows that $kh\varepsilon_A \leq \varepsilon_A$, hence $kh \leq \mathrm{id}_{\mathfrak{F}A}$ since $Im(\varepsilon_A)$ generate $\mathfrak{F}A$, and therefore $kh = \mathrm{id}_{\mathfrak{F}A}$ by the regularity of the frames involved. Finally, since k is dense this implies k is an isomorphism, again by regularity. □

For nearness semilattices A and B we let $i_A : \mathfrak{F}A \to \mathfrak{D}A$ be the identical embedding, a semilattice homomorphism by the properties of $\mathfrak{F}A$, and $\nu_B : \mathfrak{D}B \to \mathfrak{F}B$ the quotient homomorphism defining $\mathfrak{F}B$. Then we have:

Lemma 4.2. (1) *For any inform extended semilattice homomorphism $\varphi : A \multimap B$, $\nu_B\varphi : A \to \mathfrak{F}B$ is a uniform semilattice homomorphism.*
(2) *For any uniform semilattice homomorphism $\psi \to A\mathfrak{F}B$, $i_B\psi : A \multimap B$ is a uniform extended semilattice homomorphism.*

Proof. First a general observation: a cover $\mathfrak{C}$ of $\mathfrak{F}A$ is uniform iff $\bigcup \mathfrak{C}$ is a uniform cover of A. It is clear that $\bigcup \mathfrak{C}$ is a cover of A for any cover $\mathfrak{C}$ of $\mathfrak{F}A$, and if $\mathfrak{C}$ is uniform then $\tilde{E} \subseteq \mathfrak{C}$ for some $E \in \mathfrak{N}A$ which implies $E \subseteq \bigcup \mathfrak{C}$. Conversely, if $E = \bigcup \mathfrak{C}$ is uniform then $\mathfrak{C}$ is uniform because $\tilde{E} \leq \mathfrak{C}$.

(1) $\nu_B\varphi$ is a semilattice homomorphism because ν_B is a nucleus such that $\nu_B(\downarrow 0) = \downarrow 0$. Further, for any $C \in \mathfrak{N}A$,

$$\varphi[C] = \bigcup\{\varphi(s) \mid s \in C\} \subseteq \bigcup\{\nu_B\varphi(s) \mid s \in C\} = \bigcup(\nu_B\varphi)[C],$$

the second step since ν_B is a closure operator, and since $\varphi[C]$ is uniform by hypothesis the same holds for $(\nu_B\varphi)[C]$ by the above observation.

(2) $i_B\psi$ is clearly a semilattice homomorphism. Further, for any $C \in \mathfrak{N}A$,

$$i_B\psi[B] = \bigcup\{i_B\psi(s) \mid s \in C\} = \bigcup\{\psi(s) \mid s \in C\} = \bigcup \psi[C]$$

is uniform, by our initial observation, because $\psi[C] \in \mathfrak{N}(\mathfrak{F}B)$ by hypothesis. □

Proposition 4.2 *The correspondence $A \mapsto \mathfrak{F}A$ extends to a full functor $\mathfrak{F}$: **ENearS** $\to$ **CSNearF**.*

Proof. For any $\varphi : A \multimap B$ in **ENearS**, $\nu_B\varphi$ is a uniform semilattice homomorphism by Lemma 4.2 and hence determines a uniform frame homomorphism $\bar{\varphi} : \mathfrak{F}A \to \mathfrak{F}B$, unique such that $\bar{\varphi}\varepsilon_A \leq \nu_B\varphi$, by Corollary 4.1. We claim the correspondence $\varphi \mapsto \bar{\varphi}$ has the required properties.

First, for any identity map $\eta_A : A \to \mathfrak{D}A$, $\nu_A\eta_A = \varepsilon_A$ so that $\bar{\eta}_A\varepsilon_A \leq \varepsilon_A$, and this implies $\bar{\eta}_A = \mathrm{id}_{\mathfrak{F}A}$ by reasoning already encountered.

Regarding composition, given any $\varphi : A \multimap B$ and $\psi : B \multimap C$, we have $\bar{\varphi} : \mathfrak{F}A \to \mathfrak{F}B$ and $\bar{\psi} : \mathfrak{F}B \to \mathfrak{F}C$ such that $\bar{\varphi}\varepsilon_A \leq \nu_B\varphi$ and $\bar{\psi}\varepsilon_B \leq \nu_C\psi$, and then for any $a \in A$

$$\begin{aligned} \bar{\psi}\nu_B\varphi(a) &= \bar{\psi}(\bigvee\{\downarrow x \mid x \in \varphi(a)\}) = \bigvee\{\bar{\psi}\varepsilon_B(x) \mid x \in \varphi(a)\} \leq \\ &\leq \bigvee\{\nu_C\psi(x) \mid x \in \varphi(a)\} = \nu_C(\bigcup\{\psi(x) \mid x \in \varphi(a)\}) = \nu_C(\psi \cdot \varphi)(a), \end{aligned}$$

the inequality by the definition of $\bar{\psi}$. Further, $\bar{\psi}\bar{\varphi}\varepsilon_A \leq \bar{\psi}\nu_B\varphi$ by the definition of $\bar{\varphi}$ and hence $\bar{\psi}\bar{\varphi}\varepsilon_A \leq \nu_C(\psi \cdot \varphi)$, showing that $\overline{\psi \cdot \varphi} = \bar{\psi} \cdot \bar{\varphi}$.

It follows that putting $\mathfrak{F}\varphi = \bar{\varphi}$ defines the desired functor.

Further, for any uniform homomorphism $h : \mathfrak{F}A \to \mathfrak{F}B$, $h\varepsilon_A : A \to \mathfrak{F}B$ is a uniform semilattice homomorphism and hence $i_Bh\varepsilon_A : A \multimap B$ a uniform extended semilattice homomorphism by Lemma 4.2. Now

$$\mathfrak{F}(i_Bh\varepsilon_A)\varepsilon_A \leq \nu_Bi_Bh\varepsilon_A = h\varepsilon_A$$

and hence $\mathfrak{F}(i_Bh\varepsilon_A) = h$, showing $\mathfrak{F}$ is full. □

If A is the underlying nearness semilattice of a nearness frame L then the definition of $\mathfrak{F}A$ shows that this is exactly the *completion* of L and consequently the functor $\mathfrak{F}$ is also onto on objects, up to isomorphism. The following makes the nature of $\mathfrak{F}$ more precise.

Proposition 4.3. *Modulo order $\mathfrak{F}$ is an equivalence between* **ENearS** *and* **CSNearF**.

Proof. Of course, the partial order between the maps in the second category is trivial, by the regularity of nearness frames, and hence what has to be shown is that, for any $\varphi, \psi : A \multimap B$ in **ENearS**, $\mathfrak{F}\varphi = \mathfrak{F}\psi$ iff $\varphi \sim \psi$. Now, if $\varphi \leq \psi$ then $(\nu_B\varphi)^o = (\nu_B\psi)^o$ by Lemma 4.1, hence $(\mathfrak{F}\varphi)\varepsilon_A = (\mathfrak{F}\psi)\varepsilon_A$ and therefore $\mathfrak{F}\varphi = \mathfrak{F}\psi$. Conversely, if $\mathfrak{F}\varphi = \mathfrak{F}\psi$ then $(\nu_B\varphi)^o = (\nu_B\psi)^o$, and we have the relations

$$\varphi \leq i_B\nu_B\varphi \geq i_B(\nu_B\varphi)^o = i_B(\nu_B\psi)^o \leq i_B\nu_B\psi \geq \psi$$

where all maps are uniform extended homomorphisms $A \multimap B$ by Lemma 4.2, and hence $\varphi \sim \psi$. □

Corollary 4.3. *The functors* $\mathfrak{F}\mathfrak{L}\mathbf{S}$: **ApprSyst** $\to$ **CSNearF** *and* $\mathfrak{F}\mathfrak{L}$: **StrInfSyst** $\to$ **CSNearF** *are dual equivalences modulo order.*

Proof. Regarding $\mathfrak{F}\mathfrak{L}\mathbf{S}$ it should be noted that, although **S** and hence $\mathfrak{L}\mathbf{S}$ are only lax functors, the latter since $\mathfrak{L}$ is clearly order preserving on maps, $\mathfrak{F}$ transform inequalities into equalities and hence $\mathfrak{F}\mathfrak{L}\mathbf{S}$ is indeed a functor. □

Remark 4.1. Each of the above three functors has the property that it makes uniform entities correspond to uniform entities, inducing the same relation between the subcategories involved.

Remark 4.2. To provide some added perspective it should be pointed out that the functor $\mathfrak{F}$ as such really is not an equivalence, saying: there do exist different $\varphi, \psi : A \multimap B$ for which $\varphi \leq \psi$. Let A be the underlying semilattice of an arbitrary regular frame with the (strong!) nearness of all covers and consider $\varphi : A \multimap A$ such that $\varphi(a) = \downarrow(a^{**})$ where $(\ )^*$ indicates pseudocomplement. This is clearly an extended semilattice homomorphism, uniform since $a^{**} \geq a$. Further, $\eta_A \leq \varphi$ for the same reason, but examples where $\eta_A \neq \varphi$ abound: this holds for the frame of ideals of any complete Boolean algebra which has non-principal ideals, and by the Principle of Dependent Choices this is the case for any infinite Boolean algebra.

5 Points

Following Banaschewski-Pultr [5] it is natural to consider as the *points* of an approximation system X its regular Cauchy filters, that is, the down-directed upsets $P \subseteq X$ which meet every uniform cover and contain, for each $x \in P$, some $z \triangleleft x$ (listing the properties in reverse order). We let PtX be the nearness space of these P with basic open sets

$$\mathfrak{W}_x = \{P \in PtX \mid x \in P\},$$

equipped with the nearness determined by the open covers

$$(\#) \qquad \{\mathfrak{W}_x \mid x \in C\}, \qquad C \in \mathfrak{N}X.$$

That the latter do indeed form a nearness basis for the topology considered here readily follows from the fact that

(1) if $x \leq z$ in X then $\mathfrak{W}_x \subseteq \mathfrak{W}_z$,

(2) if $x \triangleleft z$ in X then $\mathfrak{W}_x \triangleleft \mathfrak{W}_z$ relative to the covers (#), and

(3) if $P \in \mathfrak{W}_x$ then $P \in \mathfrak{W}_z$ for some $\mathfrak{W}_z \triangleleft \mathfrak{W}_x$

Since (1) is obvious we only consider (2) and (3). Regarding (2), if $\mathfrak{W}_x \cap \mathfrak{W}_y \neq \emptyset$ for some $y \in C$ where $x \triangleleft_C z$ then $x, y \in P$ for some point P, hence $\{x, y\}$ is a cluster and therefore $y \leq z$, showing that $\mathfrak{W}_y \subseteq \mathfrak{W}_z$, as desired. Further, (3) follows immediately from this because $P \in \mathfrak{W}_x$, that is: $x \in P$, implies $z \triangleleft x$ for some $z \in X$.

In the following, $\mathbf{F}X = \mathfrak{F}\mathfrak{L}\mathbf{S}X$, the complete nearness frame associated with the nearness semilattice $\mathfrak{L}\mathbf{S}X$; also, recall that, for any nearness frame L, ΣL is the frame spectrum of L equipped with the nearness induced by that of L in the manner analogous to the above (#).

Proposition 5.1. *For any approximation system X, $PtX \cong \Sigma\mathbf{F}X$.*

Proof. Let $\lambda : X \to \mathbf{F}X$ be the composite map $X \to \mathfrak{L}\mathbf{S}X \to \mathbf{F}X = \mathfrak{F}\mathfrak{L}\mathbf{S}X$ for which

$$x \mapsto [\{x\}] \mapsto \downarrow[\{x\}] = \{[u] \in \mathfrak{L}\mathbf{S}X \mid \ell b(u) \subseteq \downarrow s\} \cup \{\bot\}$$

where the equality holds because $[u] \leq [\{x\}]$ iff $u \mapsto \{x\}$ iff $\ell b(u) \subseteq \downarrow x$ for any cluster u of X and $x \in X$.

Now, λ is a partial order embedding: $x \leq z$ in X implies $\downarrow x \subseteq \downarrow z$, hence $\ell b(u) \subseteq \downarrow x$ implies $\ell b(u) \subseteq \downarrow z$ and therefore $\lambda(x) \subseteq \lambda(z)$. Conversely, given the latter, $[\{x\}] \in \downarrow[\{z\}]$, hence $\downarrow x = \ell b(\{x\}) \subseteq \downarrow z$ and thus $x \leq z$.

Further, the nearness $\mathfrak{N}(\mathbf{F}X)$ has a basis the covers $\lambda[C]$, $C \in \mathfrak{N}X$, and for any such C, $x \triangleleft_C z$ iff $\lambda(x) \triangleleft_{\lambda[C]} \lambda(z)$. Regarding the forward implication, if $\lambda(s) \wedge \lambda(x) \neq \{\bot\}$ for some $s \in C$ then there exist $u \in ClX$ such that $\ell b(u) \subseteq \downarrow s$ and $\ell b(u) \subseteq \downarrow x$, hence $s \wedge x \neq 0$ so that $s \leq z$ by hypothesis and therefore $\lambda(s) \leq \lambda(z)$, showing $\lambda(x) \triangleleft_{\lambda[C]} \lambda(z)$. The converse follows by the same kind of argument.

As a result, $\lambda[P]$ is a regular Cauchy filter basis in $\mathbf{F}X$ for any point P of X, and by the completeness of $\mathbf{F}X$ it follows that the filter generated by $\lambda[P]$ in $\mathbf{F}X$ is *completely prime*, that is, belongs to $\Sigma\mathbf{F}X$ (Banaschewski [2]). Conversely, for any $Q \in \Sigma\mathbf{F}X$, $\lambda^{-1}[Q]$ is a regular Cauchy filter in X: it is trivially an upset and downdirected because $\bigcup\{\lambda[C] \mid C \in \mathfrak{N}X\}$ is join-dense in $\mathbf{F}X$, as a familiar consequence of the admissibility condition, obviously Cauchy, and regular since $\lambda(x) \triangleleft \lambda(z)$ in $\mathbf{F}X$ implies $x \triangleleft z$ in X, as already noted. Furthermore, $\lambda[\lambda^{-1}[Q]] = Q \cap \lambda[X]$ generates Q, and hence the correspondence $P \mapsto \lambda[P]$ induces a one-one onto map between PtX and $\Sigma\mathbf{F}X$. Finally, that this is an isomorphism of nearness spaces is obvious since it makes $\mathfrak{W}_x$ correspond to $\Sigma_{\lambda(x)}$. □

Given the functoriality of Σ we have as an immediate consequence

Corollary 5.1. *The correspondence $X \mapsto PtX$ is functorial, providing a covariant functor from* **ApprSyst** *to the category of complete strong nearness spaces and uniformly continuous maps.*

In analogy with terminology used for uniform frames, we call an approximation system *of countable type* whenever its nearness has a countable basis and apply the same terminology to nearness frames and nearness spaces. Regarding the complete nearness frames of this kind there is the fundamental fact that they are *spatial* so that the spectrum functor effects a dual equivalence on them. As a result we have

Proposition 5.2. *Modulo order, the functor Pt induces an equivalence between the category of approximation systems of countable type and the category of complete strong nearness spaces of countable type and uniformly continuous maps.*

There is a special case of this proposition which merits particular mention.

Corollary 5.2. *Modulo order, the functor Pt induces an equivalence between the category of uniform approximation systems of countable type and the category of complete metric spaces and uniformly continuous maps.*

Proof. This follows immediately from the proposition, by the classical result that any uniform T_0-space of countable type is metrizable, that is, has a metric which determines its uniformity (Bourbaki [7], Ch. IX, §2). □

These results should be viewed as providing a concrete representation of approximation systems of countable type: their elements do indeed amount to approximations of actual entities, namely the points of the associated nearness space, in the form of basic open sets of this space, such that the abstract partial order " is a better approximation than" is realized by set inclusion. Moreover, in the uniform case, the levels of precision are represented in a particularly suggestive manner, namely by a metric.

By way of contrast, it should be pointed out that, at the other extreme, there obviously exist approximation systems which are completely beyond such concrete representation. For instance, if X is given by the non-zero elements of an atomless complete Boolean algebra M, with the nearness of all covers (which is a uniformity), then $\mathbf{F}X \cong M$ and $PtX \cong \Sigma M = \emptyset$.

Appendix 1. The Decimal Approximation System

The present aim is to verify that the structure D described in Example 2.1(1) is indeed a uniform approximation system and to add some remarks placing it into a wider context.

In order to avoid confusion it will be desirable to distinguish between the partial order of D, as given in the Introduction, and the usual order of the rationals; we shall continue to use $\leq$ for the former and then $\sqsubseteq$ for the latter. Also, $\alpha, \beta, \ldots$ will always stand for the decimal expansions involved and never by equal to $\top$, the added top.

We proceed in the obvious steps to prove the first assertion.

(1) *Each* $C_n = \{\alpha \in D \mid \mathrm{ord}\alpha = n\}$ *is a cover of* D.

It has to be shown that (i) there is no γ such that $C_n \subseteq {\downarrow}\gamma$ (the cover condition for $\top$) and (ii) if ${\downarrow}\alpha \cap {\downarrow}C_n \subseteq {\downarrow}\gamma$ then $\alpha \leq \gamma$, where (i) is obvious: for any γ, if the rationals represented by γ and $\alpha \in C_n$ are sufficiently far apart then clearly $\alpha \not\leq \gamma$.

Regarding (ii), let $\alpha = b_m \cdots b_0.a_1 \cdots a_k$ and consider first $n \leq k$. Then $\alpha \in {\downarrow}C_n$ since $\alpha \leq b_m \cdots b_0.a_1 \cdots a_n$, and hence trivially $\alpha \leq \gamma$. On the other hand, if $k < n$ then $\beta \leq \gamma$ for any $\beta = b_m \cdots b_0.a_1 \cdots a_k \cdots a_n$ and for $\ell =$ ordγ this says

$$\gamma - 10^{-\ell} \sqsubseteq b_m \cdots b_0.a_1 \cdots a_k \cdots a_n - 10^{-n},$$
$$b_m \cdots b_0.a_1 \cdots a_k \cdots + 10^{-n} \sqsubseteq \gamma + 10^{-\ell}.$$

Here, for $a_{k+1} = \cdots = a_n = 0$, we have $\gamma - 10^{-\ell} \sqsubseteq \alpha - 10^{-n} \sqsubseteq \alpha - 10^{-k}$.

Similarly, for $a_{k+1} = \cdots = a_n = 9$ we obtain $\alpha + 10^{-k} \sqsubseteq \gamma + 10^{-\ell}$, and in all this shows $\alpha \leq \gamma$, as claimed.

(2) *For any $\alpha, \beta \in D$, $\{\alpha, \beta\}$ is a cluster iff*

$$(\alpha - 10^{-\mathrm{ord}\alpha}, \alpha + 10^{-\mathrm{ord}\alpha}) \cap (\beta - 10^{-\mathrm{ord}\beta}, \beta + 10^{-\mathrm{ord}\beta}) \neq \emptyset.$$

Any non-void rational open interval contains an interval of the form $(\gamma - 10^{-\mathrm{ord}\gamma}, \gamma + 10^{-\mathrm{ord}\gamma})$ provided $\mathrm{ord}\gamma$ is large enough. Hence the latter implies the former, and the converse is obvious.

(3) *Admissibility.*

Note first that $\alpha \triangleleft \beta$ iff $\beta - 10^{-\mathrm{ord}\beta} \sqsubset \alpha - 10^{-\mathrm{ord}\alpha}$ and $\alpha + 10^{-\mathrm{ord}\alpha} \sqsubset \beta + 10^{-\mathrm{ord}\beta}$. Given the latter then, for any γ of sufficiently large order n,

$$(\gamma - 10^{-n}, \gamma + 10^{-n}) \cap (\alpha - 10^{-\mathrm{ord}\alpha}, \alpha + 10^{-\mathrm{ord}\alpha}) \neq \emptyset$$

implies $(\gamma - 10^{-n}, \gamma + 10^{-n}) \subseteq (\beta - 10^{-\mathrm{ord}\beta}, \beta + 10^{-\mathrm{ord}\beta})$, and by (2) this shows: if $\{\gamma, \alpha\}$ is a cluster then $\gamma \leq \beta$ and hence $\alpha \triangleleft \beta$. Conversely, if the former holds then certainly $\alpha \leq \beta$ but both, $\beta - 10^{-\mathrm{ord}\beta} = \alpha - 10^{-\mathrm{ord}\alpha}$ and $\alpha + 10^{-\mathrm{ord}\alpha} = \beta + 10^{-\mathrm{ord}\beta}$, are clearly excluded and hence the stated condition holds.

(4) *For each n, $C_{n+1} \leq^* C_n$.*

By (2), if $\alpha \in C_{n+1}$ and $\beta \in C_n$ then $\alpha \triangleleft_{C_{n+1}} \beta$ whenever

$$\beta - 10^{-n} \sqsubseteq \alpha - 3 \cdot 10^{-n-1} \text{ and } \alpha + 3 \cdot 10^{-n-1} \sqsubseteq \beta + 10^{-n},$$

or equivalently

$$\alpha - 7 \cdot 10^{-n-1} \sqsubseteq \beta \sqsubseteq \alpha + 7 \cdot 10^{-n-1},$$

and if $\alpha = b_m \cdots b_0.a_1 \cdots a_{n+1}$ then simple calculations show that this holds for $\beta = b_m \cdots b_0.a_1 \cdots a_n$ if $a_{n+1} = 0, \ldots, 7$ and for $\beta = b_m \cdots b_0.a_1 \cdots a_n + 10^{-n}$ if $a_{n+1} = 8, 9$.

This completes the proof that D is a uniform approximation system.

The wider context refereed to at the beginning of this section is that of the *frame* $\mathfrak{L}(\mathbf{R})$ *of the reals*. We briefly indicate, without going into any detail, how D is related to this. For the relevant background we refer to Johnstone [8], Chapter IV, 1, or to Banaschewski [3].

Recall that $\mathfrak{L}(\mathbf{R})$ is defined by generators and relations where the generators are the ordered pairs (p, q) of rational numbers and the relations are

$$\begin{aligned}
&(p,q) \wedge (r,s) = (p \vee r, q \wedge s), \\
&(p,q) \vee (r,s) = (p,s) \text{ whenever } p \leq r < q \leq s, \\
&(p,q) = \bigvee\{(r,s) \mid r < r < s < q\}, \text{ and} \\
&e = \bigvee\{(p,q) \mid \text{ all } p,q\}.
\end{aligned}$$

Now, straightforward checking shows that the assignment

$$(p,q) \mapsto \bigvee \left\{\lambda(\alpha) \mid \alpha \in D,\ p \sqsubset \alpha - 10^{-\mathrm{ord}\alpha}, \alpha + 10^{-\mathrm{ord}\alpha} \sqsubset q\right\}$$

for the map $\lambda : D \to \mathbf{F}D$ discussed in general in the last section turns these relations into identities in the frame $\mathbf{F}D$ and hence defines a frame homomorphism $\mathfrak{L}(\mathbf{R}) \to \mathbf{F}D$. Further, this maps any $(p, 0)$ to 0, and as a result it induces a homomorphism $h :\uparrow (-,0) \to \mathbf{F}D$ on the closed quotient of $\mathfrak{L}(\mathbf{R})$ determined by

$$(-,0) = \bigvee\{(p,0) \mid \text{ all } p < 0\}$$

which is otherwise known as the *frame* $\mathfrak{L}(\mathbf{R}^+)$ *of the non-negative reals.* Next, h turns out to be dense, onto, and a uniform surjection for the natural uniformity of $\mathfrak{L}(\mathbf{R}^+)$ induced by that of $\mathfrak{L}(\mathbf{R})$ which is given by the basic covers

$$\{(p,q) \mid 0 < q - p < \tfrac{1}{n}\}, \quad n = 1, 2, \ldots.$$

Consequently, by the completeness of $\mathbf{F}D$ (Corollary 4.2), $h : \mathfrak{L}(\mathbf{R}^+) \to \mathbf{F}D$ *is an isomorphism of uniform frames.*

As an added observation one notes that the partial order embedding

$$D \xrightarrow{\lambda} \mathbf{F}D \xrightarrow{h^{-1}} \mathfrak{L}(\mathbf{R}^+)$$

makes $\alpha \in D$ correspond to the element of $\mathfrak{L}(\mathbf{R}^+)$ determined by the pair $(\alpha - 10^{-\mathrm{ord}\alpha}, \alpha + 10^{-\mathrm{ord}\alpha})$.

Finally, it now follows by Proposition 5.1 that $PtD \cong \mathbf{R}^+$ as uniform spaces, given the general fact that $\Sigma(\mathfrak{L}(\mathbf{R})) \cong \mathbf{R}$ and consequently $\Sigma(\mathfrak{L}(\mathbf{R}^+)) \cong \mathbf{R}^+$.

Regarding the isomorphism $PtD \cong \mathbf{R}^+$, it should not be passed over that this could also be obtained directly: each point of D determines a Cauchy filter basis in $\mathbf{R}^+$ which converges to some $\lambda \in \mathbf{R}^+$, and this provides a map $PtD \to \mathbf{R}^+$ which can then be shown to be an isomorphism of uniform spaces. On the other hand, though, the initial isomorphism $\mathfrak{L}(\mathbf{R}^+) \cong \mathbf{F}D$ is more fundamental, in particular since it is constructively valid in the sense of topos theory.

Appendix 2. An alternative view of the relation between approximation systems and nearness frames

The aim here is to provide a closer look at the situation considered in Example 2.2(3).

We use the following notation. For any complete strong nearness frame L, $L^\#$ will be the approximation system determined by $L - \{0\}$, and for any uniform homomorphism $h : M \to L$ of such nearness frames we put $h^\# : L^\# \to M^\#$ for the approximation map introduced in that example, that is, $xh^\# y$ iff $x \leq h(y)$. Then we have:

The correspondences $L \mapsto L^\#$ and $h \mapsto h^\#$ constitute a contravariant embedding of **CSNearF** *into* **ApprSyst** *which is replete and full, and hence a dual equivalence, modulo order.*

Proof. For the functoriality, if $g : N \to M$, $h : M \to L$, $x \in L^\#$, and $z \in N^\#$ then

$$x(h^\# \circ g^\#)z \quad \text{iff} \quad x \leq h(y) \text{ and } y \leq g(z)$$

for some $y \in M^\#$ so that $x \leq hg(z)$ and hence $x(hg)^\# z$; conversely, given this then $x(h^\# \circ g^\#)z$ holds with $y = g(z)$, and hence $(hg)^\# = g^\# h^\#$. Further, it is clear that $\mathrm{id}_L^\# = \leq_{L^\#}$, as required.

Next, this functor is faithful. Consider any $g, h : M \to L$ such that $g^\# = h^\#$. Then, trivially, $g(y) = h(y)$ whenever $g(y) \neq 0 \neq h(y)$. Further, if $g(y) \neq 0$ then $g(y) \in L^\#$ and $g(y)g^\# y$, hence $g(y)h^\# y$ so that $g(y) \leq h(y)$ and consequently also $h(y) \neq 0$. By symmetry it follows that $g(y) \neq 0$ iff $h(y) \neq 0$ and hence $g = h$.

For the next step we show that $X \cong (\mathbf{F}X)^\#$ modulo order, for any approximation system X, where $\mathbf{F}X = \mathfrak{F}\mathfrak{L}\mathbf{S}X$ as in Section 5. For this consider the following binary relations:

$$f \subseteq X \times (\mathbf{F}X)^{\#} \quad , \quad xfU \text{ iff } \lambda(x) \subseteq U,$$
$$g \subseteq (\mathbf{F}X)^{\#} \times X \quad , \quad Ugx \text{ iff } U \subseteq \lambda(x),$$

where $\lambda : X \to \mathbf{F}X$ is the partial order embedding introduced in the proof of Proposition 5.1. It is obvious that (AM1)–(AM3) hold in either case; on the other hand, for any $C \in \mathfrak{N}X$,

$$f^{-1}[\lambda[C]] = \{x \in X \mid \lambda(x) \leq \lambda(s) \text{ for some } s \in C\} = \downarrow C$$

and

$$g^{-1}[C] = \{U \in (\mathbf{F}X)^{\#} \mid U \subseteq \lambda(s) \text{ for some } s \in C\} = \downarrow \lambda[C],$$

and since $\mathfrak{N}((\mathbf{F}X)^{\#})$ is generated by the $\lambda[C]$ for the $C \in \mathfrak{N}X$ not containing 0, by the proof of Proposition 5.1, this proves (AM4) is either case. Thus we have maps $f : X \to (\mathbf{F}X)^{\#}$ and $g : (\mathbf{F}X)^{\#} \to X$, and simple calculations then show $f \circ g = \text{id}_X$ and $g \circ f \leq \text{id}_{(\mathbf{F}X)^{\#}}$; consequently, f is an isomorphism modulo order.

Finally, for any map $f : L^{\#} \to M^{\#}$, consider $\varphi : M \to L$ given by

$$\varphi(y) = \bigvee\{x \in L \mid y_1 \wedge \cdots \wedge y_n \leq y \text{ and } xfy_i \text{ for each } i\}.$$

It is clear that $\varphi(0) = 0$, $\varphi(e) = e$, and φ preserves $\leq$; in addition, simple calculation shows $\varphi(y) \wedge \varphi(z) \leq \varphi(y \wedge z)$ and hence φ preserves $\wedge$. Further, if $C \in \mathfrak{N}M$ then $f^{-1}[C] \leq \varphi[C]$ since xfs implies $x \leq \varphi(s)$ for any $x \in L^{\#}$ and $s \in C$ ($s \neq 0$ without loss of generality), showing that $\varphi[C] \in \mathfrak{N}L$ since this holds for $f^{-1}[C]$ by hypothesis. In all, φ is a uniform semilattice homomorphisms, hence $h = \varphi^o$ is a regular uniform semilattice homomorphism, and by a fundamental property of complete nearness frames this makes $h : M \to L$ a uniform frame homomorphism.

Now, let $g \subseteq L^{\#} \times M^{\#}$ be such that

$$xgy \quad \text{iff} \quad xfz \text{ for some } z \triangleleft y.$$

It is easily seen that this is a map $g : L^{\#} \to M^{\#}$: the first three conditions hold quite obviously, and the fourth one follows from the fact that $g^{-1}[C] \supseteq f^{-1}[\check{C}]$ for each $C \in \mathfrak{N}(M^{\#})$. Furthermore, if xgy and hence xfz for some $z \triangleleft y$ then $x \leq \varphi(z) \leq \varphi^o(y) = h(y)$, showing that $g \leq h^{\#}$; on the other hand, $g \leq f$ trivially, and consequently $h^{\#} \sim f$. □

It should be pointed out that, although full modulo order, *the above functor is not full per se.* To see this, let L be any complete strong nearness frame and $f : L^{\#} \to L^{\#}$ such that xfy iff $x \leq y^{**}$, which is easily checked to define a map as indicated. Now, if $f = h^{\#}$ for some $h : L \to L$ then $h(y) \leq y^{**}$ for all $y \in L$, and since

$$h(y) = \bigvee \{h(z) \mid z \triangleleft y\}$$

and $z^{**} \leq y$ whenever $z \triangleleft y$ it follows that $h(y) \leq y$, showing $h = \mathrm{id}_L$. Further, $y^{**} \leq y^{**}$ implies $y^{**} f y$, hence $y^{**} \leq y$ and then $y = y^{**}$, for all $y \in L$, and this holds iff L is *Boolean.*

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Department of Mathematics, McMaster University, 1200 Main St. W, Hamilton, Ontario L8S 4K1, Canada

Department of Applied Mathematics and ITI, MFF, Charles University, Malostranské nám.25, CZ 11800 Praha 1, Czech Republic

E-mail address: pultr@kam.ms.mff.cuni.cz