The 7-cycle C_7 is light in the family of planar graphs with minimum degree 5

T. Madaras[*†a], R. Škrekovski[‡*b,c] and H.-J. Voss [d]

[a] Department of Geometry and Algebra, University of P.J.Šafárik, Jesenná 5, 041 54 Košice, Slovak Republic
madaras@science.upjs.sk

[b] Charles University, Faculty of Mathematics and Physics, DIMATIA and Institute for Theoretical Computer Science (ITI)
Malostranské nám. 2/25, 118 00, Prague, Czech Republic
skreko@kam.mff.cuni.cz

[c] Department of Mathematics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia
skreko@fmf.uni-lj.si

[d] Department of Mathematics, Technical University of Dresden
D-01062 Dresden, Germany
voss@math.tu-dresden.de

Abstract

A connected graph H is said to be light in the class of graphs $\mathcal{H}$ if there exists a positive integer k such that each graph $G \in \mathcal{H}$ that contains an isomorphic copy of H contains a subgraph K isomorphic to H that satisfies the inequality $\sum_{v \in V(K)} \deg_G(v) \leq k$. It is known that an r-cycle C_r is light in the family of planar graphs with minimum degree 5 if $3 \leq r \leq 6$, and not light for $r \geq 11$. We prove that C_7 is also light in this family

MSC: 05C10, 05C38

Keywords: planar graphs, light graphs, light cycles

[*]Supported in part by the Ministry of Education of Czech republic, Project LN00A056.
[†]Supported in part by Slovak VEGA grant 1/7467/20.
[‡]Supported by the Ministry of Science and Technology of Slovenia, Project Z1-3129.

1 Introduction and preliminaries

By Euler's famous theorem, we know that each planar graph has a vertex of degree ≤ 5. A beautiful theorem of Kotzig [8] states that every 3-connected planar graph contains an edge with degree sum of its endvertices being at most 13. We say that the path with one or two vertices is *light* in the class of those graphs. In general, let $\mathcal{H}$ be a family of graphs and let H be a connected graph. Denote by $w(H, \mathcal{H})$ the smallest integer such that each graph $G \in \mathcal{H}$ containing a subgraph isomorphic with H, contains also a subgraph K, $K \cong H$ such that

$$\sum_{v \in V(K)} deg_G(v) \leq w(H, \mathcal{H}).$$

The sum on the left side is called the *weight* of subgraph K in G. If such a finite number does not exist or there is no graph in $\mathcal{H}$ which contains H as a subgraph, then we put $w(H, \mathcal{H}) = +\infty$.

We say that a graph H is *light* in the family $\mathcal{H}$ provided $w(H, \mathcal{H})$ is finite, otherwise we call it *heavy*. The integer $w(H, \mathcal{H})$ is called the *weight* of H in the family $\mathcal{H}$.

For the family of polyhedral graphs, only the paths are light [2]. The same holds for the family of polyhedral graphs with minimum degree 4 [3] and minimum face size 4 [4]. On the other hand, in the family of planar graphs of minimum degree 5, there are graphs other than paths which are light, see [1, 6].

The complete characterization of light cycles in the family of all planar triangulations of minimum degree 5 was given in [7] and [9] (see also [1] and [6]). It was proved that a k-cycle is light in this family if and only if $3 \leq k \leq 10$. For the family of all planar graphs of minimum degree 5, it is known that C_3 [1], C_4, C_5 [6] (also R. Soták, private communication) and C_6 [10] are light. Moreover, C_r is heavy for $r \geq 11$ [7]. A survey of results on light subgraphs can be found in [5].

In this paper we prove the following theorem:

Theorem 1.1 *The cycle C_7 is light in the family of all planar graphs with minimum degree 5. Moreover, each graph of that family contains a light cycle C_7.*

For proving of this result, the Discharging method is used. We consider a hypothetical counterexample G with vertex set $V(G)$ and face set $F(G)$.

We may assume that G is connected. We assign initial charge to every vertex $v \in V(G)$ and every face $f \in F(G)$ of the graph G in the following way:

$$c(v) := d(v) - 6 \quad \text{and} \quad c(f) := 2r(f) - 6, \tag{1}$$

where $d(v)$ and $r(f)$ stand for the degree of v and the size of f, respectively. Hence, we can rewrite Euler's formula in the following form:

$$\sum_{v \in V(G)} c(v) + \sum_{f \in F(G)} c(f) = -12.$$

Thus, the total sum of the charges of the vertices and faces of G is negative. We will redistribute the charges of the vertices and faces of G by applying some rules without changing the total sum of all charges. Denote by $c^*(x)$ the charge of a vertex x or a face x after applying the rules. It will be also called the *final charge* of x. We will prove that each face and vertex of G has a nonnegative final charge. Since the total sum must be negative, it will be a contradiction and complete the proof.

An i-vertex and a j-face is a vertex of degree i and a face of size j, respectively.

2 The lightness of C_7

In this section we prove Theorem 1.1.

Proof. Suppose that the theorem is false. Then, we may assume that for each integer m there exists a planar graph of minimum degree 5 which has no 7-cycle or has at least one 7-cycle and each 7-cycle of G contains a vertex of degree $\geq m$. So, let G be a graph which satisfies the above assumptions for $m = 360$. For the purpose of the proof, a vertex of degree at least 360 is called *big*, a vertex of degree between 6 and 359 is called *intermediate*. A face of size at least 4 is called *big*. Moreover, a 7-cycle whose vertices are of degree ≤ 359 is called a *light* 7-cycle.

The local redistribution of charges preserving their total sum is performed by the rules given below. In order to describe and deal easily with these rules we introduce the following definitions. If a 3-face $[x_1, x_2, x_3]$ and a big face f have the common edge x_1x_2, then we say that the vertex x_3 is *diagonally incident* with f (at x_1x_2). If $[x_1, x_2, x_3]$ and $[x_1, x_2, x_4]$ are two incident 3-faces with $x_3 \neq x_4$, the vertices x_3, x_4 are *diagonally adjacent* (at

x_1x_2). If $[x_1, x_2, x_4, x_5]$ and $[x_1, x_2, x_3]$ are a 4-face and a 3-face, respectively, then, we say that x_3 and x_5 are *squarely adjacent* (at x_1x_2). If x_3 is a 5-vertex and some charge is sent to x_3 from f or from x_5, then we use to say that this charge is sent through the edge x_1x_2.

Rule R1: Each big face sends $\frac{1}{2}$ to each of its incident 5-vertices. The remaining positive charge of the face is equally distributed to its diagonally incident 5-vertices.

Rule R2: Each big vertex sends $\frac{1}{2} + \frac{1}{12}$ to each adjacent 5-vertex or intermediate vertex.

Rule R3: Each big vertex sends $\frac{2}{5}$ to each diagonally adjacent 5-vertex.

Rule R4: Each big vertex sends $\frac{1}{5}$ to each squarely adjacent 5-vertex, where the big vertex is in the 4-face and the 5-vertex is in the 3-face.

Rule R5: Each intermediate vertex of degree ≥ 7 sends $\frac{1}{5}$ to each adjacent 5-vertex if the edge joining them is incident with two 3-faces.

Rule R6: Let $[v, v_1, v_2]$ be a 3-face, v be big, v_1 be a 5-vertex, and v_2 be intermediate. Then the intermediate vertex v_2 sends $\frac{1}{4}$ to the 5-vertex v_1.

Note that the rules R5 and R6 can independently be applied. So, if in rule R6 the edge v_2v_1 is in two triangles and v_2 has a degree ≥ 7 then v_1 receives at least $\frac{1}{4} + \frac{1}{5}$ from v_2.

A 5-vertex is called *overcharged*, if after applying rules R1 – R6 its charge is positive, and *undercharged*, if it is negative.

Rule R7: Each overcharged 5-vertex equally distributes its positive charge to its adjacent undercharged 5-vertices.

Now, we will prove that for every $x \in V(G) \cup F(G)$, the final charge $c^*(x) \geq 0$. To these purposes, several cases have to be considered. Firstly, observe that if f is a 3-face of G then $c^*(f) = c(f) = 0$. And, if f is a big face of size $d \geq 4$, then after sending $\frac{1}{2}$ to each of its incident 5-vertices, the remaining charge of f is at least $2d - 6 - \frac{d}{2} \geq 0$. Hence, we conclude that the final charge of each face is nonnegative.

Now, we determine the charge which f sends to each diagonally incident 5-vertex. If f is a 4-face incident with at most three (or at most two) 5-vertices, then f sends at least $\frac{1}{8}$ (or at least $\frac{1}{4}$, respectively) to each diagonally incident 5-vertex. Suppose now that f is a 5-face. If f is not incident with a big vertex then it is diagonally incident with at most two

5-vertices; otherwise G would have a light 7-cycle. (Note: at most one vertex, which is diagonally incident with f, can also be incident with f, i.e., it belongs to the boundary of f). In that case each such 5-vertex receives at least $\frac{3}{4}$ from f. If f is incident with a big vertex then f has at least $\frac{2}{5}$ to send to each diagonally incident 5-vertex. Finally, if f is of size $d \geq 6$, then f sends at least $\frac{2d-6-d/2}{d} \geq \frac{1}{2}$ to each diagonally incident 5-vertex. We conclude that a face of size ≥ 5 sends to each diagonally incident vertex at least $\frac{2}{5}$.

Suppose that v is a vertex of degree d. We denote the neigbours of v around v by $v_1, \ldots, v_d$. The face incident with the subwalk $v_i v v_{i+1}$ is denoted by f_i. If f_i is a 3-face, then let f_i' be the other face incident with $v_i v_{i+1}$ (indices modulo d). Moreover, if f_i' is a 3-face, then we denote by x_i the third vertex incident with f_i'.

Case A: *v is an intermediate vertex.* Let k be the number of big neighbours of v and let s be the number of 5-neighbours of v receiving a positive charge from v; $k + s \leq d$. If $d = 6$ then $s = 0$. If $d = 7$ then $s \leq 5$; otherwise v is only incident with 3-faces, and the six 5-neighbours and v would generate a light 7-cycle. Each of the k big neighbours of v sends $\frac{1}{2} + \frac{1}{12}$ to v by R2, v sends $\frac{1}{4}$ to each of at most $2k$ neighbours by R6, and v sends $\frac{1}{5}$ to each of the s receiving 5-neighbours of v. With the values of s, the new charge $c^*(v) \geq d - 6 - \frac{s}{5} + k(\frac{1}{2} + \frac{1}{12}) - 2k \cdot \frac{1}{4} \geq 0$.

Case B: *v is a big vertex.* By rules R3 and R4, it sends at most $\frac{2}{5}d$ to its diagonally and squarely adjacent 5-vertices and by rule R2 it sends at most $(\frac{1}{2} + \frac{1}{12})d$ to its 5-neighbours or intermediate neighbours. Since $d \geq 360$, the final charge is $c^*(v) > d - 6 - \frac{2}{5}\,d - \left(\frac{1}{2} + \frac{1}{12}\right)d = \frac{d}{60} - 6 \geq 0$.

Case C: *v is a 5-vertex.* First, we introduce the following definitions. We say that the face f_i' is a *twisted* 3-face (*around* v), if it is a 3-face and x_i coincides with v_{i+3}. We say that the faces f_i', f_j' are *kissing* 3-faces (*around* v) if they are 3-faces with $x_i = x_j$ (Observe that $j \notin \{i-1, i, i+1\}$). By the planarity, at most one triangle is twisted or at most one couple of triangles is kissing, but both these cases are not possible at the same time.

Suppose that $v_i v_{i+1}$ is an edge of G. If, by the rules above, a charge $\geq \frac{13}{40}$ is sent to v through $v_i v_{i+1}$, then we say that it is a *good* edge, and otherwise we say that it is *bad*. If f_i' is a triangle then $v_i v_{i+1}$ is bad only if x_i is not big. If f_i' is of size ≥ 5 then it is diagonally incident with v. So, f_i' sends at least $\frac{2}{5}$ to v and consequently $v_i v_{i+1}$ is always good. Suppose now that $f_i' = [v_i, a, b, v_{i+1}]$ is a 4-face. If a or b is big, then by R4 and R1, at least $\frac{1}{5} + \frac{1}{8} = \frac{13}{40}$ is sent to v through the edge v_iv_{i+1}; and so it is good. Thus, f_i' is bad only if f_i' is a 3-face and x_i is not big or f_i' is a 4-face and both vertices of $V(f_i') \setminus \{v_i, v_{i+1}\}$ are not big.

If v is incident with two big faces, then $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{2} = 0$. If v is incident with three big faces then $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$, and v is overcharged with $\geq \frac{1}{2} > \frac{1}{12}$. Observe that if v is adjacent to two big vertices or it is incident with one big face and adjacent to one big vertex, then $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{1}{2} = \frac{1}{12} > 0$; thus it is overcharged with $\geq \frac{1}{12}$. Hence, we have to consider the three cases given below. In each of these cases, we assume that there is no 4-face $[v_i, v_{i+1}, x, y]$, $i = 2, 3, 4$ with x, y not big and not being neighbours of v; otherwise, a light C_7 can be found.

Case 1: *There is precisely one big face incident with v and there is no big vertex adjacent to v.* We may assume that the big face is f_3. Then, f_3 sends $\frac{1}{2}$ to v. Now, we consider the charge sent to v through each of the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_1$. Consider first when the edge v_2v_3 is bad. It follows that f_2' is a 3-face and x_2 is not big or $f_2' = [v_2, a, b, v_3]$ is a 4-face, and a, b are not big. Since every 7-cycle of G contains a big vertex and G is a simple graph, one can easily show that whenever v_2v_3 is bad, one of the next two conditions is satisfied:

- f_2' is a 3-face with $x_2 = v_4$ or $x_2 = v_5$;
- f_2' is the 4-face $[v_2, v_3, v_4, v_5]$ or a 4-face $[v_2, x, v_5, v_3]$ with some vertex $x \notin \{v, v_1, v_2, \ldots, v_5\}$.

By similar arguments one can show: if v_1v_2 is bad, then f_1' is twisted. By the symmetry arguments, similar necessary conditions could be found for the badness of the edges v_3v_4 and v_4v_5. We claim: *there are at most two bad edges.* Suppose that this is false. As it was stated above there is at most one twisted triangle around v. So, we may assume that v_1v_2, v_2v_3, and v_4v_5 are bad edges. Then $f_1' = [v_1, v_2, v_4]$ is a twisted triangle. The 3-cycle $[v, v_2, v_4]$ separates the vertices v_3 and v_5, and so v_2v_3 is a good edge. Thus the claim is proved. Hence, we obtain that at least two of the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_1$ are good; and so, $c^*(v) \geq -1 + \frac{1}{2} + 2 \cdot \frac{13}{40} = \frac{3}{20} > 0$. We can conclude that if a 5-vertex is incident with precisely one big face (and is not adjacent with a big vertex), then it is overcharged with $\geq \frac{3}{20} > \frac{1}{12}$.

Remark. We have proved: *if a 5-vertex x is incident with at least one big* *face then it is overcharged with $\geq \frac{3}{20}$ besides the case that x is incident with precisely two big faces.*

Case 2: *All incident faces of v are triangles and v has no big neighbours.* Suppose that an edge $v_i v_{i+1}$ is bad. Then, f_i' is a 4-face or a 3-face. If f_i' is a 4-face then all its vertices are not big and precisely three of them are neighbours of v. If f_i' is a 3-face and x_i is not a neighbour of v, then x_i is not big. In both cases, we easily find a light C_7. Thus, we conclude that f_i' is a twisted triangle. Since there is at most one twisted triangle, four edges $v_j v_{j+1}$ are good and $c^*(v) \geq -1 + 4 \cdot \frac{13}{40} > 0$.

Case 3: *All incident faces of v are triangles, and v has only one big neighbour.* We may assume that the big neighbour is v_1. This vertex sends $\frac{1}{2} + \frac{1}{12}$ to v.

In the forthcoming part of the proof we often encounter a quadrangle f_i' containing more than two neighbours of v. There are precisely two different essential possibilities depicted in Fig. 1.

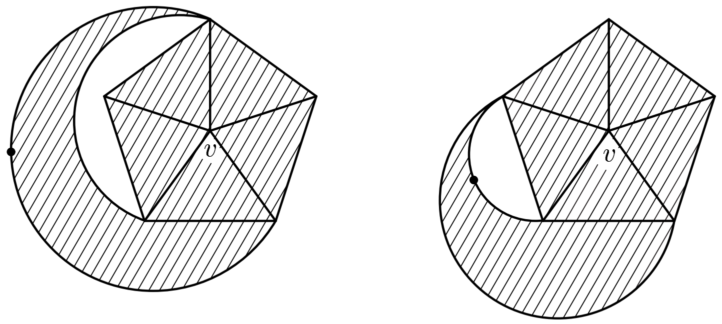

Fig. 1

Case 3.1: *There is a twisted triangle (around v).* Regarding which triangle f_i' is twisted, we consider several subcases:

Case 3.1.1: *f_1' or f_5' is a twisted triangle*, say f_1'. Suppose first that $v_4 v_5$ is a bad edge. Then, f_4' is a triangle and x_4 is not big. If $v_2 v_3$ or $v_3 v_4$ is bad, then we obtain a vertex x which is not big and which is adjacent to both v_2, v_3 or to both v_3, v_4; a light C_7 can be found which contains both vertices x, x_4. Hence both edges $v_2 v_3$ and $v_3 v_4$ are good, we obtain that $c^*(v) > 0$.

Suppose now that v_4v_5 is a good edge. We may assume that each of v_2, v_5 is a 5-vertex. Otherwise, v_2 or v_5 sends $\frac{1}{4}$ to v by R6, and $c^*(v) \geq -1+\frac{1}{2}+\frac{1}{12}+\frac{13}{40}+\frac{1}{4} > 0$. If one of v_2v_3 and v_3v_4 is good, then also $c^*(v) > 0$. So, assume that these two edges are bad. Observe that one of f_2', f_3' is a 3-face (but not both since we would obtain a light C_7).

If f_2' is a 3-face, then $f_3' = [v_3, x, v_2, v_4]$ for some vertex x. Note that $x \neq x_2$, otherwise this vertex is of degree 2. This implies that v_2 has degree ≥ 6, which contradicts the assumption that v_2 is a 5-vertex.

If f_3' is a 3-face, then $f_2' = [v_4, x, v_3, v_2]$ for some vertex x. Note that x and x_3 are different vertices. Then v_2 has degree 4, which contradicts the assumption that v_2 is a 5-vertex.

Case 3.1.2: *f_3' is a twisted triangle.* Then, v_1 sends additional $\frac{2}{5}$ to v by R3. Note that one of the edges v_2v_3 and v_4v_5 is good. Otherwise, we obtain that f_2', f_4' are 3-faces and x_2, x_4 are not big vertices; and so we encounter a light C_7. Thus, through v_2v_3 or v_4v_5 is sent at least $\frac{13}{40}$ to v. Now, we easily infer that the final charge of v is positive.

Case 3.1.3: *f_2' or f_4' is a twisted triangle*, say f_4'. Regarding the degree of v_2, we consider the following possibilities:

$d(v_2) = 5$**:** Note that f_1', f_2' are of size ≥ 4. Observe that v_5 is incident with f_1' and v_3, v_4 are incident with f_2'. This implies (see Case 1) that they are not undercharged 5-vertices. Since v_2 is adjacent with one big vertex and two big faces, it follows that it is an overcharged 5-vertex with more than $\frac{1}{2}$ extra charge. By rule R7, this charge is sent to v. Thus, $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{1}{2} > 0$.

$d(v_2) = 6$**:** By rule R6, v_2 sends $\frac{1}{4}$ to v. Note that f_1' or f_2' is a big face. If f_1' or f_2' is a face of size ≥ 5, then it sends at least $\frac{2}{5}$ to v. And if f_1' is a 4-face, then it sends at least $\frac{1}{4}$ to v (since v_1 and v_2 are of degree ≥ 6). In both of these cases, it follows that $c^*(v) \geq 0$. Thus, assume that f_1' is a triangle and $f_2' = [v_2, v_3, y, v_4]$ is a quadrangle.

If f_2' has at most two incident 5-vertices, then it sends $\frac{1}{4}$ to v; and so $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{1}{4} + \frac{1}{4} > 0$. We may assume now that v_4, y, v_3 are 5-vertices. If v_3 is overcharged with at least $\frac{1}{12}$ then v_3 sends at least $\frac{1}{24}$ to v by R7, and f_2' sends $\frac{1}{8}$ to v. Consequently, $c^*(v) \geq 0$.

Next let v_3 be a 5-vertex being not overcharged with at least $\frac{1}{12}$. By our remark at the end of Case 1 the vertex v has only 5-neighbours and intermediate neighbours, and v_3 is incident with three 3-faces and two big faces, one of them is f_2'. Since v_4 is a 5-vertex the boundary of the region formed by the three 3-faces and the 4-face f_2' is a light C_7. So we arrive at a contradiction, and v_3 is overcharged with at least $\frac{1}{12}$ in any case. Thus the proof of the subcase $d(v_2) = 6$ is complete.

$d(v_2) \geq 7$**:** By rules R6 and R7, the vertex v_2 sends the charge $\frac{1}{4} + \frac{1}{5}$ to v. Thus, $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{1}{4} + \frac{1}{5} > 0$.

Case 3.2: *There is no twisted triangle and no kissing couple (around* v*).* We can assume that at least two of the edges v_2v_3, v_3v_4, v_4v_5 are bad; otherwise $c^*(v) > 0$. Suppose that v_iv_{i+1} and v_jv_{j+1} $(i, j \in \{2, 3, 4\})$ are two bad edges. If f_i', f_j' are triangles or f_i', f_j' are quadrangles, then we obtain a light C_7 in G (In the case of quadrangles see Fig. 1). So, assume that f_i' is a 3-face and f_j' is a 4-face. Note that v_kv_{k+1} is good for $k \in \{2, 3, 4\} \backslash \{i, j\}$. If f_j' is not incident with v_i or f_j' is not incident with v_{i+1}, we again encounter a light C_7 in G. Assume now that v_i, v_{i+1} are vertices of f_j'.

If $i = 2$, then $j = 3$ and $f_3' = [v_3, v_4, v_2, y]$ for some vertex y. Since v_3 has a degree ≥ 5 the vertex $y \neq x_2$. So, the vertex v_2 is of degree ≥ 6 and it sends $\frac{1}{4}$ by R6. Thus, $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{13}{40} + \frac{1}{4} > 0$. We argue similarly, if $i = 4$.

Suppose now that $i = 3$. Then, $j = 2$ or 4, say $j = 2$. Then, $f_2' = [v_2, v_3, y, v_4]$ for some vertex $y \neq x_3$. So, vertex v_4 is of degree ≥ 6 and hence f_2' sends $\geq \frac{1}{8}$ to v by R1. We infer that $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{13}{40} + \frac{1}{8} > 0$.

Case 3.3: *There is a kissing couple of 3-faces (around* v*).* Note that there is no twisted triangle in this case. Note also that the common vertex x of these two triangles is not a big vertex; otherwise $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + 2 \cdot \frac{2}{5} > 0$. Regarding which faces are kissing, we consider the following cases:

Case 3.3.1: f_3' *is one of the kissing 3-faces.* We may assume that the other face is f_1'. Then, v_4v_5 is a good edge; otherwise f_4' is a 3-face with x_4 not big, or a 4-face not incident with a big vertex. In both cases, a light C_7 containing $v, v_2 \ldots v_5$ can be found. From the same reasons, v_2v_3 is a good edge; thus,
$c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + 2 \cdot \frac{13}{40} > 0$.

Case 3.3.2: f_1' *is one of the kissing 3-faces.* By the previous case, we may assume that f_4' is the other 3-face from the kissing couple. If f_2' is a 3-face and x_2 is not a big vertex, or a 4-face without a big vertex not containing v_4, then one can easily find a light C_7.

If f_2' is a 4-face without a big vertex containing v_4, then $f_2' = [v_2v_3yv_2]$, and there exists a light C_7 through x and y. Hence v_2v_3 is a good edge.

By the same arguments the edge v_3v_4 is good, too. In that case, $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + 2 \cdot \frac{13}{40} > 0$.

Case 3.3.3: *f_2' and f_4' are kissing 3-faces.* Then $x = (x_2 = x_4)$ is not big. In order to complete the proof, we consider the following two possibilities:

f_3' is not a 4-face: If f_3' has a size ≥ 5 then at least $\frac{2}{5}$ is sent to v through v_3v_4. If f_3' is a 3-face then $x_3 = x$ because v_3 and v_4 have a degree ≥ 5. If x_3 is not big then through x and x_3 a light C_7 can be found. Hence x_3 is big, and also in this case at least $\frac{2}{5}$ is sent to v through v_3v_4. If at least one of v_2, v_5 is of degree ≥ 6, then R6 is applied and $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{2}{5} + \frac{1}{4} > 0$. So, assume that both v_2, v_5 are 5-vertices. If all faces incident with v_2 are triangles, then the neighbours of v_2 lie on the cycle $[vv_1zxv_3]$. The vertex $z \neq v_5$, because otherwise v_1 would have degree 3. There exists a C_7 through $x, z, v, v_2, v_3, v_4, v_5$ avoiding v_1. Consequently, z is a big vertex. This big vertex sends $\frac{2}{5}$ to v through v_1v_2 and, subsequently, $c^*(v) > 0$. The same holds for v_5. Thus, each of v_2, v_5 is incident with a big face and the big vertex v_1. Hence, these two vertices are overcharged with $\frac{1}{12}$, and each of them sends at least $\frac{1}{48}$ to v. Thus, $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{2}{5} + 2 \cdot \frac{1}{48} > 0$.

f_3' is a 4-face: Let $f_3' = [v_3, v_4, z, y]$. Observe that f_3' is incident with a big vertex, say z; otherwise a light C_7 is found. If f_3' is incident with less than three 5-vertices, then at least $\frac{1}{5} + \frac{1}{4} > \frac{2}{5}$ is sent to v by R4 and R1, afterwards use a similar argument as in previous case to deduce that $c^*(v) > 0$. Now, assume that y, v_3, v_4 are 5-vertices. Note that f_3' and z together send $\frac{13}{40}$ to v. Vertex v_3 cannot be incident with four 3-faces; otherwise a light C_7 exists. Hence, x, v_3 are not undercharged 5-vertices. Since v_4 is incident with one big vertex and one big face, it is an overcharged 5-vertex.

If at least one of v_2, v_5 is of degree ≥ 6, then R6 is used and $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{13}{40} + \frac{1}{4} > 0$. Otherwise, as in the previous case, each of v_2,v_5 is an overcharged 5-vertex which sends $\frac{1}{48}$ to v. In this case v is the only possible undercharged 5-neighbour of v_4, so v_4 sends $\frac{1}{12}$ to v by R7. Thus, we infer that $c^*(v) \geq -1 + \frac{1}{2} + \frac{1}{12} + \frac{13}{40} + \frac{1}{12} + 2 \cdot \frac{1}{48} > 0$.

□

References

[1] O. V. Borodin, *Solution of problems of Kotzig and Grünbaum concerning the isolation of cycles in planar graphs*, Mat. Zametki **46** (1989) 9–12.

[2] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar 3-connected graphs*, Graphs Comb. **13**, No.3 (1997) 245–250.

[3] I. Fabrici, E. Hexel, S. Jendrol', and H. Walther, *On vertex-degree restricted paths in polyhedral graphs*, Discrete Math. **212**, No.1-2 (2000) 61–73.

[4] J. Harant, S. Jendrol', and M. Tkáč, *On* 3*-connected plane graphs without triangular faces*, J. Comb. Theory, Ser. B **77**, No.1 (1999) 150–161.

[5] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in plane and in the projective plane – a survey –*, Preprint Inst. of Algebra MATH-AL-2-2001, TU Dresden.

[6] S. Jendrol' and T. Madaras, *On light subgraphs in plane graphs of minimum degree five*, Discuss. Math. Graph Theory **16** (1996) 207–217.

[7] S. Jendrol', T. Madaras, R. Soták, and Z. Tuza, *On light cycles in plane triangulations*, Discrete Math. **197/198** (1999) 453–467.

[8] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat. Čas. SAV (Math. Slovaca) **5** (1955) 111–113.

[9] T. Madaras and R. Soták, *The* 10*-cycle* C_{10} *is light in the family of all plane triangulations with minimum degree five*, Tatra Mountains Math. Publ. **18** (1999) 35–56.

[10] B. Mohar, R. Škrekovski, and H.-J. Voss, *Light subgraphs in planar graphs of minimum degree* 4 *and edge degree* 9, Preprint Ser. Univ. of Ljubljana **38** (2000), 722.