Feasible Sets of Pattern Hypergraphs

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Abstract

A pattern $(\Pi_1, \dots, \Pi_\lambda)$-hypergraph H is a hypergraph $(V, \mathcal{E})$ with edge types Π_i; each of Π_i is a set of equivalences. Each edge of H is assigned one of the edge types Π_i. A coloring c of H is proper if for each Π_i-edge E the color classes of c on E form an equivalence which is contained in the type Π_i. The feasible set of H is the set of all k's for which there is a proper coloring using exactly k colors.

We prove a simple sufficient and necessary condition on $(\Pi_1, \dots, \Pi_\lambda)$ for the existence of a pattern $(\Pi_1, \dots, \Pi_\lambda)$-hypergraph H whose feasible set is not an interval of integers.

1 Introduction

Coloring problems are among the most intensively studied combinatorial problems both from the theoretical and the practical point of view. Generalizations of usual graph and hypergraph coloring, e.g., channel assignment problems, are widely applied in practice. Quite a new concept of mixed hypergraphs has attracted lots of interest [1, 6–14, 16, 19–22]: A mixed hypergraph is a hypergraph with two types of edges, $\mathcal{C}$-edges and $\mathcal{D}$-edges. A coloring of a mixed hypergraph is proper if no $\mathcal{C}$-edge is polychromatic and no $\mathcal{D}$-edge is monochromatic. However, mixed hypergraphs have some very surprising properties. The most striking results include: For any finite set

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of integers I such that $1 \notin I$, there is a mixed hypergraph which can be colored by precisely k of colors iff $k \in I$ [7], e.g., there exists a mixed hypergraph on 6 vertices which is 2-colorable and 4-colorable and which is not 3-colorable. Even more stronger result is the following: For any sequence $s_1, \ldots, s_k$ of integers such that $s_1 = 0$, there exists a mixed hypergraph which has precisely $s_{k'}$ proper colorings using k' colors, $1 \leq k' \leq k$, and no proper coloring using more than k colors [10]. These results led to lots of papers, e.g., [1,6,8,11–14,16,21,22], which study for which subclasses of mixed hypergraphs results of this kind may be obtained. In this paper, we provide a full characterization of edge types of hypergraphs which cause that the numbers of colors which may be used by proper colorings do not form intervals. We introduce a concept of pattern hypergraphs which generalizes concept of usual graph, usual hypergraphs and mixed hypergraphs and which allows several different types of edges. In addition, pattern hypergraphs appear naturally in certain types of constraint satisfaction problems and our characterization gives also an interesting result in this area as discussed further.

An *edge type* is a non-empty set Π of equivalences on the same ordered set; the *size of the edge type* Π is the size of the set on which the equivalences of Π are defined. An *equivalence* is a reflexive, transitive and symmetric relation; we write $x \sim_\pi y$ if x and y are π-equivalent. An equivalence partitions a support set into *classes*. A **pattern hypergraph** H consists of a *type* which is a sequence $(\Pi_1, \ldots, \Pi_\lambda)$ of edge types, a *vertex set* $V(H)$ and an *edge set* $\mathcal{E}(H)$ which contains ordered tuples of its vertices (an edge may contain each vertex at most once); each tuple is assigned one of the λ edge types $\Pi_1, \ldots, \Pi_\lambda$; if an edge E is of type Π_i, we say that E is a Π_i*-edge*. The size of an Π_i-edge has to be equal to the size of Π_i. If H is a pattern hypergraph of type $(\Pi_1, \ldots, \Pi_\lambda)$, we say that H is $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph and in case $\lambda = 1$, briefly a Π_1-hypergraph. An example of a pattern hypergraph can be found in Figure 1.

Let H be a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph. A k*-coloring* c is a mapping of the vertices $V(H)$ to the set $[1, k]$; we write $[a, b]$ for the interval of integers between a and b (inclusively). Unless we need to emphasize the number of used colors, we briefly use a term *coloring*. If π is an equivalence, we say that a coloring c is π*-consistent* on an edge $E = (v_1, \ldots, v_l)$ if $c(v_i) = c(v_j)$ iff $i \sim_\pi j$. If Π is an edge type, a coloring c is Π*-consistent* on E if it is π-consistent for some $\pi \in \Pi$. A coloring c is *proper* if c is Π_i-consistent with all Π_i-edges E. Alternatively: A coloring c defines an equivalence $\sim$ such that $v \sim w$ iff $c(v) = c(w)$. The classes of the equivalence $\sim$ are called *color*

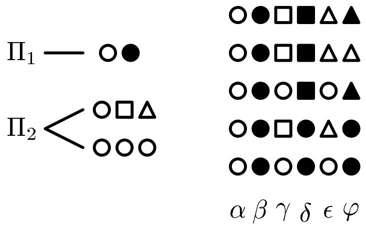

Figure 1: An example of a pattern hypergraph with the vertex set $\{\alpha, \beta, \gamma, \delta, \epsilon, \varphi\}$. There are two edge types: Π_1 is of size two and it contains only a single equivalence. Π_2 is of size three and it contains the two equivalences depicted above. The pattern hypergraph has ten edges: eight Π_1-edges $\{\alpha, \beta\}$, $\{\alpha, \delta\}$, $\{\alpha, \varphi\}$, $\{\gamma, \beta\}$, $\{\gamma, \delta\}$, $\{\gamma, \varphi\}$, $\{\epsilon, \beta\}$ and $\{\epsilon, \delta\}$ and two Π_2-edges $\{\alpha, \gamma, \epsilon\}$ and $\{\beta, \delta, \varphi\}$. All the proper colorings are depicted above. The feasible set of the pattern hypergraph is $\{2, 4, 5, 6\}$.

classes. A coloring c is proper iff the equivalence induced by $\sim$ on any edge E is contained in its edge type.

A coloring c is a *strict k-coloring* if it is proper and it uses all the colors $1, \ldots, k$. The *feasible set* $\mathcal{F}(H)$ is the set of all the integers k for which there is a strict k-coloring of H. If $\mathcal{F}(H)$ is non-empty, a pattern hypergraph is *colorable.* If H is colorable, the least element of $\mathcal{F}(H)$ is called the *chromatic number* of H and it is denoted by $\chi(H)$. Then, the largest element of $\mathcal{F}(H)$ is called the *upper chromatic number* of H and it is denoted by $\bar{\chi}(H)$. If $\mathcal{F}(H) = [\chi(H), \bar{\chi}(H)]$ or $\mathcal{F}(H) = \emptyset$, the feasible set is said to be *unbroken* or *gap-free.* Otherwise, the feasible set is said to be *broken.* A spectrum of a pattern hypergraph H is a sequence $s_1, \ldots, s_{\bar{\chi}(H)}$ such that s_k is the number of different strict k-colorings, $1 \le k \le \bar{\chi}(H)$ (two colorings are different if they define different color classes). The results mentioned in the first paragraph may be now stated as follows: For any finite set of integers I avoiding 1, there is a mixed hypergraph H with a feasible set equal to I [7]; for any sequence $s_1, \ldots, s_k$ with $s_1 = 0$, there is a mixed hypergraph with the spectrum $s_1, \ldots, s_k$ [10].

An equivalence is the *universal equivalence* if it has a single class; an equivalence is the *trivial equivalence* if each of its classes are singletons. $\mathcal{C}_l$ is the edge type containing all the equivalences on l elements except for the trivial equivalence. $\mathcal{D}_l$ is the edge type containing all the equivalences on l elements except for the universal equivalence. We show several examples of pattern hypergraphs: $\mathcal{D}_2$-hypergraphs are usual graphs and proper colorings of $\mathcal{D}_2$-hypergraphs are exactly proper colorings of the corresponding graph. Similarly, $\mathcal{D}_l$-hypergraphs are l-uniform hypergraphs and proper colorings of $\mathcal{D}_l$-hypergraphs are exactly proper colorings of the corresponding l-uniform hypergraph and $(\mathcal{D}_2, \ldots, \mathcal{D}_l)$-hypergraphs are just hypergraphs with maximum edge size l. The notions of a chromatic number for pattern hypergraphs and corresponding (hyper)graphs coincide. A $(\mathcal{D}_2, \mathcal{C}_{l+1})$-hypergraphs may be used to model an l-choosability problems (a similar construction can be found in [15] for mixed hypergraphs): In an l-choosability problem, you are given a graph $G = (V, E)$ together with lists $\Lambda(v)$ of colors of size l. The goal is to find a coloring c of the vertices of V such that $c(v) \in \Lambda(v)$ and $c(u) \neq c(v)$ whenever $uv \in E$. The construction of a $(\mathcal{D}_2, \mathcal{C}_{l+1})$-hypergraph H for a given l-choosability problem is the following: The vertex set of H is $V \cup \Lambda$ where Λ is the union of $\Lambda(v)$ taken over all $v \in V$. Any pair of u and v such that $uv \in E$ is a $\mathcal{D}_2$-edge of H. Any pair of colors of Λ also forms a $\mathcal{D}_2$-edge of H. The $\mathcal{C}_{l+1}$-edges of H are $(l+1)$-tuples $\{v\} \cup \Lambda(v)$ for $v \in V$. It is easy to check that any proper coloring of H is also a valid coloring of the given l-choosability problem. Besides this, pattern hypergraphs generalize the concept of mixed hypergraphs as it is explained in a detail in Section 2. They also generalize a concept of coloring of co-hypergraphs as studied, e.g., in [3, 9, 17, 18].

We introduce four closure concepts for the edge types Π. An equivalence π is *finer* than π', if $x \not\sim_{\pi'} y$ implies that $x \not\sim_{\pi} y$. If π is finer than π', then π' is *coarser* than π. The equivalence $\rho(\Pi)$ is the equivalence such $x \sim_{\rho(\Pi)} y$ iff $x \sim_\pi y$ for all $\pi \in \Pi$; it is easy to check that a relation defined in this way is indeed an equivalence relation. The equivalence $\rho(\Pi)$ may be viewed as an intersection of all the equivalences of Π. An equivalence π' is a *refinement* of π with respect to $\rho(\Pi)$ if π' is coarser than $\rho(\Pi)$ and π' can be obtained from π by splitting one of the equivalence classes to two (in particular, π' is finer than π). The following four closure concepts are considered in this paper (cf. Figures 2–5; equivalence classes are described by shapes):

- The edge type Π is *simply-closed* if it contains all the equivalences π which has only classes of size one with possible exception for a single class. In particular, Π contains both the universal and the trivial equivalence. The unique inclusion-wise smallest edge type which is simply-closed is denoted by Π_{simple}. Note that Π is simply closed iff $\Pi_{\mathrm{simple}} \subseteq \Pi$.

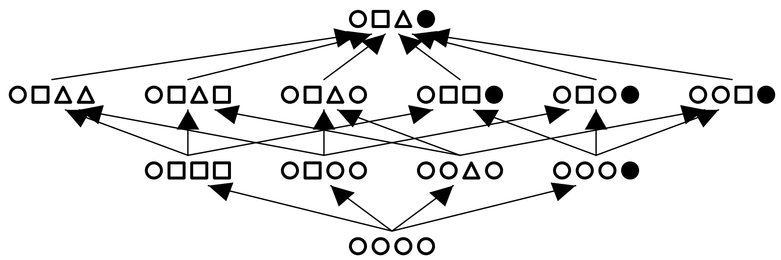

Figure 2: The smallest simply-closed edge type Π_{simple} for the edge size 4.

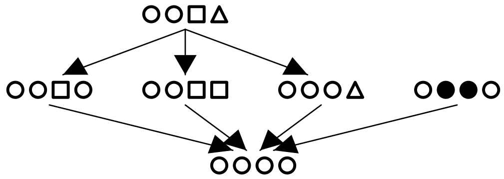

Figure 3: An edge type which is down-closed but which is neither simply-closed, up-closed nor up-group-closed.

- The edge type Π is *down-closed* if for any $\pi \in \Pi$ the edge type Π contains all relations π' which are coarser than π. In particular, Π contains the universal equivalence.
- The edge type Π is *up-closed* if for any $\pi \in \Pi$ the edge type Π contains all the relations π' which can be obtained from π by choosing an element x and introducing a new single element class containing only x. In particular, Π contains the trivial equivalence.
- The edge type Π is *up-group-closed* if for any $\pi \in \Pi$ the edge type Π contains all the refinements π' of the equivalence π with respect to $\rho(\Pi)$.

If all the edge types $\Pi_1, \ldots, \Pi_\lambda$ are simply-closed (Lemma 1), down-closed (Lemma 2), up-closed (Lemma 3) or up-group-closed (Lemma 4), then any $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph has an unbroken feasible set. The main theorem of this paper, Theorem 4 in Section 4, states that these sufficient conditions are also necessary with respect for feasible sets being unbroken:

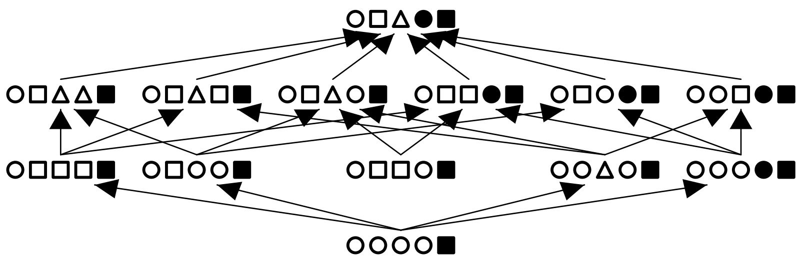

Figure 4: An edge type which is up-closed but which is neither simply-closed, down-closed nor up-group-closed.

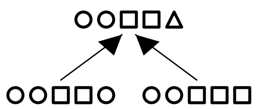

Figure 5: An edge type which is up-group-closed but which is neither simply-closed, down-closed nor up-closed.

There is a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set iff all the edge types $\Pi_1, \ldots, \Pi_\lambda$ are not simply-closed, down-closed, up-closed or up-group-closed. This provides a full characterization of edge types which may cause that the feasible set of a pattern hypergraph is broken.

There is a close relation between pattern hypergraphs and certain types of constraint satisfaction problems. A *constraint satisfaction problem* consists of variables $x_1, \ldots, x_n$, a domain set X and several types of predicates $P_i : X^{r_i} \to \{0, 1\}$ where r_i is the arity of the i-th predicate. One demands that the predicate P_i is satisfied, for prescribed r_i-tuples of $x_1, \ldots, x_n$, i.e., the value of P_i is one for the prescribed tuples. The goal is to find an assignment $\sigma : \{x_1, \ldots, x_n\} \to X$ which satisfies all the predicates. One of possible constraint satisfaction problems is radio frequency assignment: The variables $x_1, \ldots, x_n$ represent radio transmitters, the set X consists of available frequencies and predicates P_i represent possible types of interferences. In this example, the predicates P_i are often invariant to permuting the elements of X, i.e., the predicates are essentially sets of equivalences. Then, an existence of a desired proper assignment σ depends only on the cardinality of the domain set X and such constraint satisfaction problems may be viewed as pattern hypergraphs. There exists a proper assignment σ for X of size $k = |X|$ iff the corresponding pattern hypergraph is k-colorable. The problem which we study in this paper may reformulated as the following question: For which types of predicates can one conclude from the fact that there is a proper assignment σ for X of size k and no proper assignment σ for X of size $k-1$ that there is no proper assignment σ for X of size at most $k-1$?

The paper is structured as follows: We first discuss relation of the concept of pattern hypergraphs to mixed hypergraphs in Section 2. We show that our new general results for pattern hypergraphs provide new interesting results also for mixed hypergraphs. The sufficiency and necessity of the closure properties are studied in Section 3 and Section 4. We show that several possible modifications of definition of pattern hypergraphs do not lead to more general concepts, we discuss possible directions for future research and we post several problems in Section 5.

2 Mixed Hypergraphs

Mixed hypergraphs were introduced in [19,20]. A mixed hypergraph has two types of edges: $\mathcal{C}$-edges and $\mathcal{D}$-edges. A $\mathcal{C}$-edge of size l is in a terminology of pattern hypergraphs a $\mathcal{C}_l$-edge and a $\mathcal{D}$-edge of size l is a $\mathcal{D}_l$-edge. A mixed hypergraph is a mixed bihypergraph if each edge is simultaneously a $\mathcal{C}$-edge and a $\mathcal{D}$-edge. The definitions of a feasible set, a spectrum, a chromatic number, an upper chromatic number, etc., for pattern and mixed hypergraphs coincide. A hypergraph H is spanned by a graph G if $V(G) = V(H)$ and any edge of H induce a connected subgraph of G. The following results on feasible sets of mixed hypergraphs were obtained:

- For any finite integer set I such that $1 \notin I$, there exists a mixed hypergraph H with $\mathcal{F}(H) = I$ [7]. Moreover, there is such a hypergraph H which has only one strict k-coloring for any $k \in I$. A similar result may be obtained for l-uniform mixed bihypergraphs for $l \geq 3$.
- Any mixed hypergraph on at most five vertices has an unbroken feasible set [7]. This bound on the number of vertices is sharp.
- For any sequence of integers $s_1, \ldots, s_k$ such that $s_1 = 0$, there exists a mixed hypergraph H with $O(k + \sum_{i=1}^{k} \log(s_i + 1))$ vertices with the spectrum $s_1, \ldots, s_k$ [10].

- Any mixed hypergraph spanned by a path [1], a tree [11, 12], a cycle [21, 22] or a strong cactus [13] has an unbroken feasible set. There are mixed hypergraphs spanned by weak cacti with a broken feasible set [13].
- For any non-planar graph G with at least six vertices, there is a mixed hypergraph H spanned by G with a broken feasible set [13].
- There are mixed hypergraphs with the maximum vertex degree three with broken feasible sets. Any mixed hypergraph with the maximum vertex degree at most two has an unbroken feasible set [14].
- There are planar mixed hypergraphs with broken feasible sets but the gap in such sets may be only for 3 colors [6, 8, 16]. There is no planar mixed bihypergraph with a broken feasible set [4, 6, 8, 16].

Theorem 4 allows us to enhance this list of results by the following theorems:

Theorem 1 *For any $l_1 \geq 3$ and $l_2 \geq 2$, there exists a mixed hypergraph H with $\mathcal{C}$-edges only of size l_1 and $\mathcal{D}$-edges only of size l_2 such that $\mathcal{F}(H)$ is broken.*

Proof: The edge type $\mathcal{C}_{l_1}$ is neither simply-closed, up-closed nor up-group-closed; the edge type $\mathcal{D}_{l_2}$ is not down-closed. Hence Theorem 4 may be applied. $\square$

In a similar fashion, one may also reprove the following theorem of [7]:

Theorem 2 *For any $l \geq 3$, there exists an l-uniform mixed bihypergraph H with a broken feasible set.*

Proof: The edge type $\mathcal{C}_l \cap \mathcal{D}_l$ is neither simply-closed, down-closed, up-closed nor up-group-closed. Then Theorem 4 can be applied. $\square$

3 Sufficiency of the Condition

We prove the sufficiency of the closure properties on the edge types in this section:

Lemma 1 *Let* $(\Pi_1, \ldots, \Pi_\lambda)$ *be a fixed type of a pattern hypergraph. If all the types* $\Pi_1, \ldots, \Pi_\lambda$ *are simply-closed, then any* $(\Pi_1, \ldots, \Pi_\lambda)$*-hypergraph* H *has an unbroken feasible set.*

Proof: Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H with n vertices. Let $1 \leq k \leq n$. Color $k-1$ vertices with mutually different colors and the remaining vertices all with the same color different from the $k-1$ colors. This coloring is a strict k-coloring because all the edge types are simply-closed. Hence, $\mathcal{F}(H) = [1, n]$. $\Box$

Lemma 2 *Let* $(\Pi_1, \ldots, \Pi_\lambda)$ *be a fixed type of a pattern hypergraph. If all the types* $\Pi_1, \ldots, \Pi_\lambda$ *are down-closed, then any* $(\Pi_1, \ldots, \Pi_\lambda)$*-hypergraph* H *has an unbroken feasible set.*

Proof: Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H. If H is uncolorable, then its feasible set is not broken. Otherwise, let c be a strict k-coloring of H. Since all the edge types are down-closed, the coloring c' defined as $c'(v) := c(v)$ for $c(v) < k$ and $c'(v) := k-1$ for $c(v) = k$ is a strict $(k-1)$-coloring. Hence, $\mathcal{F}(H) = [1, \bar{\chi}(H)]$. $\Box$

Lemma 3 *Let* $(\Pi_1, \ldots, \Pi_\lambda)$ *be a fixed type of a pattern hypergraph. If all the types* $\Pi_1, \ldots, \Pi_\lambda$ *are up-closed, then any* $(\Pi_1, \ldots, \Pi_\lambda)$*-hypergraph* H *has an unbroken feasible set.*

Proof: Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H with n vertices. If H is uncolorable, then its feasible set is not broken. Otherwise, let c be a strict k-coloring of H for $k < n$. Assume that the color k is used to color at least two vertices and one of them is a vertex w. Since all the edge types are up-closed, the coloring c' defined as $c'(v) := c(v)$ for $v \neq w$ and $c'(w) := k+1$ is a strict $(k+1)$-coloring. Hence, $\mathcal{F}(H) = [\chi(H), n]$. $\Box$

Lemma 4 *Let* $(\Pi_1, \ldots, \Pi_\lambda)$ *be a fixed type of a pattern hypergraph. If all the types* $\Pi_1, \ldots, \Pi_\lambda$ *are up-group-closed, then any* $(\Pi_1, \ldots, \Pi_\lambda)$*-hypergraph* H *has an unbroken feasible set.*

Proof: Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H. If H is uncolorable, then its feasible set is not broken. Otherwise, consider the following relation $\sim'$ on the vertices of H: $v \sim' w$ iff $v \sim_{\rho(\Pi_i)} w$ for some Π_i-edge. Let $\sim$ be the equivalence closure of the relation $\sim'$ and k_0 the number of its classes. In any proper coloring c of H, if $v \sim w$, then $c(v) = c(w)$. Thus $\bar{\chi}(H) \leq k_0$. Let c be a strict k-coloring of H for $k < k_0$. Assume that the color k is used to color at least two different classes of $\sim$ and one of the vertices colored with the color k is a vertex w. Consider the coloring c' defined as $c'(v) := c(v)$ for $v \not\sim w$ and $c'(v) := k+1$ for $v \sim w$. Since all the edge-types are up-group-closed and $\rho(\Pi_i)$ is on Π_i-edge E finer than $\sim$, c' is a strict $(k+1)$-coloring. Hence, $\mathcal{F}(H) = [\chi(H), k_0] = [\chi(H), \bar{\chi}(H)]$. $\Box$

4 Necessity of the Condition

In the beginning of this section, we consider the special case when the pattern hypergraph has a single edge type. First, several lemmas dealing with the case that this edge type contains the trivial or the universal equivalence are presented:

Lemma 5 *Let Π be a fixed edge type which contains both the trivial and the universal equivalence and which is not simply-closed. Then there exists a Π-hypergraph H whose feasible set is broken.*

Proof: Let l be the edge size of Π and consider a hypergraph H with $n = l^2$ vertices such that all possible l-tuples form edges of H. Clearly, $1 \in \mathcal{F}(H)$ and $n \in \mathcal{F}(H)$. Assume for the sake of contradiction that $l \in \mathcal{F}(H)$. Let c be a strict l-coloring of H; assume that the color 1 is used the most times, i.e., there are at least l vertices $v_1, \ldots, v_l$ are colored with the color 1. Let u_i for $2 \leq i \leq l$ be any vertex colored with the color i. Let π be an equivalence of Π_{simple}; the tuple containing some of vertices $v_1, \ldots, v_l$ in the positions of the largest class of π and some of vertices $u_2, \ldots, u_l$ in the positions of the single-element classes of π is an edge of H and thus $\pi \in \Pi$. But this is impossible for all the $\pi \in \Pi_{\mathrm{simple}}$ because Π is not simply-closed. $\Box$

Lemma 6 *Let Π be a fixed edge type which contains the trivial equivalence, which does not contain the universal equivalence and which is not up-closed. Then there exists a Π-hypergraph H whose feasible set is broken.*

Proof: Let l be the edge size of Π and consider a hypergraph H with $n = l^2(l+1)$ vertices v_{ij} for $1 \leq i \leq l$ and $1 \leq j \leq l(l+1)$. Take an l-coloring c_0 such that $c_0(v_{ij}) = i$. The edges of H are all l-tuples Π-consistent with the coloring c_0. Clearly, $l \in \mathcal{F}(H)$ and $n \in \mathcal{F}(H)$. Assume for the sake of contradiction that $l+1 \in \mathcal{F}(H)$; let c be a strict $(l+1)$-coloring and ξ_i be the color used by c to color most of vertices v_{ij}, $1 \leq j \leq l(l+1)$. We may assume w.l.o.g. that $c(v_{i1}) = \ldots = c(v_{il}) = \xi_i$ for each $1 \leq i \leq l$.

We first prove that $\xi_i \neq \xi_{i'}$ for all $i \neq i'$. Assume that $\xi_1 = \xi_2$. Let π be the equivalence of Π such that the size of the largest class of π is as large as possible; let l_0 be the size of largest class of π. Note that $l_0 < l$ because Π does not contain the universal equivalence. Consider an edge E of H which corresponds to a tuple π-consistent with c_0 which contains vertices $v_{11}, \ldots, v_{1l_0}$ and a vertex v_{21}. Then there is an equivalence of Π such that the size of its largest class is at least l_0+1 because c is a proper coloring and E contains l_0+1 vertices of the same color. This contradicts the choice of π. This argumentation may be easily extended for the case $\xi_i = \xi_{i'}$ for any $i \neq i'$.

Next, we prove that Π is up-closed. Since c is a strict $(l+1)$-coloring, we may assume w.l.o.g. that $c(v_{1,l+1}) \neq c(v_{i1})$ for all $1 \leq i \leq l$. Let π be an equivalence of Π which is not the trivial equivalence and π' be an equivalence obtained from π by creating a single element class by separating an element w from a class of π. Consider the edge E of H corresponding to a tuple π-consistent with c_0 such that w is $v_{1,l+1}$, the remaining elements of the class of w in π are some of the vertices $v_{11}, \ldots, v_{1l}$ and other classes of π are formed by vertices $v_{i1}, \ldots, v_{il}$. Since c is a proper coloring, we have obtained $\pi' \in \Pi$. Hence Π is up-closed. $\Box$

We state as a proposition a deficit version of Hall's theorem [2] which we use in the proof of the next lemma:

Proposition 1 *Let A_i be a system of sets for $1 \leq i \leq n$; let $0 \leq \Delta \leq n$. If it holds that $|\bigcup_{i \in I} A_i| \geq |I| - \Delta$ for any $I \subseteq [1,n]$, then there exists a set $I_0 \subseteq [1,n]$ of size $n - \Delta$ and elements a_i for $i \in I_0$ such that $a_i \in A_i$ and all the elements $a_i, i \in I_0$ are mutually different.*

Lemma 7 *Let Π be a fixed edge type which contains the universal equivalence, which does not contain the trivial equivalence and which is not down-closed. Then there exists a Π-hypergraph H whose feasible set is broken.*

Proof: Let l be the edge size and consider a hypergraph H with $n = l^2(l+1)^2$ vertices v_{ij} for $1 \leq i \leq (l+1)^2$ and $1 \leq j \leq l^2$; set $L = (l+1)^2$. Consider a coloring c_0 such that $c_0(v_{ij}) = i$ for $1 \leq i \leq L$. The edges of

H are all l-tuples Π-consistent with the coloring c_0. Clearly, $1 \in \mathcal{F}(H)$ and $L \in \mathcal{F}(H)$. We prove $L-1 \notin \mathcal{F}(H)$.

Assume for the sake of contradiction that there is a strict $(L-1)$-coloring c of H. We prove that $c(v_{ij}) = c(v_{ij'})$ for all $1 \le i \le L$ and $1 \le j, j' \le l^2$. Assume that, e.g., $c(v_{11}) \ne c(v_{12})$. Let $C_i = \{c(v_{ij}), 1 \le j \le l^2\}$. We distinguish two cases:

- **There exists a set $I \subseteq [1, L]$ such that $|I| = L - (l+1)$ and $|\bigcup_{i \in I} C_i| \le l$.**

  Let I be such a set and $I' = [1, L] \setminus I$. The size of $\bigcup_{i \in I'} C_i$ is at least $L - 1 - l = L - (l+1)$ because c is a strict $(L-1)$-coloring. Since $|I'| = l+1$, there exists $i_0 \in I'$, such that $|C_{i_0}|$ is at least $(L-(l+1))/(l+1) = l$. Consider an edge E of H which contains vertices $v_{i_0 j}$ for $1 \le j \le l^2$ such that their colors are mutually different; such an edge E exists because Π contains the universal equivalence. Since c is proper, Π has to contain the trivial equivalence due to E.

- **It holds that $|\bigcup_{i \in I} C_i| \ge l+1$ for each set $I \subseteq [1, L]$ such that $|I| = L - (l+1)$.**

  We apply Proposition 1 to the sets C_i for $1 \le i \le L$ with $\Delta = L-(l+1)$. If size of $I \subseteq [1, L]$ is at most $L-(l+1)$, then $|I| - \Delta \le 0$. If size of $I \subseteq [1, L]$ is at least $L-(l+1)$, then $|\bigcup_{i \in I} C_i| \ge l+1 = L - \Delta \ge |I| - \Delta$. Hence there exists a set I_0 of size $l+1$ such that the colors of vertices $v_{ij_i}, i \in I_0$ are mutually different. We may assume w.l.o.g. that all the colors of the vertices $v_{11}, v_{21}, \ldots, v_{l1}$ and of a vertex v_{12} are different. Let π be an equivalence of Π with the largest number of classes; let l' be this number. Note that $l' < l$. Consider the edge E of H corresponding to the tuple π-consistent with c_0 such that E contains all the vertices $v_{11}, \ldots, v_{l'1}$ and the vertex v_{12}. Since c is proper, there must be an equivalence of Π with at least $l'+1$ classes due to the presence of the edge E. But this is impossible because of the choice of l'.

Let ξ_i be further the common color of the vertices v_{ij} for $1 \le j \le l^2$. Assume w.l.o.g. that $\xi_1 = \xi_2$ and all the colors ξ_i for $i \ge 2$ are mutually different. Consider an equivalence $\pi \in \Pi$ and an equivalence π' obtained from π by an union of two classes of π. Let E be the edge of H corresponding to the tuple π-consistent with c_0 such that the united classes of π contain vertices v_{1j} and v_{2j}. Since c is proper, $\pi' \in \Pi$ due to the presence of the edge E in H. Hence Π is down-closed. $\Box$

We focus on edge types avoiding both the universal and the trivial equivalence:

Lemma 8 *Let Π be a fixed edge type which contains neither the universal nor the trivial equivalence. There exists a Π-hypergraph whose feasible set is broken iff there exists a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph whose feasible set is broken.*

Proof: It is enough to prove that if there is a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with a broken feasible set, then there is a Π-hypergraph H' with a broken feasible set because the opposite implication is trivial. Let l be the edge size of Π.

We first construct a certain special Π-hypergraph H_0: Consider a set of $2l^3$ vertices v_{ij} such that $1 \le i \le 2l$ and $1 \le j \le l^2$ and a coloring $c_0(v_{ij}) = i$. The edges of H_0 are all the possible tuples of vertices Π-consistent with c_0. We claim that c_0 is the only proper coloring of H_0. Consider any proper coloring c of H_0. Let $C_i = \{c(v_{ij}), 1 \le j \le l^2\}$ for $1 \le i \le 2l$ and I the set of i for which $|C_i| \le l$. If $|I| \le l$, then there is an edge E such that all its vertices are colored by c with mutually different colors: Consider $\pi \in \Pi$. Choose vertices v_{ij} with the same $i \notin I$ to form a color class of π and for different color classes choose vertices with different indices i; due to the definition of I, it is possible to choose these vertices in such a way that all of them are colored by c with mutually different colors. This edge is π-consistent with c_0, but it is not Π-consistent with c because Π does not contain a trivial equivalence. Since c is proper, we may conclude that it must be that $|I| \ge l + 1$.

We assume w.l.o.g. that $[1, l+1] \subseteq I$ and $c(v_{i1}) = \ldots = c(v_{il})$ for each $i \in I$. Let $\xi_i = c(v_{i1})$ for $i \in I$ and $V = \{v_{ij}, c(v_{ij}) = \xi_i, 1 \le i \le l\} \supseteq \{v_{ij}, 1 \le i \le l+1, 1 \le j \le l\}$. The colors $\xi_i, i \in I$ are mutually different: Assume $\xi_1 = \xi_2$. Consider the equivalence $\pi \in \Pi$ such that it has the largest size l' of a class among all the equivalences in Π and an edge E formed by some of the vertices of V which is π-consistent with c_0 such that the largest class of π is formed by some of the vertices v_{1j} with $1 \le j \le l$ and one of the other classes is formed by the vertices v_{2j} with $1 \le j \le l$. But E contains more than l' vertices colored by c with the same color which is impossible because of the choice of π. Observe that we have actually proved that $\xi_1 \ne c(v_{ij})$ for any $2 \le i \le 2l$ and $1 \le j \le l^2$.

We first prove that $I = [1, 2l]$ and later we prove that V actually contains all the vertices of H_0. Assume that $2l \notin I$. We may assume w.l.o.g. $c(v_{2l,1}) \ne c(v_{2l,2})$ and that both these colors are different from all

$\xi_1, \dots, \xi_{l+1}$ (this is because of the observation at the end of the previous paragraph). Consider an equivalence of Π with the largest number l' of classes and an edge E which is π-consistent with c_0 and one of its classes contain both the vertices $v_{2l,1}$ and $v_{2l,2}$ and E contains only the vertices of V except for the vertices $v_{2l,j}$ for $1 \le j \le l^2$. The vertices of E are colored by c with more colors than l' which is impossible because of the choice of π. Thus $I = [1, 2l]$.

We assume that V does not contain all the vertices of H_0; let w be a vertex of H_0 not contained in V. The color $c(w)$ is different from all ξ_i, $1 \le i \le 2l$, due to the observation at the end of the last but one paragraph. Assume that $w = v_{i,l+1}$. Consider again an equivalence of Π with the largest number l' of classes and an edge E which is π-consistent with c_0 and one of its classes contains both the vertices v_{i1} and $v_{i,l+1}$ and E contains only the vertices of V except for $v_{i,l+1}$. The vertices of E are colored by c with more colors than l' which is impossible because of the choice of π. Hence V contains all the vertices of H_0.

We have proved that the special Π-hypergraph H_0 has the desired properties, namely, c_0 is its only proper coloring. Consider now a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with a broken feasible set; let $u_1, \dots, u_n$ be the vertices of H. We now construct a Π-hypergraph H' with a broken feasible set. The vertex set of H' consists of vertices u_{ij} for $1 \le i \le n$ and $1 \le j \le l^2$ and vertices u^*_{ij} for $2 \le i \le 2l$ and $1 \le j \le l^2$. We form on these vertices several copies of H_0: The first copy is formed on the vertices u^*_{ij} setting $v_{ij} = u^*_{ij}$. Other copies are formed on the vertices $u_{i_0 j}$ for a fixed $1 \le i_0 \le n$ and u^*_{ij} for $2 \le i \le 2l$ setting $v_{1j} = u_{i_0 j}$ and $v_{ij} = u^*_{ij}$ for $2 \le i \le 2l$. The just formed copies of H_0 force that any proper coloring colors all the vertices u^*_{ij}, $1 \le j \le l^2$, for a fixed i by the same color ξ^*_i and it colors all the vertices u_{ij}, $1 \le j \le l^2$, for a fixed i also by the same color ξ_i. Besides this, $\xi^*_i \ne \xi^*_{i'}$ for any $i \ne i'$ and $\xi^*_i \ne \xi_{i'}$ for any i, i'.

If the vertices u_{i_1} and u_{i_2} form a $\mathcal{D}_2$-edge in H, we add a copy of H_0 on the vertices $u_{i_1 j}$, $u_{i_2 j}$ and u^*_{ij} for $3 \le i \le 2l$ setting $v_{1j} = u_{i_1 j}$, $v_{2j} = u_{i_2 j}$ and $v_{ij} = u^*_{ij}$ for $3 \le i \le 2l$. This forces that $\xi_{i_1} \ne \xi_{i_2}$. If the vertices u_{i_1} and u_{i_2} form a $\mathcal{C}_2$-edge in H, we add the following copy of H_0 on the vertices $u_{i_1 1}$, $u_{i_2 j}$ and u^*_{ij} for $2 \le i \le 2l$ setting $v_{11} = u_{i_1 1}$, $v_{1j} = u_{i_2 j}$ (this for $2 \le j \le l^2$) and $v_{ij} = u^*_{ij}$ for $2 \le i \le 2l$. This forces $\xi_{i_1} = \xi_{i_2}$. Finally, if the vertices $u_{i_1}, \dots, u_{i_l}$ form a Π-edge in H, we add an Π-edge $u_{i_1,1}, \dots, u_{i_l,1}$ to H'. This completes the construction of H'. Proper colorings of H and H' one-to-one correspond through setting $c(u_i) = \xi_i$. Hence we may conclude

$\mathcal{F}(H') = \{k+2l | k \in \mathcal{F}(H)\}$ and the feasible set of H' is broken. $\square$

We define a projection of an edge type Π: An edge type Π' is a $\mathcal{C}$-*projection* of Π if there exist α and β such that $\Pi' = \{\pi | \pi \in \Pi \wedge \alpha \sim_\pi \beta\}$. An edge type Π' is a $\mathcal{D}$-*projection* of Π if there exist α and β such that $\Pi' = \{\pi | \pi \in \Pi \wedge \alpha \not\sim_\pi \beta\}$. A *projection* of Π is any edge type which may be obtained from Π by a sequence of $\mathcal{C}$-projections and $\mathcal{D}$-projections.

We first state the following proposition to show the reasons for introducing the previous definition:

Proposition 2 *Let Π be an edge type containing neither the trivial nor the universal equivalence. Let Π' be any projection of Π. If there is Π'-hypergraph H' with a broken feasible set, then there is Π-hypergraph with a broken feasible set.*

Proof: It is enough to show the proposition for Π' which is either a $\mathcal{C}$-projection or a $\mathcal{D}$-projection of Π (in a general case, this proof is inductively applied to the sequence of projections needed to obtain Π' from Π). We construct a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with a broken feasible set which is sufficient due to Lemma 8. The hypergraph H has the same vertex set as H'; the Π-edges of H are exactly the same one as Π'-edges of H'. In case that Π' is a $\mathcal{C}$-projection of Π, we place $\mathcal{C}_2$-edges in each edge on the positions of the pairs defining the projection Π'. In case that Π' is a $\mathcal{D}$-projection, we place $\mathcal{D}_2$-edges in a similar way. Clearly, $\mathcal{F}(H) = \mathcal{F}(H')$ and the feasible set of H is broken. $\square$

We now prove several little technical lemmas:

Lemma 9 *Let Π be a fixed edge type which contains neither the universal nor the trivial equivalence such that $\rho(\Pi) \notin \Pi$. Then there exist (not necessarily distinct) elements α, β, γ and δ which satisfy for some projection Π_0 of Π the following:*

- *There is $\pi \in \Pi_0$ such that $\alpha \sim_\pi \beta$.*
- *It holds that $\alpha \not\sim_{\rho(\Pi)} \beta$, $\alpha \not\sim_{\rho(\Pi)} \gamma$, $\alpha \not\sim_{\rho(\Pi)} \delta$, $\beta \not\sim_{\rho(\Pi)} \gamma$ and $\gamma \not\sim_{\rho(\Pi)} \delta$.*
- *For any $\pi \in \Pi_0$, if $\alpha \not\sim_\pi \beta$, then $\gamma \sim_\pi \delta$.*

Proof: Let $\pi_0 = \rho(\Pi)$ and α and β be any two elements such that $\alpha \not\sim_{\pi_0} \beta$ and such that there is an equivalence $\pi \in \Pi$ for which $\alpha \sim_\pi \beta$. Consider the $\mathcal{D}$-projection Π' with respect to the pair α and β. Assume first that $\pi_0 \neq \rho(\Pi')$. Let γ and δ be such that $\gamma \not\sim_{\pi_0} \delta$, but $\gamma \sim_{\rho(\Pi')} \delta$. If $\alpha \sim_{\pi_0} \gamma$ and $\beta \sim_{\pi_0} \gamma$, then $\alpha \sim_{\pi_0} \beta$ which is impossible due to the choice of α and β. Similarly for $\alpha \sim_{\pi_0} \delta$ and $\beta \sim_{\pi_0} \delta$ **or** $\alpha \sim_{\pi_0} \gamma$ and $\alpha \sim_{\pi_0} \delta$ **or** $\beta \sim_{\pi_0} \gamma$ and $\beta \sim_{\pi_0} \delta$. If $\alpha \sim_{\pi_0} \gamma$ and $\beta \sim_{\pi_0} \delta$, then $\gamma \sim_{\rho(\Pi')} \delta$ implies $\alpha \sim_{\rho(\Pi')} \beta$ which is impossible due to the choice of Π'. Similarly for $\alpha \sim_{\pi_0} \delta$ and $\beta \sim_{\pi_0} \gamma$. Hence we may assume w.l.o.g. that $\alpha \not\sim_{\pi_0} \gamma$, $\alpha \not\sim_{\pi_0} \delta$ and $\beta \not\sim_{\pi_0} \gamma$. We get $\alpha \not\sim_{\pi_0} \beta$ from the choice of α and β and $\gamma \not\sim_{\pi_0} \delta$ from the choice of γ and δ. Then the elements α, β, γ and δ satisfy the conditions of the statement of the lemma.

If $\rho(\Pi') = \rho(\Pi)$, we apply the above described procedure to Π'. Since each time the number of equivalences contained in Π is decreased (we have chosen α and β such that $\alpha \not\sim_{\pi_0} \beta$), we either end with the quadruple α, β, γ and δ and a projection Π_0 of Π obtained as through the above sequence of $\mathcal{D}$-projections such that Π_0 satisfies the conditions of the statement of the lemma or we end with a projection of Π consisting only of the equivalence π_0, but then $\pi_0 = \rho(\Pi) \in \Pi$. $\Box$

Lemma 10 *Let Π be a fixed edge type which contains neither the universal nor the trivial equivalence. If there exist (not necessarily distinct) elements α, β, γ and δ which satisfy the following:*

- *There is $\pi \in \Pi$ such that $\alpha \sim_\pi \beta$.*
- *It holds that $\alpha \not\sim_{\rho(\Pi)} \beta$, $\alpha \not\sim_{\rho(\Pi)} \gamma$, $\alpha \not\sim_{\rho(\Pi)} \delta$, $\beta \not\sim_{\rho(\Pi)} \gamma$ and $\gamma \not\sim_{\rho(\Pi)} \delta$.*

And one of the following two conditions holds:

- *For any $\pi \in \Pi$, if $\alpha \not\sim_\pi \beta$, then $\gamma \sim_\pi \delta$.*
- *For any $\pi \in \Pi$, if $\alpha \sim_\pi \beta$, then $\gamma \sim_\pi \delta$.*

Then there exists a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H whose feasible set is broken.

Proof: Let l be the edge size of Π (note that $l \geq 3$) and consider the following $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H on the vertices v_{ijk} for $1 \leq i \leq l+2$, $1 \leq j \leq l+4$ and $1 \leq k \leq l$. For simplicity we say that the vertices with the same first index form rows and the vertices with the same second index form columns. Let c_1 be the $(l+2)$-strict coloring defined as $c_1(v_{ijk}) = i$ and c_2 the $(l+4)$-strict coloring defined as $c_2(v_{ijk}) = j$. The hypergraph H contains all the tuples Π-consistent with both c_1 and c_2 as Π-edges, the pairs $\mathcal{D}_2$-consistent with both c_1 and c_2 as $\mathcal{D}_2$-edges and the pairs $\mathcal{C}_2$-consistent with both c_1 and c_2 as $\mathcal{C}_2$-edges. Hence, two vertices v_{ijk} and $v_{i'j'k'}$ form a $\mathcal{D}_2$-edge iff $i \neq i'$ and $j \neq j'$ and two vertices v_{ijk} and $v_{i'j'k'}$ form a $\mathcal{C}_2$-edge iff $i = i'$ and $j = j'$. Consider any proper coloring c of H and let ξ_{ij} be the common color of the vertices v_{ijk} for $1 \leq k \leq l$. Observe that α, β and γ are mutually different and δ is either β or different from both α and β (note that δ cannot be γ).

We first deal with **the case that $\alpha \not\sim_\pi \beta$ implies $\gamma \sim_\pi \delta$ for all $\pi \in \Pi$.** In this case, the feasible set of H is always broken. Let π_1 be the relation of Π such that $\gamma \not\sim_{\pi_1} \delta$ (and hence $\alpha \sim_{\pi_1} \beta$) and π_2 the relation of Π such that $\alpha \not\sim_{\pi_2} \beta$ (and hence $\gamma \sim_{\pi_2} \delta$). We distinguish several cases with respect to the relation of π_1, π_2, α, β, γ and δ from the statement of the theorem:

- $\beta = \delta$

  Assume that there exists i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. Due to the construction of H and the existence of π_1 and π_2, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{i'j'1}$ for all $i' \neq i$. But then $\xi_{i'j'} = \xi_{ij'}$ for all $i \neq i'$. Similarly, with setting $\alpha = v_{ij'1}$ and $\beta = v_{ij1}$ we may conclude that $\xi_{i'j} = \xi_{ij}$ for all $i \neq i'$. Let j'' be any number different from both j and j'. Then $\xi_{ij''} \neq \xi_{ij}$ or $\xi_{ij''} \neq \xi_{ij'}$. Hence, we may conclude that $\xi_{ij''} = \xi_{i'j''}$ for all $i \neq i'$. Since the choice of j'' was arbitrary, we may conclude that each column is monochromatic with respect to c. In such case, c has to be a strict $(l+4)$-coloring due to the presence of $\mathcal{D}_2$-edges.

  On the other hand, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c consists of monochromatic rows and c is a strict $(l+2)$-coloring. Finally, $\mathcal{F}(H) = \{l+2, l+4\}$ and this feasible set is clearly broken.

- $\beta \neq \delta$, either γ or δ is π_1-equivalent to α (and hence to β as well) and either α or β is π_2-equivalent to γ (and hence to δ as well).

  We may assume w.l.o.g. that $\beta \sim_{\pi_1} \delta$ and $\beta \sim_{\pi_2} \delta$ (otherwise we exchange α and β or γ and δ). We proceed as in the previous with additional setting $\delta = v_{ij2}$ whenever β has been set to be v_{ij1} for some i and j and we conclude that $\mathcal{F}(H) = \{l+2, l+4\}$.

- $\beta \neq \delta$ and either γ or δ is π_1-equivalent to α (and hence to β as well), but $\alpha \not\sim_{\pi_2} \gamma$ and $\beta \not\sim_{\pi_2} \gamma$.

  We assume w.l.o.g. that $\alpha \sim_{\pi_1} \gamma$.

  Assume that there exist i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. Due to the construction of H and the existence of π_1 and π_2, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{ij''1}$ and $\delta = v_{i'j''1}$ for any $i' \neq i$ and $j'' \neq j, j'$. But then $\xi_{ij''} = \xi_{i'j''}$. Hence all the columns are monochromatic except possible for those with indices j and j'. Due to the presence of the $\mathcal{D}_2$-edges, we may choose now two different j and j' such that $\xi_{ij} \neq \xi_{ij'}$ to be indices of these monochromatic columns and applying a symmetric argument yields that all the columns are monochromatic. Then the coloring c has to be a strict $(l+4)$-coloring.

  On the other hand, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c has to be a strict $(l+2)$-coloring due to the $\mathcal{D}_2$-edges. Hence $\mathcal{F}(H) = \{l+2, l+4\}$ and this feasible set is clearly broken.

- $\beta \neq \delta$ and either α or β is π_2-equivalent to γ (and hence to δ as well), but $\alpha \not\sim_{\pi_1} \gamma$ and $\alpha \not\sim_{\pi_1} \delta$.

  We assume w.l.o.g. that $\beta \sim_{\pi_2} \gamma$.

  Assume that there exist i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. Due to the construction of H and the existence of π_1 and π_2, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$, $\gamma = v_{i'j'1}$ and $\delta = v_{i''j'1}$ for all $i', i'' \neq i$. But then $\xi_{i'j'} = \xi_{i''j'}$ for all $i \neq i', i''$. Similarly as in the first case, we conclude that all the columns except possibly for their elements of the i-th row are monochromatic. Using the same reasoning for another choice of i (which is possible because of the presence of $\mathcal{D}_2$-edges in H and because the columns are monochromatic with possible exception for the elements of the i-th row), one can conclude that actually the columns are completely monochromatic. Hence c is a strict $(l+4)$-coloring.

  On the other hand, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c consists of monochromatic rows and it is a strict $(l+2)$-coloring. We conclude again that $\mathcal{F}(H) = \{l+2, l+4\}$ and this feasible set is clearly broken.

- $\beta \neq \delta$, $\alpha \not\sim_{\pi_1} \gamma$, $\alpha \not\sim_{\pi_1} \delta$, $\alpha \not\sim_{\pi_2} \gamma$ and $\beta \not\sim_{\pi_2} \gamma$.

  Assume that there exist i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. Due to the construction of H and the existence of π_1 and π_2, there is an edge

E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{i'j''1}$ and $\delta = v_{i''j''1}$ for any $i \neq i', i''$ and $j'' \neq j, j'$. But then $\xi_{i'j''} = \xi_{i''j''}$. Hence all the columns except possibly for those with indices j and j' and with a possible exception for the elements of the i-th row are monochromatic. Combining technique of the previous two cases, we again conclude that all the columns are completely monochromatic. Hence the coloring c has to be a strict $(l+4)$-coloring.

On the other hand, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c consists of monochromatic rows and it is a strict $(l+2)$-coloring. Finally, $\mathcal{F}(H) = \{l+2, l+4\}$ and this feasible set is broken.

It remains to deal with **the case that** $\alpha \sim_\pi \beta$ **implies** $\gamma \sim_\pi \delta$ **for all** $\pi \in \Pi$. Due to Lemma 9, either $\rho(\Pi) \in \Pi$ or there exists other α, β, γ and δ and a projection of Π for which the already proven part of this lemma may be applied (and hence this would give an existence of a desired hypergraph with a broken feasible set due to Proposition 2). In the latter case, the proof is finished. Hence assume that $\rho(\Pi) \in \Pi$ and let $\pi_0 = \rho(\Pi)$.

We first deal with the case $\beta = \delta$. The assumptions of the lemma yield that $\alpha \not\sim_{\rho(\Pi)} \beta$, $\alpha \not\sim_{\rho(\Pi)} \gamma$ and $\beta \not\sim_{\rho(\Pi)} \gamma$. Thus $\alpha \not\sim_{\pi_0} \beta$, $\alpha \not\sim_{\pi_0} \gamma$ and $\beta \not\sim_{\pi_0} \gamma$. Let π be the equivalence of Π such that $\alpha \sim_\pi \beta$ (and thus $\beta \sim_\pi \gamma$). Consider i, j and j' such that $\xi_{ij} = \xi_{ij'}$. Due to the construction of H and the existence of π_0 and π, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{ij''1}$ for any $j'' \neq j, j'$. But then $\xi_{ij} = \xi_{ij'} = \xi_{ij''}$. Hence each row is either monochromatic or completely polychromatic. If all the rows are monochromatic, then c is a strict $(l+2)$-coloring. If any of the rows is completely polychromatic, then c uses at least $l+4$ colors. We conclude that $\mathcal{F}(H) \subseteq \{l+2\} \cup [l+4, n]$ where $n = (l+2)(l+4)$.

The remaining case is that all α, β, γ and δ are mutually different. If it holds that $\beta \sim_\pi \delta$ for all $\pi \in \Pi$, we may set $\beta = \delta$ and this case can be reduced to the case considered in the previous paragraph. Assume that $\beta \not\sim_\pi \delta$ for some $\pi \in \Pi$, i.e., $\beta \not\sim_{\rho(\Pi)} \delta$. Then for any pair of α, β, γ and δ, there is $\pi \in \Pi$ such that the elements of this pair are not π-equivalent (this follows from the assumptions of the lemma for pairs different from β and δ). Hence any two elements of α, β, γ and δ are not π_0-equivalent. Let π be the equivalence of Π such that $\alpha \sim_\pi \beta$ (and thus $\gamma \sim_\pi \delta$). If all such equivalences satisfy that $\beta \sim_\pi \gamma$, we may replace δ by β and the case considered in the previous paragraph may be applied. Assume hence further that $\beta \not\sim_\pi \gamma$.

Consider i, j and j' such that $\xi_{ij} = \xi_{ij'}$. Due to the construction of H and the existence of π_0 and π, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$, $\gamma = v_{i'j''1}$ and $\delta = v_{i'j'''1}$ for any $i' \neq i$, $j'' \neq j, j'$ and $j''' \neq j, j', j''$. But then $\xi_{i'j''} = \xi_{i'j'''}$. Hence each row except possibly for the i-th row is monochromatic with a possible exception for the elements of the j-th and j'-th column. Using a similar arguments as in the previous, we may conclude that all the rows are completely monochromatic. Hence either all the rows are completely monochromatic or there is a completely polychromatic row. Thus $\mathcal{F}(H) \subseteq \{l+2\} \cup [l+4, n]$ where $n = (l+2)(l+4)$. $\square$

Lemma 11 *Let Π be a fixed edge type. If Π is not up-group-closed, then there exist (not necessarily distinct) elements α, β, γ and δ and a projection Π_0 of Π which satisfy the following:*

- *There is $\pi \in \Pi_0$ such that $\alpha \sim_\pi \beta$.*
- *It holds that $\alpha \not\sim_{\rho(\Pi)} \beta$, $\alpha \not\sim_{\rho(\Pi)} \gamma$, $\alpha \not\sim_{\rho(\Pi)} \delta$, $\beta \not\sim_{\rho(\Pi)} \gamma$ and $\gamma \not\sim_{\rho(\Pi)} \delta$.*

Moreover, one of the following two conditions holds:

- *For any $\pi \in \Pi_0$, if $\alpha \not\sim_\pi \beta$, then $\gamma \sim_\pi \delta$.*
- *For any $\pi \in \Pi_0$, if $\alpha \sim_\pi \beta$, then $\gamma \sim_\pi \delta$.*

Proof: Assume that Π is not up-group-closed. Let π be an equivalence of Π and π' an refinement of π w.r.t. $\rho(\Pi)$ such that $\pi' \in \Pi$. If $\pi' = \rho(\Pi)$, we use Lemma 9. Otherwise there exist α and β such that $\alpha \not\sim_{\rho(\Pi)} \beta$ and $\alpha \sim_{\pi'} \beta$. Consider the $\mathcal{C}$-projection Π' of Π with respect to the pair α and β. Note that $\pi \in \Pi'$. Assume first that there are two elements γ and δ such that $\gamma \sim_{\rho(\Pi')} \delta$ and $\gamma \not\sim_{\pi'} \delta$, i.e., $\rho(\Pi')$ is not finer than π'. If $\alpha \sim_{\rho(\Pi)} \gamma$ and $\beta \sim_{\rho(\Pi)} \gamma$, then $\alpha \sim_{\rho(\Pi)} \beta$ which is impossible due to the choice of α and β. Similarly for $\alpha \sim_{\rho(\Pi)} \delta$ and $\beta \sim_{\rho(\Pi)} \delta$ **or** $\alpha \sim_{\rho(\Pi)} \gamma$ and $\alpha \sim_{\rho(\Pi)} \delta$ **or** $\beta \sim_{\rho(\Pi)} \gamma$ and $\beta \sim_{\rho(\Pi)} \delta$. If $\alpha \sim_{\rho(\Pi)} \gamma$ and $\beta \sim_{\rho(\Pi)} \delta$, then $\alpha \sim_{\pi'} \beta$ implies $\gamma \sim_{\pi'} \delta$ which is impossible due to the choice of γ and δ. Similarly for $\alpha \sim_{\rho(\Pi)} \delta$ and $\beta \sim_{\rho(\Pi)} \gamma$. Hence we may assume w.l.o.g. that $\alpha \not\sim_{\rho(\Pi)} \gamma$, $\alpha \not\sim_{\rho(\Pi)} \delta$ and $\beta \not\sim_{\rho(\Pi)} \gamma$. We get $\alpha \not\sim_{\rho(\Pi)} \beta$ from the choice of α and β and $\gamma \not\sim_{\rho(\Pi)} \delta$ from the choice of γ and δ. It is now clear that the elements α, β, γ and δ satisfy the conditions of the statement of the lemma.

If $\rho(\Pi')$ is finer than π', we apply the above described procedure to Π' (recall that $\pi \in \Pi'$). Since each time the number of equivalences contained in Π is decreased (we have chosen α and β such that $\alpha \not\sim_{\rho(\Pi)} \beta$), we either end with the quadruple α, β, γ and δ and a suitable projection Π_0 of Π or a projection Π'' of Π such that $\rho(\Pi'') = \pi'$. In the latter case, either $\pi' \in \Pi''$ or we find the desired quadruple α, β, γ and δ and a projection Π_0 of Π'' (and hence of Π) due to Lemma 9. If $\pi' \in \Pi''$, then $\pi' \in \Pi$ which is impossible due to the choice of π'. $\square$

We are now able to prove the main results:

Theorem 3 *Let Π be an edge type. There exists a Π-hypergraph with a broken feasible set iff Π is neither simply-closed, down-closed, up-closed nor up-group-closed.*

Proof: We distinguish several the following four cases:

- **Π contains both the trivial and the universal equivalence.**

  Note that if Π is down-closed, up-closed or up-group-closed, then it is simply-closed. If Π is simply closed, then any Π-hypergraph has unbroken feasible set due to Lemma 4. If Π is not simply-closed, then there is a Π-hypergraph with a broken feasible set due to Lemma 5.

- **Π contains the trivial equivalence but it does not contain the universal equivalence.**

  Π is neither simply-closed nor down-closed. Note that if Π is up-group-closed, then it is up-closed. If Π is up-closed, then any Π-hypergraph has unbroken feasible set due to Lemma 3. If Π is not up-closed, then there is a Π-hypergraph with a broken feasible set due to Lemma 7.

- **Π contains the universal equivalence but it does not contain the trivial equivalence.**

  Π is neither simply-closed nor up-closed. Note that if Π is up-group-closed, then it is down-closed. If Π is down-closed, then any Π-hypergraph has unbroken feasible set due to Lemma 2. If Π is not down-closed, then there is a Π-hypergraph with a broken feasible set due to Lemma 6.

- **Π contains neither the trivial nor the universal equivalence.**

Π is neither simply-closed nor down-closed nor up-closed. If Π is up-group-closed, then any Π hypergraph has unbroken feasible set due to Lemma 4. If Π is not up-group-closed, then there exists a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph with a broken feasible set due to Lemma 10 applied to the quadruple and a projection of Π from Lemma 11. But then, there is also a Π-hypergraph with a broken feasible set due to Proposition 2.

$\square$

Theorem 4 *Let Π_i be edge types for $1 \le i \le \lambda$. There exists a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set iff there is an edge type which is not simply-closed, an edge type which is not down-closed, an edge type which is not up-closed and an edge type which is not up-group-closed.*

Proof: Let l_i be the edge size of Π_λ. Consider the following edge type Π of the size $l_1 + \ldots + l_\lambda$: The Π consists of equivalences on the set x_{ij} for $1 \le i \le \lambda$ and $1 \le j \le l_i$; Π contains all the equivalences π such that the equivalence π_i induced by π on $x_{i1}, \ldots, x_{il}$, i.e., $x_{ij} \sim_{\pi_i} x_{ij'}$ iff $x_{ij} \sim_\pi x_{ij'}$, is contained in Π_i. Π is simply-closed iff all the edge types Π_i, $1 \le i \le \lambda$ are simply-closed. Similarly: Π is down-closed iff all the edge types Π_i, $1 \le i \le \lambda$ are down-closed. Π is up-closed iff all the edge types Π_i, $1 \le i \le \lambda$ are up-closed. Π is up-group-closed iff all the edge types Π_i, $1 \le i \le \lambda$ are up-group-closed. Hence there is a Π-hypergraph H with a broken feasible set. We turn H into $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set; each edge E of H is decomposed into λ parts of sizes $l_1, \ldots, l_\lambda$ and each of the these parts is set to be an Π_i-edge, $1 \le i \le \lambda$. $\square$

5 Conclusion

Several modification of our definition of pattern hypergraphs may be considered however they all are included in our definition:

- The edges are formed by unordered tuples. In such case, we may define a corresponding pattern hypergraph in which an edge type Π contains together with an equivalence π all the equivalences obtainable by a permutation of the elements of π.
- Cycle systems considered in [5]. In this case, an edge type Π contains together with an equivalence π all its rotations.

- The edges are ordered but the vertex may be contained in the same edge more times. In this case we may either add dummy vertices and join them by $\mathcal{C}_2$-edges to the old ones, or for each edge type Π add all the edge types Π' of smaller sizes which may be obtained by identifying two elements of the support set.

For any such modification, our main Theorem 4 may be easily reformulated.

We have made a first step for understanding why mixed hypergraphs allows such surprising results like those of [7, 10]. Let $(\Pi_1, \ldots, \Pi_\lambda)$ be a type of a pattern hypergraph. The types may be classified as proposed in the following hierarchy:

Type 0 Any $(\Pi_1, \ldots, \Pi_\lambda)$-pattern hypergraph has unbroken feasible set.

Type 1 There is a $(\Pi_1, \ldots, \Pi_\lambda)$-pattern hypergraph with a broken feasible set.

Type 2 There is a set of integers I_0 and an integer k_0 such that for any finite set of integers I with $I_0 = I \cap [1, k_0]$, there is $(\Pi_1, \ldots, \Pi_\lambda)$-pattern hypergraph H with $\mathcal{F}(H) = I$.

Type 3 There is a set of integers I_0 and an integer k_0 such that for any set of integers I with $I_0 = I \cap [1, k_0]$, there is $(\Pi_1, \ldots, \Pi_\lambda)$-pattern hypergraph H with $\mathcal{F}(H) = I$ and H has exactly one strict k-coloring for $k \in \mathcal{F}(H)$.

Type 4 There is an integer number k_0, such that for any sequence of integers $s_{k_0+1}, \ldots, s_k$, there is $(\Pi_1, \ldots, \Pi_\lambda)$-pattern hypergraph H whose spectrum is a sequence $s_1, \ldots, s_k$ for some choice of $s_1, \ldots, s_{k_0}$.

Any type of pattern hypergraph of type i is also of type i' for $1 \leq i' < i$. Hence the classification is really hierarchical. Usual graphs and hypergraphs are of type 0. The result of [7] with setting $I_0 = \emptyset$ and $k_0 = 1$ yield that mixed hypergraphs are of type 3; more precisely, even $(\mathcal{D}_3 \cap \mathcal{C}_3)$-hypergraphs are of type 3. The result of [10] yields that mixed hypergraphs are even of type 4; more precisely, even $(\mathcal{D}_2, \mathcal{D}_3, \mathcal{C}_3)$-hypergraphs are of type 4. In this paper, we have fully described the border between the type 0 and type 1 pattern hypergraphs. It remains an interesting open problem to find the full characterizations of pattern hypergraphs of type 2, type 3 and type 4. It might even be that some of the types 1, 2, 3 or 4 turn to be exactly the same. We have managed to prove that for an edge type Π containing only equivalences with at most two classes, if Π is of type 1, then it is also of type 3.

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