Coloring Face Hypergraphs on Surfaces

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Abstract

The face hypergraph of a graph G embedded in a surface has the same vertex set as G and its edges are the sets of vertices forming faces of G. A hypergraph is k-choosable if for each assignment of lists of colors of sizes k to its vertices, there is a coloring of the vertices from these lists avoiding a monochromatic edge.

We prove that the face hypergraph of a triangulation of a surface of Euler genus g is $O(\sqrt[3]{g})$-choosable. This bound matches a previously known lower bound of the order $\Omega(\sqrt[3]{g})$. If each face of the graph is incident with at least r vertices, then the face hypergraph is $O(\sqrt[r]{g})$-choosable. Separate results for small genera are presented: The bound of 3 for triangulations of surfaces of Euler genus $g = 3$ and

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the bound of $\left\lceil\frac{7+\sqrt{36g+49}}{6}\right\rceil$ for $g \geq 3$ are shown. Our results dominate the previously known bounds for all genera except for $g = 4, 7, 8, 9, 14$.

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1 Introduction

We study coloring vertices of face hypergraphs in this paper. The concept of face hypergraphs represents a way how to embed hypergraphs naturally into surfaces and this concept coincide in the plane with a previously known notion of planar hypergraphs [16]. Given an embedding of a graph $G = (V, E)$ in a surface $\mathcal{S}$, its *face hypergraph* $\mathcal{F}(G)$ is the hypergraph on the vertex set V such that $V' \subseteq V$ is an edge of $\mathcal{F}(G)$ if there is a face in G whose vertices are exactly those of V'. Face hypergraphs were introduced in [12] but several related concepts had been known before: Planar hypergraphs are studied in [3, 14, 16]; e.g., in [3, 16], it is proved that every face hypergraph of a plane graph can be colored by two colors. In case of planar hypergraphs, even special types of vertex coloring have been intensively studied, e.g., there is a series of papers on so-called planar mixed hypergraphs [5, 9, 11]. In this concept one wants to color vertices of a plane graph such that some of the faces are not monochromatic and some of the faces are not polychromatic.

We refer the reader to [7] and [13] for introduction to topological graph theory. Graphs considered in this paper are loopless and they may contain parallel edges unless stated that they are simple, i.e., without parallel edges. A *triangulation* is an embedded graph such that each its face is formed by a triangle. Triangulations are allowed to have parallel edges throughout this paper. The *degree* of a vertex is the number of edges incident with it counting edges with multiplicity. The *size of a face* is the number of vertices incident with it.

A *proper coloring* of a face hypergraph $\mathcal{F}(G)$ is a coloring of its vertices such that no edge of $\mathcal{F}(G)$ is monochromatic, i.e., G has no monochromatic face. The *chromatic number* of a face hypergraph is the least number of colors needed to properly color its vertices. We say that $\mathcal{F}(G)$ is *k-colorable* if it may be properly colored by k colors. A notion of colorings can be also generalized to a notion of list colorings as for usual graphs [10]. In *list coloring*, there is a prescribed list of available colors for each vertex. The least number k for which $\mathcal{F}(G)$ can be always properly colored from any lists of size k is called the *list chromatic number* of $\mathcal{F}(G)$. $\mathcal{F}(G)$ is said to be *k-choosable* if it might be properly colored from lists of sizes k.

We survey results on coloring face hypergraphs from [12]: All graphs considered in this paragraph has minimum face size at least three. A face hypergraph of a plane graph is 2-colorable and 3-choosable. Kündgen and Ramamurthi conjectured that all such hypergraphs are actually 2-choosable. Face hypergraphs of graphs embedded in the projective plane or on the torus are 3-choosable and this bound is tight. Each face hypergraph of a graph embedded in a surface of Euler genus $g \geq 3$ is $\lfloor \frac{9+\sqrt{1+24g}}{4} \rfloor$-choosable. They construct graphs embedded on a surface of Euler genus g whose face hypergraphs need $\Omega(\sqrt[3]{g})$ colors to be properly colored.

We find the correct order of the magnitude of the chromatic number and the list chromatic number of face hypergraphs for graphs embedded in a surface of Euler genus g: Face hypergraphs of triangulations (and hence also of graphs with minimum face size at least three) are $3\left\lceil \sqrt[3]{\frac{g-2}{2}} \right\rceil$-colorable and $\left(3\lceil \sqrt[3]{3e(g-2)} \rceil + 6\right)$-choosable (Corollary 1 and Corollary 2). Our proofs actually give more: Vertices of a graph embedded in a surface of genus g can be colored from lists of size $\Theta(\sqrt[r]{g})$ in such a way that no face of size at least r is monochromatic (for $r \geq 3$). This together with the classical result of Heawood [8] combines to: Vertices of a graph embedded in a surface of genus g with a minimum face size r can be colored from lists of size $O(\sqrt[r]{g})$ in such a way that none of its faces is monochromatic. In the case $r = 2$ and $r = 3$, the bounds were proven to be asymptotically tight in [8, 12]. In the remaining cases, there is a lower bound of $\Omega(\sqrt[r]{g})$, too: It can be easily shown that there is an embedding of an n-vertex graph in a surface of genus $2r\binom{n}{r}$ such that each r-tuple of its vertices form a face. Since more than n/r colors are needed to color its vertices in order to avoid a monochromatic face of size at least r, we have the desired lower bound $\Omega(\sqrt[r]{g})$.

Separate proofs for coloring and list-coloring of the asymptotically optimal upper bounds are included in Section 2. The proof of the bound for the ordinary coloring is simpler, provides a better upper bound and avoids using Lovász Local Lemma which is used in the list-coloring proof. In either of the two proofs, we did not try to tune the multiplicative and additive constants in the obtained bounds.

The bounds of Section 2 are good for large values of g. Still, we show that neither for small values of g the previous bound $\lfloor \frac{9+\sqrt{1+24g}}{4} \rfloor$ for triangulations is best possible. The bound of $\left\lceil\frac{7+\sqrt{36g+49}}{6}\right\rceil$ for list-coloring (Theorem 3) is proved using a careful precoloring technique with assistance of the probability method. We deal separately with the case of $g = 3$ in Section 4 and the statement that face hypergraphs of triangulations of a surface of genus 3 are 3-choosable is proven. Our bounds are smaller than those of [12] by at least one except for $g = 4, 7, 8, 9, 14$. However, even our bounds can be improved, e.g., it is possible to prove that face hypergraphs of graphs embedded in a surface of genus 23 are 6-choosable (the bound of Theorem 3 gives 7-choosability and the bound of [12] gives only 8-choosability).

2 Asymptotically Optimal Upper Bounds

As in the case of the usual coloring of graphs, one may also remove vertices of small degrees when coloring of face hypergraphs [12]:

Proposition 1 *Let G be a graph embedded in a surface $\mathcal{S}$ and v any of its vertices. Every coloring of $G-v$ can be extended to a coloring of G avoiding a monochromatic face incident with v if the set of available colors for v has size at least $\lceil(\deg_G(v)+1)/2\rceil$.*

First, we prove an asymptotically tight upper bound for colorings:

Theorem 1 *Let G be a graph embedded in a surface $\mathcal{S}$ of Euler genus $g \geq 5$ and let $r \geq 3$ be an integer. The vertices of G can be colored by at most $3\left\lceil\sqrt[r]{\frac{g-2}{2}}\right\rceil$ colors avoiding a monochromatic face of size r or more.*

Proof: For the sake of simplicity, let further $l = 3\left\lceil\sqrt[r]{\frac{g-2}{2}}\right\rceil \geq 6$ and $l_0 = 2l/3$. Suppose that the claim is false and G is a counterexample with n vertices and m faces such that its number of vertices is as small as possible. The minimum degree of G is at least $2l$ due to Proposition 1 and hence the average degree $2m/n \geq 2l$. By Euler's formula, we have $m \leq 3(n+g-2)$ which combine to $n \leq 3(g-2)/(l-3)$. Let f_r be the number of faces of G of size at least r; using Euler's formula, we have $f_r \leq 6(n+g-2)/r$ (note that it might be that $f_r > 2m/r$, e.g., G may contain isolated vertices).

Next, color the vertices of G by choosing a color for each vertex from $1, \ldots, l_0$ independently and uniformly. The probability that a face of size at least r is monochromatic is at most $1/l_0^{r-1}$ and hence the expected number

E of monochromatic faces of sizes at least r is at most f_r/l_0^{r-1}. It holds that $E \leq l_0$ which we prove later.

Choose a coloring of G with at most l_0 monochromatic faces and find a minimum set W of vertices of G such that W contains at least one vertex of each of monochromatic faces. Clearly $|W| \leq l_0 = 2(l - l_0)$. Recolor first two vertices of W with the color $l_0 + 1$, next two vertices with the color $l_0 + 2$, etc. We have finally constructed a coloring of vertices of G avoiding a monochromatic face of size r or more. This contradicts the choice of G.

We prove the missing inequality $E \leq l_0$:

$$\begin{aligned}
E \leq \frac{f_r}{l_0^{r-1}} \leq \frac{6(n+g-2)}{rl_0^{r-1}} &\leq l_0 \\
6(n+g-2) &\leq rl_0^r \\
6\left(\frac{3(g-2)}{l-3} + g - 2\right) &\leq r2^{r-1}(g-2) \\
6\left(1 + \frac{3}{l-3}\right) &\leq r2^{r-1} \\
6\left(1 + \frac{3}{6-3}\right) &\leq 12 \\
12 &\leq 12
\end{aligned}$$

■

We recall the statement of Lovász Local Lemma [1] in order to prove the next theorem:

Proposition 2 *Let $A_1, \ldots, A_n$ be events. A graph $G = (V, E)$ on a vertex set $V = \{1, \ldots, n\}$ is a dependency graph for $A_1, \ldots, A_n$ if the event A_i is mutually independent of all events A_j such that $ij \notin E$. Suppose that there exists $0 \leq x_1, \ldots, x_n < 1$ satisfying the following inequality for all $1 \leq i \leq n$:*

$$P(A_i) \leq x_i \prod_{j:ij \in E} (1 - x_j)$$

Then $P(\cap_{1 \leq i \leq n} \overline{A_i}) > 0$.

Note that it is not enough that A_i is independent from each A_j individually in the statement of Proposition 2 but it must be independent of any combination of outcomes of A_j's

We are now ready to prove an asymptotically optimal upper bound for choosability:

Theorem 2 *Let G be a graph embedded in a surface $\mathcal{S}$ of Euler genus $g \geq 3$ and let $r \geq 3$ be an integer. The vertices of G can be colored from lists of sizes at least $3\lceil \sqrt[r]{re(g-2)} \rceil + 6$ avoiding a monochromatic face of size r or more.*

Proof: For the sake of simplicity, let $l = 3\lceil \sqrt[r]{re(g-2)} \rceil + 6 \geq 12$ and $l_0 = 2l/3$. Suppose that the claim is false and G is a counterexample with n vertices and m faces such that n is the least possible. As in the previous theorem, one may again obtain that $m \leq 3(n+g-2)$ and $n \leq 3(g-2)/(l-3)$. Let f_r be the number of faces of G of size at least r; using Euler's formula, we have $f_r \leq 6(n+g-2)/r$.

Let $\Delta = \frac{4(g-2)}{l-l_0}$ and W a minimum set of vertices of G such that $G' := G - W$ has maximum degree at most Δ. Clearly, $|W| \leq \frac{m}{\Delta}$. We prove that $|W| \leq l - l_0$:

$$\begin{aligned}
\frac{m}{\Delta} \leq \frac{3(n+g-2)}{\Delta} &\leq l - l_0 \\
3(n+g-2) &\leq 4(g-2) \\
3(g-2)\left(1 + \frac{3}{l-3}\right) &\leq 4(g-2) \\
3\left(1 + \frac{3}{9}\right) &\leq 4
\end{aligned}$$

Color the vertices of W by mutually different colors from their lists and remove these colors from the lists of the remaining vertices; the sizes of the new lists are at least l_0.

Color the vertices of G' by colors from the reduced lists randomly and uniformly. We use Lovász Local Lemma (Proposition 2) to prove that at least one of the random colorings avoids monochromatic faces of size r or more. Construct a dependency graph where the events are that a face F is monochromatic where F is a face of size r or more. It is enough to join by an edge in this graph the events corresponding to faces F and F' such that F and F' are not vertex disjoint; let us write $F \sim F'$ if the corresponding vertices in the dependency graph are adjacent. The degree $d(F)$ of a vertex of the dependency graph corresponding to a face F of size

r' is at most $r'(\Delta-1)$. Set $x(F)=\frac{1}{r'(\Delta-1)+1}$ where F is a face of size r'. The probability $p(F)$ that such a face is monochromatic is at most $\frac{1}{l_0^{r'-1}}$. We only need to establish the inequality (1) for $r'\geq r$ in order to be able to use Proposition 2:

$$\begin{aligned}
p(F) &\leq x(F)\prod_{F'\sim F}(1-x(F')) \qquad (1)\\
\frac{1}{l_0^{r'-1}} &\leq \frac{1}{r'(\Delta-1)+1}\prod_{F'\sim F}(1-x(F'))\\
\frac{r'(\Delta-1)+1}{l_0^{r'-1}} &\leq \left(1-\frac{1}{r(\Delta-1)+1}\right)^{d(F)}\\
\frac{r'4(g-2)}{(l-l_0)l_0^{r'-1}} &\leq e^{-\frac{d(F)}{r(\Delta-1)}}\\
\frac{r'8(g-2)}{l_0^{r'}} &\leq e^{-\frac{r'(\Delta-1)}{r(\Delta-1)}}\\
\frac{r'8(g-2)}{2^{r'}(re(g-2))^{r'/r}} &\leq e^{-\frac{r'}{r}}\\
\frac{8r'}{2^{r'}(re)^{r'/r}} &\leq e^{-\frac{r'}{r}}\\
8r' &\leq 2^{r'}r^{r'/r}
\end{aligned}$$

Hence there exists the desired coloring of vertices of G'.

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Immediate corollaries of Theorems 1 and 2 for coloring face hypergraphs are the following:

Corollary 1 *Let G be a graph embedded in a surface $\mathcal{S}$ of Euler genus $g\geq 5$ such that its minimum face size is at least $r\geq 3$. Then, the face hypergraph $\mathcal{F}(G)$ is $3\left\lceil\sqrt[r]{\frac{g-2}{2}}\right\rceil$-colorable.*

Corollary 2 *Let G be a graph embedded in a surface $\mathcal{S}$ of Euler genus $g\geq 3$ such that its minimum face size is at least $r\geq 3$. Then, the face hypergraph $\mathcal{F}(G)$ is $\left(3\lceil\sqrt[r]{re(g-2)}\rceil+6\right)$-choosable.*

3 Triangulations of Surfaces of Small Genera

Before stating and proving the first lemma in this section, let us recall that triangulations are allowed to have parallel edges but we prohibit them to have loops.

Proposition 3 *Let G be a graph embedded in a surface $\mathcal{S}$ such that all its faces are triangles except for a face F. If F is a 2-cell face of an embedded graph such that its boundary is a walk formed by at least three different vertices (the boundary walk may contain some vertices several times), then it is possible to add some edges to the interior of the face F to get a triangulation of the surface $\mathcal{S}$.*

Proof: The proof proceeds by induction on the length of the boundary walk $v_1 \ldots v_k$. Assume $k \geq 4$. Since the walk contains at least three different vertices, it contains three consecutive different vertices, say v_1, v_2 and v_3. If v_2 is also one of the vertices $v_4, \ldots, v_k$, we add an edge v_1v_3 and apply the induction to the new face $v_1v_3 \ldots v_k$. If v_2 does not appear anywhere else in the boundary walk, we add edges v_2v_4, …, v_2v_k and we obtain a triangulation.

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Lemma 1 *A minimal (in the number of vertices) triangulation G of a surface $\mathcal{S}$ whose face hypergraph $\mathcal{F}(G)$ is not l-choosable satisfy the following:*

1. *The minimum degree of G is at least $2l$.*
2. *If G is $2l$-regular, then it is a complete graph on $2l+1$ vertices.*
3. *G has at least $2l+1$ vertices.*
4. *If G has exactly $2l+1$ vertices, then each possible triple of the vertices form a face.*

Proof: We deal with one case after another:

1. Let v be a vertex of G of degree smaller than $2l$. Remove v from G: If v is adjacent to only two different vertices, say v' and v'', proceed as follows (note that it still might be that the degree of v is greater than 2 because of parallel edges vv' or vv'' which may be in the graph): Contract all the edges vv' to a vertex v' and keep removing from each

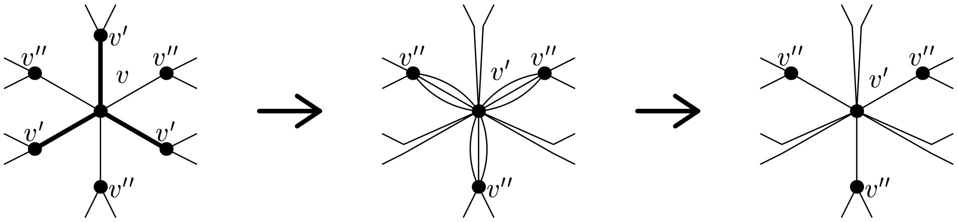

Figure 1: Reducing a vertex of degree 6 with only 2 neighbors.

newly created digon formed by two edges $v'v''$ one of the edges until there are some such digons (cf. Figure 1); the contracted edges are drawn bold in the figure. One clearly ends with a triangulation. On the other hand, if v is adjacent to at least 3 different vertices, it is always possible to subtriangulate the face of $G - v$ which contains v in G to get again a triangulation of the surface due to Proposition 3.

The obtained triangulation is l-choosable by minimality of G and hence we have a coloring of $G - v$ avoiding a monochromatic triangular face. By Proposition 1, such a coloring can be extended to G.

2. If G is $2l$-regular and it is not a complete graph on $2l + 1$ vertices, then G is $2l$-choosable [6, 15]. Hence it may be colored from the lists in such a way that there is no monochromatic face (Theorem 4.1 of [12]).

3. If G has at most $2l$ vertices, it can be colored from the lists in such a way that each color is used at most twice. Such a coloring clearly avoids a monochromatic face.

4. Let $L(v)$ denote the list of the colors available for a vertex v. If there are two vertices u and v such that $L(u) \neq L(v)$, then it is possible to color the vertices of G using each color at most twice (and such a coloring avoids a monochromatic face). If the lists of all the vertices are the same, then color the triple of vertices which do not form a face by the same color, then any other two vertices by another color, then other two vertices by another color, etc. This coloring avoids a monochromatic face.

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Lemma 2 *Let G be a triangulation and suppose that there are given lists of sizes at least l of available colors for each of the vertices. If each color appears in at most $2l$ lists, then $\mathcal{F}(G)$ can be colored from these lists.*

Proof: We use a modified Hall's matching theorem [4]: Let A_i be a system of sets such that for each index set I, $|\cup_{i\in I} A_i| \geq |I|/2$, then there exist a_i such that $a_i \in A_i$ and there are no three different indexes i_1, i_2 and i_3 such that $a_{i_1} = a_{i_2} = a_{i_3}$. The lists of colors satisfy this property and hence there exists a coloring of vertices of G such that each color is used at most twice. Such a coloring clearly avoids a monochromatic face.

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Lemma 3 *Let G be a triangulation with some vertices precolored. Let A be the set of the precolored vertices and $f(A)$ the number of faces containing two or more vertices of different colors. For arbitrary vertex $v \notin A$ such that $N(v) \cap A \neq \emptyset$ and $N(v) \not\subseteq A$, every extension of the precoloring assigning a vertex v a color different from the colors of all its neighbors has at least $f(A) + 2$ faces containing two or more vertices of different colors.*

Proof: Let d be the degree of the vertex v and $v_1, \ldots, v_d$ its neighbors in the cyclic ordering induced by the embedding. We may assume that $v_1, \ldots, v_k \in A$ and $v_{k+1}, v_d \notin A$ for some $1 \leq k < d$. After coloring the vertex v, the sought new two faces with two different precolored vertices are vv_dv_1 and vv_kv_{k+1}.

■

Lemma 4 *Let G be a triangulation of minimum degree at least l and suppose that there are given lists of available colors for each of the vertices of sizes at least l and a color c_0 contained in all but at most l lists. If G is not a complete multigraph or c_0 is not contained in all the lists, then it is possible to color some of the vertices of G in the following way:*

- *Each face containing two or more colored vertices contains at least two vertices with different colors and there are at least $2(l-1)$ such faces.*
- *No vertex is colored by the color c_0.*

- *All the vertices whose lists do not contain the color c_0 are colored.*

Proof: We find a special ordering $v_1, \ldots, v_{l+k-1}$ of some of the vertices of G for some $k \geq 1$ which has the properties 1–4 described below. Let N_i be the set of the neighbors of v_i preceding it in this ordering. This ordering should satisfy the following conditions:

1. $|N_i| \leq l-1$ for all $1 \leq i \leq l+k-1$.
2. If $|N_i| = l-1$, then the list of v_i does not contain the color c_0.
3. $|N_i| > 0$ for all $1 \leq i \leq l+k-1$ except for at most k values of i.
4. Each vertex of G whose list does not contain the color c_0 is among the vertices $v_1, \ldots, v_{l+k-1}$.

Let us first show that it is enough to find such an ordering. Assume that we have such an ordering in hand. Sequentially for i from 1 up to $l+k-1$, assign v_i a color which is contained in its list, which is different from c_0 and which is not assigned to any neighbor of v_i in N_i. The conditions 1 and 2 assure that it is possible to assign colors to all the vertices $v_1, \ldots, v_{l+k-1}$. Note that the obtained coloring is a proper coloring (in the usual sense) of the subgraph of G induced by the colored vertices. For a vertex v_i with $|N_i| > 0$, Lemma 3 can be used (keeping in mind that minimum degree of G is at least l) and hence at least $2(l-1)$ faces contain two or three vertices of $v_1, \ldots, v_{l+k-1}$ (and these vertices have different colors because the coloring is proper). The condition 4 gives that all the vertices whose lists do not contain the color c_0 are colored.

We show how to find the desired ordering in the rest. Assume first that the color c_0 is contained in all the lists. Since G is not complete, there are two non-adjacent vertices w' and w'' with a common neighbor w. Set $v_{l-2} = w$, $v_{l-1} = w'$ and $v_l = w''$. For i from $l-3$ down to 1, set v_i to be any neighbor of v_{i+1} not already contained in the sequence. We check that the constructed sequence has the desired properties (with $k = 1$):

1. $|N_i| \leq l-1$ for all $1 \leq i \leq l$ is clearly satisfied.
2. The only i for which it might be $|N_i| = l-1$ is $i = l$. But since w_{l-1} and w_l are not adjacent, then $|N_l| < l-1$.
3. $|N_i| > 0$ for all $2 \leq i \leq l$ because $v_{i-1} \in N_i$ for $2 \leq i \leq l-1$ and $v_{l-2} \in N_l$.

4. There are no vertices whose lists do not contain the color c_0.

The remaining case is that there are some vertices whose lists do not contain the color c_0. Let W' be the set of all such vertices and $l' = |W'|$. Let w_0 be any vertex of W'. If $l' = l$, set $W = W'$. If $l' < l$, then choose vertices $w_1, \ldots, w_{l-l'}$ such that w_i is a neighbor of w_{i-1} for $1 \leq i \leq l-l'$ and w_i is neither in W' nor among $w_1, \ldots, w_{i-1}$. Set $W = W' \cup \{w_1, \ldots, w_{l-l'}\}$ in this case.

Let $C_1, \ldots, C_k$ be the sets of the vertices of the components induced by the set W in G; we may assume that $w_0 \in C_1$. Note that $k \leq l'$. Choose from each C_i a single vertex w_i^C; we may also assume that $w_1^C = w_{l-l'}$. Further choose mutually different vertices u_i for $1 \leq i < k$ such that the vertices u_i and w_{i+1}^C are neighbors and $u_i \notin W$. If $k = 1$, there are no vertices u_i. Now, we construct the sought sequence of vertices v_i for $1 \leq i \leq l+k-1$: Set $v_i = u_i$ for $1 \leq i < k$ and $v_k = w_{l-l'}, v_{k+1} = w_{l-l'-1}, \ldots, v_{k+l'-l} = w_0$. As the rest of the sequence, set the vertices of $W' \setminus \{w_0\}$ in the order determined by traversing a spanning tree of each C_i started in w_i^C.

Traversing a tree is the following process: You start in a vertex of a tree and you mark this vertex. Then you move to any neighbor of this vertex and mark it. Then move to any neighbor of it, mark it and continue until you reach a leaf. Once a leaf is reached, you move to the last (by time of marking) marked vertex which has an unmarked neighbor and from it you move to any unmarked neighbor of it and mark this neighbor. And again, you continue to moving to and marking unmarked neighbors unless a leaf is reached. Once a leaf is reached, you move to the last marked vertex which has an unmarked neighbor and continue in the above fashion. The process ends when all the vertices are marked. The order of vertices is determined by the time of assigning a mark to a vertex. Note that except for the first vertex each vertex has exactly one tree-neighbor before it in the ordering.

We verify that the constructed sequence has the desired properties:

1. $|N_i| \leq l-1$
   We distinguish several cases:
   - $1 \leq i < k+l'-l$
     Since $k \leq l'$, any of the vertices v_i for $1 \leq i < k+l'-l$ is preceded by at most $k+l'-l-2 \leq 2l'-l-2 \leq l-2$ vertices.
   - Other vertices, i.e., $i \geq k+l'-l$.
     Let C_j be the component in which the vertex v_i is contained.

The vertices which may precede v_i in the sequence are only the vertices $u_1, \ldots, u_{k-1}$ and the vertices of C_j. Note that $|C_j| \leq l - k + 1$ because there are k components and their union is the set W of size l. Hence the vertex v_i has at most $(k-1) + (l - k + 1) - 1 = l - 1$ predecessors.

2. If $|N_i| = l - 1$, then the list of v_i does not contain the color c_0.
   The only vertices for which it might be that $|N_i| = l - 1$ are those of C_j's, i.e., the vertices of W'.

3. $|N_i| > 0$ for all but k vertices in the sequence.
   Any vertex except for $u_i, 1 \leq i < k$, and $w_{l-l'}$ has at least one neighbor preceding it in the sequence. The vertices w_i for $0 \leq i < l - l'$ are preceded by w_{i+1}. The vertices w_i^C for $i > 1$ are preceded by u_i and the remaining vertices are preceded because the order of the vertices in each of the components C_i was determined by traversing a spanning tree of it (see above for analysis of it).

4. Any vertex whose list does not contain the color c_0 is among the vertices $v_1, \ldots, v_{l+k-1}$.
   This follows from the choice of W'.

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Lemma 5 *Let G be a triangulation such that G is a complete multigraph on at least $l + 1 \geq 5$ vertices and suppose that there are given lists of available colors for each of the vertices whose sizes are at least l. In addition, suppose that there is a pair of vertices of G which is joined just by a single edge. Then, for a given color c_0 contained in all the lists, it is possible to color some of the vertices of G in the following way:*

- *Each face containing two or more colored vertices contains at least two vertices with different colors and there are at least $2(l-1)$ such faces.*
- *No vertex is colored by the color c_0.*

Proof: If there are two vertices whose lists are not the same, then it is possible to color l vertices of G by mutually different colors avoiding the color c_0. By Lemma 3, used for coloring one vertex after another, we have that there are at least $2(l-1)$ faces with two vertices colored by different colors and other faces have at most one colored vertex.

The remaining case is that all the lists are the same. Choose vertices v_1 and v_2 which are joined just by a single edge. Let v' and v'' be those two vertices such that v_1v_2v' and v_1v_2v'' are faces of G.

We first deal with the case that $v' \neq v''$. Choose $v_3 = v'$, $v_4 = v''$ and $v_5, \ldots, v_l$ arbitrarily. Then, color the vertices v_1 and v_2 by the same color and the remaining $l-2$ vertices by mutually different colors different from the common color of v_1 and v_2. There are at least six faces containing at least two of the vertices v_1, v_2, v_3 and v_4. Each such face contains at least two vertices of different colors due to the choice of $v_3 = v'$ and $v_4 = v''$. Next, for each of the vertices $v_i, i \geq 5$, one may use Lemma 3 setting $A = \{v_1, \ldots, v_{i-1}\}$. Hence there are at $2(l-1)$ faces containing two vertices colored with different colors and any other face contains at most one colored vertex.

If $v' = v''$, choose $v_3 = v' = v''$ and $v_4, \ldots, v_l$ arbitrarily. Then, color the vertices v_1 and v_2 by the same color and the remaining $l-2$ vertices by mutually different colors different from the common color of v_1 and v_2. There are at least four faces containing at least two of the vertices v_1, v_2 and v_3. Each such face contains at least two vertices of different colors due to the choice of $v_3 = v' = v''$. Next, for each of the vertices $v_i, i \geq 4$, one may use again Lemma 3 setting $A = \{v_1, \ldots, v_{i-1}\}$ and the claim is established in this case, too.

■

Theorem 3 *Let G be a triangulation of a surface $\mathcal{S}$ of Euler genus $g \geq 3$. Then, the face hypergraph $\mathcal{F}(G)$ is $\left\lceil \frac{7+\sqrt{36g+49}}{6} \right\rceil$-choosable.*

Proof: Fix $g \geq 3$ throughout the whole proof. Let G be a counterexample with the smallest number of vertices, n the number of its vertices, $m = 3(n+g-2)$ the number of its edges and $f = 2(n+g-2)$ the number of its faces. Let further $l = \left\lceil \frac{7+\sqrt{36g+49}}{6} \right\rceil$.

The average degree $2m/n$ of G is at least $2l$ by Lemma 1. Since $\binom{2l+1}{3} > 3((2l+1)+g-2)$, by Lemma 1 we have also $2l+2 \leq n$ and that the average degree is more than $2l$. Using $m = 3(n+g-2)$, we get from $2l < 2m/n$ that $n < \frac{3g-6}{l-3}$. We show that $n \leq 3l+1$:

$$\begin{aligned} n < \frac{3g-6}{l-3} &\leq 3l+2 \\ 0 &\leq 3l^2 - 7l - 3g \end{aligned} \qquad (2)$$

$$0 \quad \leq \quad 3\left(l-\frac{7+\sqrt{36g+49}}{6}\right)\left(l-\frac{7-\sqrt{36g+49}}{6}\right)$$

We prove having (2) and using $f = 2(n+g-2) \leq 2(3l+g-1)$ the following inequality (3) which is used later:

$$\begin{aligned}
\frac{f-2(l-1)}{(l-1)^2} &< 3 &\quad (3)\\
\frac{4l+2g}{(l-1)^2} &< 3 \\
0 &< 3l^2-10l+3-2g \\
0 &< 3l^2-7l-3g+(g+3-3l) &\quad (4)
\end{aligned}$$

The inequality (4) is satisfied because of (2) if $3l < g+3$ which holds for all g's except for $g \leq 11$ and $g = 14$; the inequality (4) can be verified by calculating its value for $g = 3, 4, 5, 7, 8, 9, 10, 11, 14$. In case that $g = 6$, we have $l = 4$, $n \leq \frac{3\cdot 6-6}{4-3} = 12$ and $f \leq 32$. Then:

$$\frac{f-2(l-1)}{(l-1)^2} \leq \frac{32-6}{9} < 3$$

Hence, the inequality (2) holds also in this case.

Recall that $2l+2 \leq n \leq 3l+1$. If each of the colors appears in the lists of at most $2l$ vertices, then $\mathcal{F}(G)$ is l-choosable by Lemma 2. Let c_0 be a most common color in the lists, i.e., a color contained in the most number of lists, and n_0 be the number of lists in which c_0 is contained. Note that $n_0 \geq 2l+1$. Let W be the set of vertices whose lists do not contain the color c_0; note that $|W| = n - n_0 \leq l$.

If $n_0 < n$ or G is not a complete graph, by Lemma 4, we may precolor without using the color c_0 some superset of vertices of W in such a way that each face either contains at most one precolored vertex or at least two vertices precolored by different colors and the number of faces which contain at least two precolored faces is at least $2(l-1)$. If G is a complete graph, such a precoloring exists by Lemma 5 because $m \leq 3(3l+g-1) < 2\binom{2l+2}{2} \leq 2\binom{n}{2}$.

Let W_0 be the precolored vertices. Note that the list of any vertex $V(G) \setminus W_0$ contains the color c_0. We remove this color from all the lists and color the vertices of $V(G) \setminus W_0$ independently and uniformly by a random color from their cut lists of size $l-1$. The faces with two or three precolored vertices are definitely not monochromatic and the probability that a face with no or one precolored vertex is monochromatic is at most $1/(l-1)^2$. Thus, the expected number of monochromatic faces is at most $\frac{f-2(l-1)}{(l-1)^2}$. By (3), there exists a coloring of vertices of G with at most two monochromatic faces which does not use the color c_0 at all. Consider such a coloring. Recolor a single vertex from each monochromatic face by c_0 (each monochromatic face contains at least two vertices with c_0 in their lists). Since there are now at most two vertices colored by the color c_0, no new monochromatic face was created and there are no monochromatic faces in the obtained coloring at all.

■

4 Triangulations of the Surface of Genus Three

We first state the following proposition restricting a way of embedding K_7 to a surface of Euler genus three:

Proposition 4 *Let K_7 be embedded in the surface of Euler genus 3. If all the faces are not 2-cells, then exactly one of the faces of is homeomorphic to a 2-cell containing a cross-cap and the remaining faces are 2-cells. Moreover, replacing the interior of the face containing the cross-cap with an open disc gives an embedding of K_7 on the torus.*

Proof: Cut a non-2-cell face. Afterwards along each boundary cycle of this face paste a disc. We obtained an embedding of K_7 in a surface with Euler genus at most two. The torus is the only surface with such a genus which admits embedding of K_7; moreover, an embedding of K_7 in the torus has to be a triangulation [13]. Hence the surgery drops the Euler genus by exactly one. Then the cut face was a disk containing the cross-cap (otherwise the genus would drop by two or more).

■

Lemma 6 *Let G be an embedding of a complete multigraph with seven vertices in a surface of Euler genus 3 such that its minimum face size is at least three. Then the face hypergraph of G is 3-choosable.*

Proof: Note that there are at most $2(7+g-2)=16$ faces in the embedding because of Euler's formula. If all the lists are the same, then we color three vertices not forming a face by the same color (there is such a triple of vertices because $\binom{7}{3} > 16$) and the remaining two pairs of vertices each with the same color. Since the only three vertices with the same color are those which do not form a face of G, this coloring is a proper coloring of the face hypergraph of G. On the other hand, if all the lists are not the same, then there exists a coloring such that each color is used to color at most two vertices by Hall's theorem. Such a coloring is also a proper coloring of the face hypergraph.

■

In the following, K_8^- denotes a multigraph with eight vertices such that all the pairs of vertices are joined by an edge except for a single pair of them.

Lemma 7 *Let G be an embedding of K_8^- in a surface of genus 3 such that its minimum face size is at least three. Then the face hypergraph of G is 3-choosable.*

Proof: If at most six of the lists are the same, then there is a coloring using each color at most twice by Hall's theorem. Hence we may assume that either all the lists are the same or seven of the lists are the same and the last one is different. In the latter case, remove the vertex whose list is different in the manner similar as in the proof of Lemma 1, i.e., subtriangulate the new face. Color the new triangulation, now. We can extend this coloring to the removed vertex by assigning it a color which is not in any other list (and hence not used to color any other vertex).

Assume that all the lists are the same. By Euler's formula, there are at most $2(8 + g - 2) = 18$ faces. We find two disjoint triples of vertices such that none of the two triples form a face of G. There is at least a single triple, say u_1, u_2 and u_3, which does not form a face of G because $\binom{8}{3} > 18$. Let v_1, v_2, v_3, v_4 and v_5 be the remaining five vertices. If all the triples of the vertices v_1, v_2, v_3, v_4 and v_5 form faces, there are at least $\binom{5}{3} = 10$ faces formed by them. There has to be among these five vertices two vertices, say v_1 and v_2, such that $u_iv_1v_2$ is not a face of G for some $i \in \{1, 2, 3\}$ because $3\binom{5}{2} > 18$. Assume, e.g., $i = 1$. If all the triples of the vertices u_2, u_3, v_3, v_4 and v_5 form faces of G, then there are at least $\binom{5}{3} = 10$ faces formed by them. These faces are completely different from the above 10 faces except for the face $v_3v_4v_5$. Hence G has at least 19 faces which is impossible. One may conclude that there exist two disjoint triples of the vertices such that the vertices of neither of the two triples form a face.

We color vertices of each of these two triples by the same color and the remaining two vertices by the remaining color. This coloring is clearly a proper coloring of the face hypergraph.

■

Next, we state several lemmas on extensions of coloring of face hypergraphs:

Lemma 8 *Consider an embedding of a complete multigraph with seven vertices in a surface of genus* 2 *with a minimum face size three. Let three of the vertices which form a face be precolored. If the precolored vertices do not form a monochromatic face, then the remaining vertices may be colored from any lists of sizes three such that this embedding has no monochromatic face.*

Proof: Under assumptions of the lemma, the embedded graph can be only a simple complete graph with seven vertices and the embedding is a triangulation of the torus. Let u_1, u_2 and u_3 be vertices precolored with colors c_1, c_2 and c_3, respectively. Let v_1, v_2, v_3 and v_4 be the remaining vertices and L_1, L_2, L_3 and L_4 their lists, respectively. Note that there are exactly $2(7+g-2)=14$ faces in the embedding.

First, assume that all the lists L_1, L_2, L_3 and L_4 are not the same; say $L_1 \neq L_4$. We find a coloring of the vertices of K_7 such that the color of any of the vertices v_1, v_2, v_3 and v_4 is used at most twice. Such a coloring avoids a monochromatic face: No monochromatic face can contain any of the vertices v_1, v_2, v_3 and v_4 because the color of each of them is used at most twice. But neither the precolored vertices themselves can form a monochromatic face due to the assumption of the lemma.

Let $c \in L_1 \setminus L_4$. If c is not used to color more than one of u_1, u_2 and u_3, then color v_1 with c; otherwise color v_1 with a color from its list which is used at most once. Afterwards, color v_2 with any color from its list which is used so far at most once (this is possible because they are only four colored vertices at the moment). Then color v_3 with any color from its list which is used so far at most once (they are only five colored vertices at the moment). Color now the vertex v_4 with any color from its list which is used so far at most once. This is possible: There are six colored vertices but at least one of them is colored with the color $c \notin L_4$.

We now focus on the remaining case $L_1 = L_2 = L_3 = L_4$. If any of the vertices u_1, u_2 and u_3 is colored with a color not contained in L_1, we proceed as in the previous paragraph, i.e., we can find a coloring using each color at most twice by a sequential coloring the vertices v_1, v_2, v_3 and v_4. If $\{c_1, c_2, c_3\} \not\subseteq L_1$, we can proceed in the same way, too. We distinguish two cases in the rest:

- All the colors c_1, c_2 and c_3 are mutually different.
  Hence $L_1 = L_2 = L_3 = \{c_1, c_2, c_3\}$. If there are indices i, j and j' such that the vertices u_i, v_j and $v_{j'}$ do not form a face, we color the vertices v_j and $v_{j'}$ by the color c_i and the remaining of the vertices of v_1, v_2, v_3 and v_4 by the colors different from c_i. The only triple of the vertices colored with the same color is u_i, v_j and $v_{j'}$ and these vertices do not form a face. Hence such a coloring avoids a monochromatic face.

  If there is no such indices i, j and j', then any u_i, v_j and $v_{j'}$ form a face. Then, K_7 has at least $3 \cdot \binom{4}{2} = 18$ faces which is impossible.

- Two of the colors c_1, c_2 and c_3 are the same.
  Assume $c_1 = c_2 \neq c_3$ and let c_4 be the third color contained in the lists $L_1 = L_2 = L_3 = L_4$.

  As in the previous case, one may argue that for any j and j' the vertices u_3, v_j and $v_{j'}$ form a face and that for any j the vertices u_1, u_2 and v_j form a face. Then K_7 has at least $\binom{4}{2} + 2 \cdot 4 = 14$ faces. But this is impossible because there is also a face $u_1 u_2 u_3$.

■

Lemma 9 *Let G be a plane multigraph with three of its vertices precolored such that its minimum face size is at least three. If the precolored vertices do not form a monochromatic face, then the remaining vertices can be colored from any lists of sizes three such that G has no monochromatic face.*

Proof: Let G be a counterexample with the smallest number of vertices and v_1, v_2 and v_3 the precolored vertices of G. An easy application of Euler's formula yields that each plane graph has at least four vertices of degree at most five. Choose a vertex w of degree at most five which is different from v_1, v_2 and v_3. The graph $G - w$ with precolored vertices v_1, v_2 and v_3 may be colored from the lists avoiding a monochromatic face different from $v_1 v_2 v_3$ and this coloring may be extended to G due to Proposition 1.

■

We are now ready to prove the main theorem of this section:

Theorem 4 *Let G be a triangulation of a surface $\mathcal{S}$ of Euler genus 3. Then, the face hypergraph $\mathcal{F}(G)$ is 3-choosable.*

Proof: Let G be a triangulation of $\mathcal{S}$ with the smallest number of vertices whose face hypergraph is not 3-choosable. The minimum degree of G is at least six due to Proposition 1 and G is not a 6-choosable graph[12]. Then G contains a copy of K_7 as a subgraph [2]. We distinguish two cases:

- The embedding of K_7 on $\mathcal{S}$ is a 2-cell embedding.
  G cannot have more than 8 vertices because of Euler's formula and thus G has to be either K_7 or K_8^-. Then, Lemma 6 or Lemma 7 states that its face hypergraph is 3-choosable.

- The embedding of K_7 on $\mathcal{S}$ is not a 2-cell embedding.
  Exactly one of the faces of K_7 is homeomorphic to a 2-cell containing a cross-cap (Proposition 4). Let G_0 be the copy of K_7, G_1 be the part of G restricted to the face with the cross-cap and G_i parts of G restricted to other faces. By Proposition 4, G_0 may be considered as an embedding of K_7 on the torus and hence G_0 is a triangulation. Embedding of G_1 to the face with the cross-cap is actually embedding of G_1 to a projective plane and hence the face hypergraph of G_1 is 3-choosable [12]. This coloring gives a precoloring of 3 vertices of K_7 which may be extended to the whole G_0 by Lemma 8. This coloring yields a precoloring of 3 vertices of any G_i, $i \geq 2$, which can be extended to the whole G_i by Lemma 9. All the colorings together form a proper coloring of the face hypergraph of G.

■

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