Folding towards the infinite
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FOLDING

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ABSTRACT. We define folding of a directed graph as a coloring (or a homomorphism) which is injective on all the *down sets* of a given depth. While in general foldings are as complicated as homomorphisms for some some classes they present an useful tool to study colorings and homomorphisms. Our main result yields for any proper minor closed class $\mathcal{K}$ a folding (of any prescribed depth) using a fixed number of colors. This in turn yields (for any $\mathcal{K}$) the existence of a K_k-free graph which bounds all K_k-free graphs belonging to $\mathcal{K}$. This has been conjectured in [9] and elsewhere and solved for $k = 3$ in [10]. Particularly, we prove (without using 4CT) the existence of a graph H which satisfies $\chi(H) \leq 5$, $\omega(H) \leq 4$ and such that any planar graph G is homomorphic to H. This is sandwiched between 4CT and 5CT for planar graphs and the general case has bearing to Hadwiger Conjecture.

1. INTRODUCTION

For a directed graph $\vec{G}, x \in V(\vec{G}), d$ a positive integer, we denote by $D^d_{\vec{G}}(x)$ the set of all those vertices y of $\vec{G}$ for which there exists a directed path of length $\leq d$ from y to x; we also call $D^d_{\vec{G}}(x)$ *d-down set* in $\vec{G}$ (we are motivated by the terminology of the theory of partially ordered sets). Thus $D^1_{\vec{G}}(x)$ is just closed in-neighbourhood of x in $\vec{G}$.

A homomorphism $f : \vec{G} \to \vec{H}$ is said to be *d-folding* if f restricted to any d-down set in G is injective. (Recall: A *homomorphism* $f : \vec{G} \to \vec{H}$ is a mapping $f : V(\vec{G}) \to V(\vec{H})$ satisfying $(f(x), f(y)) \in E(\vec{H})$ whenever $(x, y) \in E(\vec{G})$; see [9] for background and some related motivation). If $\vec{K}_k$ is an orientation of K_k, a d-folding $\vec{G} \to \vec{K}_k$ is then a d-folding which uses k colors. In this paper we study foldings of minor closed classes and we

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establish the existence of a bounded folding for any such class. This result can be stated as follows:

Let $\mathcal{K}$ be a minor closed class of graphs. We say that $\mathcal{K}$ is *proper* if $\mathcal{K}$ does not contain all graphs. Alternatively, $\mathcal{K}$ is proper if there exists at least one forbidden minor. We have the following which may be seen as the main result of this paper:

Theorem 1.1. *Let $\mathcal{K}$ be a proper minor closed class of graphs, d a positive integer. Then there exists a positive integer $k = k(d, \mathcal{K})$ and an orientation $\vec{K}_k$ of K_k, such that for any graph $G \in \mathcal{K}$ there exists an acyclic orientation $\vec{G}$ and a d-folding $f : \vec{G} \to \vec{K}_k$ (i.e. a d-folding using at most k colors).*

It is well known that the graphs belonging to a proper minor closed class $\mathcal{K}$ have a bounded density and thus a bounded chromatic number. The celebrated Hadwiger's Conjecture asserts that this bound is identical to the maximal size of the complete graph in $\mathcal{K}$. See Section 5 below for a discussion of relevance of our results to Hadwiger's Conjecture.

Our Theorem 1.1 presents a structural refinement of this bound. This extension is not straighforward and as a consequence the present bound for $k(\mathcal{K})$ is very large (double exponential function). We do not optimize at this point. Note that the minor closed property is used repeatedly in the proof and that Therem 1.1 fails to be true for bipartite 2-degenerated graphs

We discovered Theorem 1.1 in the context of graphs and their homomorphisms. We write $G \leq H$ if there exists a homomorphism $G \to H$. This quasiorder is called *homomorphism* or *coloring order*. It is well known *Grötsch's Theorem* (see e.g. [7] and [14] for the best proof) that every triangle free planar graph is 3-colorable. Using homomorphism order this means that $G \leq K_3$ for any triangle free planar graph G. In yet another way we can also say that the class $\mathcal{P}_3$ of all K_3-free graphs is bounded by K_3. The following problem has been formulated in [9] and elsewhere:

Problem 1.1. Does there exists a triangle free graph H such that $G \leq H$ for any triangle free planar graph? In other words, is the class $\mathcal{P}_3$ bounded by a triangle free graph H?

Is the class $\mathcal{P}_4$ of all K_4-free planar graphs bounded by a K_4-free graph?

Here we give an affirmative answer to this problem. In fact for $k = 3, 4$ one can prove that the classes $\mathcal{P}_k$ are bounded by a k-colorable K_k-free graph and we also prove an analogous result for any proper minor closed class $\mathcal{K}$:

Theorem 1.2. *Let $\mathcal{K}$ be a minor closed class of graphs all of which are t-colorable, let k be a positive integer. Then the class $\mathcal{K}_k$ of all K_k-free graphs in $\mathcal{K}$ is bounded by a k-colorable K_k-free graph.*

Similar result were obtained recently in [8] and in [10]. Particularly, [10] solved the above problem for $k = 3$ (triangle-free graphs). However the case of K_k-free graphs was left open and seems more difficult. Here we treat the problem in a more general context. All results may be seen as an evidence for the following general conjecture.

Let $\mathcal{A}, B$ be classes of graphs, $\mathcal{A} \subset \mathcal{B}$. We say that the class $\mathcal{A}$ is *bounded in* $\mathcal{B}$ if there exists a graph $H \in \mathcal{B}$ such that $G \leq H$ for any $G \in \mathcal{A}$. Thus $\mathcal{A}$ is bounded in $\mathcal{A}$ iff $\mathcal{A}$ has the greatest element (with respect to $\leq$). The study of boundedness phenomena is one of the basic problems and we are pleased that in our setting it relates questions like Hadwiger conjecture to the mainstream mathematics (see Remarks).

Given a finite set $\mathcal{F}$ of graphs we denote by $\mathrm{Forb}_{\mathrm{h}}(\mathcal{F})$ the class of all graphs G with no homomorphism $F \to G$ for an $F \in \mathcal{F}$. Equivalently and more formally, $\mathrm{Forb}_{\mathrm{h}}(\mathcal{F}) = \{G; F \in \mathcal{F} \Rightarrow F \not\leq G\}$. As an example, note that $\mathrm{Forb}_{\mathrm{h}}\{K_3\}$ is the class of all triangle free graphs.

Conjecture 1. *Let $\mathcal{F}$ be any finite set of connected graphs. Then for any minor closed class of graphs $\mathcal{K}$ the class $\mathcal{K} \cap Forb_{\mathrm{h}}(\mathcal{F})$ is bounded in $Forb_{\mathrm{h}}(\mathcal{F})$.*

It has been proved in [6] (see also [3] and [8] for a different proof) that for the class $\mathcal{K}_d$ of all graphs with all their vertices bounded by d and for any finite set of graphs $\mathcal{F}$ the analogous conjecture holds. The results of this paper and [10] verify the conjecture for classes $\mathrm{Forb}_{\mathrm{h}}(\mathcal{F})$ for $\mathcal{F} = \{K_k\}$. Note that for graphs in general and even degenerated classes of graphs the analogous statement fails to be true: For $\mathcal{F} = \{K_3\}$ consider graphs $K_n^{\star\star}$ formed from K_n by subdividing each edge by two new vertices. All the graphs $K_n^{\star\star}$ are 2-degenerated yet they are not bounded by a finite triangle-free graph. Note also that this cannot be saved by (large) girth: Let G be a graph of girth ℓ with chromatic number k. Then the graph $G^{\star\star}$ has girth 3ℓ and there is no homomorphism of G into a triangle free graph with at most k vertices.

This paper is organized as follows: In Section 1 we prove Theorem 1.1. In Section 2 we prove the following result which is perhaps of an independent interest:

Theorem 1.3. *Let $t \geq 3$ be a positive integer. For any proper minor closed class $\mathcal{K}$ there exists $k = k(\mathcal{K}, t)$ such that any graph $G \in \mathcal{K}$ has a proper k-coloring with the property that any set of t color classes either contains a K_t or it induces a graph with chromatic number $< t$.*

Note that, again, an analogous statement fails to be true in general even for $k = 3$: In any k-coloring of the graph $K_n^{\star\star}$, n sufficiently large, there exists odd cycles of length ≥ 6 which are colored by at most 3 colors. Also the graphs $G^{\star\star}$ (see above) have girth 3ℓ and in any k-coloring contain odd cycles colored by at most 3 colors.(More complicated examples are provided by Ramsey theory.)

The key notion for the proof of Theorem 1.3 is the above notion of d-folding and using that we prove in Section 4 Theorem 1.2 by an universal construction given in [8] (we sketch a proof for completeness). Section 5 contains concluding remarks and open problems.

2. Existence of Foldings

In this section, we shall prove that, for any fixed k, any acyclic directed graph has a k-folding using a number of colors bounded by a function of some usual parameters.

These parameters will be:

- $\Delta^{-}(\vec{G})$ the maximum indegree of $\vec{G}$,
- $\nabla(G)$ the maximum density of a minor of G:

$$\nabla(G) = \max_{H < G} \frac{|E(H)|}{|V(H)|}$$

We will need some refinement of the concept of k-folding in order to achieve our proof. Therefore, we introduce the notion of *deep coloring*, based on coloration properties of (a, b)*-paths*:

Definition 2.1. Let $a, b, a > 0, a \geq b \geq 0$ be positive integers. An (a, b)-*path* from x to y is a path of length $a + b$ which has its first a edges oriented from x to y and its remaining b edges oriented in the reverse direction, from y to x.

Definition 2.2. A coloring of the vertex set (resp. the edge set) of a directed graph $\vec{G}$ is (a, b)-*deep* if no two vertices (resp. no two edges) of an (a', b')-path of $\vec{G}$ have the same color, whenever $0 < a' \leq a$ and $0 \leq b' \leq \min(a', b)$.

Furthermore, we denote:

- $F_v(\vec{G}, a, b)$ the minimum number of colors of an (a, b)-deep vertex coloration of $\vec{G}$,
- $F_e(\vec{G}, a, b)$ the minimum number of colors of an (a, b)-deep edge coloration of $\vec{G}$.

According to this definition, a (d, d)-deep vertex coloring is almost a d-folding. The only difference between these concepts is that, for any two colors $i \neq j$, all the edges of a folding incident to a vertex of color i and a vertex of color j are oriented the same way, what may not be the case in a (d, d)-deep vertex coloring. This difference is not as important as it could seem to be, as shown by the next two lemmas:

Lemma 2.1. *Let $\vec{G}$ be a directed graph. If $\vec{G}$ has a $(2, 1)$-deep vertex coloring with k colors, then there exists an orientation $\vec{K}_N$ of K_N (with $N = k2^{k-1}$), such that $\vec{G}$ has a 1-folding to $\vec{K}_N$.*

Proof. For any $1 \leq i \neq j \leq k$, there exists no monotone path of length 2 with vertices colored i, j. Thus, for any vertex v of color i, either v has an incoming edge from a vertex colored j and no outgoing edge to a vertex colored j, or it has no incoming edge from a vertex colored j. Let $\mu_{i,j}$ be the mapping from the subset V_i of $V(\vec{G})$ with vertices colored i to $\{0, 1\}$ be defined as follows: $\mu_{i,j}(v) = 0$ if v has no incoming edge from a vertex colored j, and $\mu_{i,j}(v) = 1$ otherwise. Then, for any vertex v with color i, the k-tuple $(i, \mu_{i,1}(v), \ldots, \mu_{i,i-1}(v), \mu_{i,i+1}(v), \ldots, \mu_{i,k}(v))$ clearly defines a 1-folding to K_N. □

Lemma 2.2. *Let $\vec{G}$ be a directed graph. If $\vec{G}$ has a 1-folding to $\vec{K}_N$ and a (d, d)-deep vertex coloring with k colors, then it has a d-folding to some orientation of K_{kN}.*

Proof. Clearly, the product coloring will work. □

First, we shall notice that $(a, 0)$-deep vertex colorations and $(a, 1)$-deep edge coloration are easy to find in an acyclic directed graph with bounded indegree:

Lemma 2.3. *Let $\vec{G}$ be an acyclic directed graph with $\Delta^-(\vec{G}) \geq 2$. Then:*

$$F_v(\vec{G}, a, 0) \leq \frac{\Delta^-(\vec{G})^{a+1} - 1}{\Delta^-(\vec{G}) - 1} < 2\,\Delta^-(\vec{G})^a \tag{1}$$

$$F_e(\vec{G}, a, 1) \leq \frac{\Delta^-(\vec{G})^{a+1} - \Delta^-(\vec{G})}{\Delta^-(\vec{G}) - 1} < 2\,\Delta^-(\vec{G})^a \tag{2}$$

Proof. Denote k the maximum indegree $\Delta^-(\vec{G})$ of $\vec{G}$. Consider a topological sort $v_1, \ldots, v_n$ of the vertices. For each vertex v_i, there are at most $\frac{k^{a+1}-1}{k-1} - 1$ vertices in the graph $D^a_{\vec{G}}(v_i) - v_i$. Hence, we may choose an unused colors to color v_i. Similarly, there are at most $\frac{k^{a+1}-k}{k-1}$ edges in the graph $D^a_{\vec{G}}(v_i) - v_i$, k of them being adjacent to v_i. Thus, we may choose k unused colors to color the edges incoming v_i. The remaining inequalities are a simple consequence of $\Delta^-(\vec{G}) \geq 2$. □

When b is at least 1, an (a, b)-deep vertex coloration induces an $(a+1, b+1)$-deep edge coloration:

Lemma 2.4. *Let $\vec{G}$ be an acyclic directed graph and let $a \geq b \geq 1$. Then:*

$$F_e(\vec{G}, a+1, b+1) \leq \binom{F_v(\vec{G}, a, b)}{2} \tag{3}$$

More precisely: let $f_V : V(\vec{G}) \to X$ be an (a, b) deep vertex coloration of $\vec{G}$, with $b \geq 1$. Then, the mapping $f_E : E(\vec{G}) \to X \times X$ defined by $f_E((x, y)) = (f_V(x), f_V(y))$ is an $(a+1, b+1)$-deep edge coloration of $\vec{G}$.

Proof. Consider any $(a+1, b+1)$-path P of $\vec{G}$ and assume two edges e and f of P gets the same image by f_E. Then, the two endpoints of these edges have the same image by f_V. As the set of the endpoints of the edges of P form an (a, b)-path, all the endpoints have different images by f_V. Hence, e and f have the same endpoint and these edges form a $(1, 1)$-path. Thus, the origins of these edges have different images by f_V, what leads to a contradiction. □

One of the main properties of deep edge colorings stands in the following property:

Lemma 2.5. *Let $a \geq b \geq 1$, let $S = (c_1, \ldots, c_a, c'_b, \ldots, c'_1)$ be a sequence of $a+b$ distinct colors of an (a, b)-deep edge coloring of an acylic directed graph $\vec{G}$, and let A be a subset of the vertex set of $\vec{G}$.*

Denote $\vec{G}_{A,S}$ the partial graph of $\vec{G}$ induced by the (a, b)-paths of $\vec{G}$ colored according to S and having all their vertices but their endpoints included in A.

Then, any path of $\vec{G}_{A,S}$ linking two distinct vertices of $V(\vec{G}) \setminus A$ includes an edge colored c_a.

Proof. Consider any path P of $\vec{G}_{A,S}$ linking two distinct vertices of $V(\vec{G}) \setminus A$. As P will start with a forward edge and end with a backward one, it should include at least one sink, that is: a vertex z with two incoming edges e anf f.

Assume e is colored c_i and f is colored c_j with $j \leq i$. Let $f = (y, z)$, let $e = (x_i, z)$ and let $x_j, x_{j+1}, \dots, x_i, z$ be the part of an (a, b)-path including e, starting with the edge of color c_j and ending with e. This part is a $(i - j + 1, 0)$ path. If y is none of the x_k $(j \leq k \leq i)$, then $x_j, \dots, x_i, z, y$ is a $(i - j + 1, 1)$ path with two edges colored c_j. If $y = x_k$ for $j < k < i$ (remark that $k \neq i$, as $e \neq f$). Then, $x_j, \dots, x_k = y, z$ is a $(k-j+1, 0)$-path including two edges colored c_j. Otherwise, $y = x_j$ (thus, $j < i$ and thus $a \geq 2$). Then, as f is included in an (a, b)-path colored according to S, there exists an edge $g = (z, t)$ colored c_{j+1}. The vertex t is none of $x_{j+1}, \dots, z$, as G is acyclic. Thus, we get a $(i - j + 1, 0)$-path $x_{j+1}, \dots, z, t$ with two edges colored c_{j+1}, a contradiction.

Similarly, e and f cannot be colored c'_i and c'_j, respectively.

Assume e is colored c_i $(i < a)$ and f is colored c'_j. Then, as e belongs to an (a, b)-path colored according to S, the edge f plus the $a+b-i$ last edges of the (a, b)-path define an $(a - i + 1, b)$-path including two edges colored c'_j, a contradiction.

Thus, e has to be colored c_a, what achieves the proof. □

Lemma 2.6. *Let $\vec{G}$ be an acyclic directed graph and let $a \geq b \geq 1$.*

Let $\mu(\vec{G}, a, b)$ be the density of the graph with vertex set $V(\vec{G})$, with edges $\{x, y\}$ for each (a, b)-path with endpoints $\{x, y\}$.

Then:

$$\mu(\vec{G}, a, b) \leq \Delta^{-}(\vec{G})^{a+b} + 4\nabla(G) F_e(\vec{G}, a, b)^{a+b} \tag{4}$$

Proof. Let $k = \Delta^{-}(\vec{G})$ and let B be any subset of the vertex set of $\vec{G}$. The number of different (a, b)-paths having their sink at a vertex in B is at most $k^{a+b}|B|$.

Moreover, for any sequence $S = (c_1, \dots, c_{a'}, c'_{b'}, \dots, c'_1)$ of $a' + b'$ colors from $F_e(\vec{G}, a, b)$, we consider the partial graph $\vec{G}_{A,S}$ of $\vec{G}$ induced by the (a', b')-paths colored according to S and having all their vertices but their endpoints included in $A = V(G) \setminus B$. According to Lemma 2.5, the contraction of the edges of $\vec{G}_{A,S}$ not colored $c_{a'}$ induces no identification on B and, moreover, two endpoints in B of an (a', b')-path which interior is included in A will be joined by an edge. After simplification, the number of edges will be at most $\nabla(G)|B|$, what is an upper bound for the number of (a, b)-path including an (a', b')-path colored according to S with endpoints in B and interior vertices outside B.

Thus:

$$\begin{aligned}\mu(\vec{G},a,b) \leq & k^{a+b} + \nabla(G)\sum_{b'=1}^{b}\sum_{a'=b'}^{a} F_e(\vec{G},a,b)^{a'+b'} \\ \leq & k^{a+b} + \nabla(G)\left(\sum_{i=0}^{a} F_e(\vec{G},a,b)^i\right)\left(\sum_{i=0}^{b} F_e(\vec{G},a,b)^i\right) \\ \leq & k^{a+b} + 4\nabla(G)F_e(\vec{G},a,b)^{a+b} \qquad \text{(as } F_e(\vec{G},a,b)\geq 2\text{)}\end{aligned}$$

$\square$

Lemma 2.7. *Let $\vec{G}$ be an acyclic directed graph and let a, b, $a \geq b \geq 1$, be integers. Then:*

$$F_v(\vec{G},a,b) \leq \left(C' 2^{a-b}\nabla(G)\,\Delta^{-}(\vec{G})^{(a-b+1)(a-b+2)}\right)^{\frac{2^{a+b}\left(\frac{a+b}{2}\right)!}{\left(\frac{a-b}{2}\right)!}} \tag{5}$$

for some constant C'

Proof. According to the definition of $\mu(\vec{G},a,b)$,

$$\begin{aligned}F_v(\vec{G},a,b) \leq & 1 + 2\frac{k^{a+1}-1}{k-1} + 2\sum_{b'=1}^{b}\sum_{a'=b'}^{a}\mu(\vec{G},a',b') \\ \leq & 1 + 2\frac{k^{a+1}-1}{k-1} + 2\sum_{b'=1}^{b}\sum_{a'=b'}^{a} k^{a'+b'} + 4\nabla(G)\sum_{b'=1}^{b}\sum_{a'=b'}^{a} F_e(\vec{G},a,b)^{a'+b'} \\ \leq & 4k^a + 4k^{a+b} + 8\nabla(G)F_e(\vec{G},a,b)^{a+b}\end{aligned}$$

Thus, if $b > 1$:

$$F_v(\vec{G},a,b) \leq 12\nabla(G)F_v(\vec{G},a-1,b-1)^{2(a+b)}$$

and

$$\begin{aligned}F_v(\vec{G},a,1) \leq & 8\,\Delta^{-}(\vec{G})^{a+1} + 2^{a+4}\nabla(G)\,\Delta^{-}(\vec{G})^{a(a+1)} \\ \leq & C2^a\nabla(G)\,\Delta^{-}(\vec{G})^{a(a+1)}\end{aligned}$$

Thus, for any $b \geq 1$:

$$F_v(\vec{G},a,b) \leq \left(C' 2^{a-b}\nabla(G)\,\Delta^{-}(\vec{G})^{(a-b+1)(a-b+2)}\right)^{\frac{2^{a+b}\left(\frac{a+b}{2}\right)!}{\left(\frac{a-b}{2}\right)!}}$$

$\square$

As a consequence of all this we obtain Theorem 1.1 in the following form:

Theorem 2.8. *There exists a function $F : \mathbb{N}^3 \to \mathbb{N}$, such that, for any integer k, any acyclic directed graph $\vec{G}$ has a k-folding to some orientation of K_N, where $N = F(k, \nabla(G), \Delta^-(\vec{G}))$ is bounded by:*

$$\log F(k, \nabla(G), \Delta^-(\vec{G})) = \mathcal{O}(4^k k! \log(\nabla(G)\, \Delta^-(\vec{G})) + \mathcal{O}(\nabla(G)\, \Delta^-(\vec{G})^6)$$

Proof. This theorem follows from Lemmas 2.1, 2.2 and 2.7. □

It follows from Theorem 2.8 that, for any proper minor closed class $\mathcal{K}$ of graphs, and for any positive integer k, there exists $N(\mathcal{K}, k)$, such that any graph $G \in \mathcal{K}$ has a k-folding using $N(\mathcal{K}, k)$ colors.

Observe that even for the class $\mathcal{F}$ of the forests, $N(\mathcal{F}, k)$ may not be bounded independently of k as, for any integer N, any sufficiently large N-ary tree will contain a downset of size at least N in any of its orientations.

3. K_k-free Colorings

Theorem 3.1. *Let a directed graph $\vec{G}$ have a 3-folding to $\vec{K}_N$. Then, for any subgraph H in G which is colored by p colors, either $\chi(H) < p$ or H includes a clique of size p.*

Proof. We proceede by induction on p. Assume the statement is proved for $p - 1$. Let $\vec{G}$ be a given graph with a 3-folding into $\vec{K}_N$. Consider the homomorphic image of $\vec{G}$ as a subgraph $\vec{K}'$ of $\vec{K}_N$ and consider the subgraph $\vec{K}''$ of $\vec{K}'$ induced by any p colors. Then, we can assume without loss of generality that $\vec{K}''$ contains a monotone path with p vertices (for otherwise $\chi(K'') \leq p - 1$ and we are done). We number the colors according to this monotone path from 1 to p for convenience. Consider now any subgraph $\vec{H}$ of $\vec{G}$ with all its vertices colored by $1, \dots, p$. If the subgraph of $\vec{H}$ induced by vertices with colors $1, \dots, p-1$ has no clique of size $p - 1$, then it has chromatic number at most $p - 2$. Hence, in this case we have $\chi(\vec{H}) \leq p - 1$. Otherwise, we may assume $\vec{H}$ is p-color critical. Let $\vec{K}$ be a clique of size $p - 1$ of this subgraph. Then, as $\vec{G}$ is acyclic oriented, so is $\vec{K}$. According to the existence of a monotone path $1, 2, \dots, p - 1$ the orientation of $\vec{K}$ follows. Let u be the vertex with color $p - 1$. Thus u has an incoming edge from the vertices of $\vec{K}$ colored $1, \dots, p - 2$. As the degree of u in $\vec{H}$ is at least $p - 1$ (because $\vec{H}$ is p-color-critical), and as u may not have other incoming edges from vertices colored $1, \dots, p-2$ other than those in K (here we use $(2, 1)$-deep vertex coloring) and as the edge of $\vec{K}'$ linking $p - 1$ to

p is oriented from $p-1$ to p (this is the last edge of the monotone path), $\vec{H}$ contains an edge (u,v) with the vertex v is colored by color p. As our coloring is also a $(3,0)$-deep vertex coloring, v may not have an outgoing edge (for otherwise we would get an oriented cycle of form $i, p-1, p, i$) and, again using $(2,1)$-deep vertex coloring, the vertex v may not have incoming edges from vertices outside of $\vec{K}$. Thus, as the degree of v is at least $p-1$ (as $\vec{H}$ is p-color-critical), the vertex v is adjacent to all the vertices of the clique $\vec{K}$ and thus $\vec{K}$ together with the vertex u form a clique of size p. □

As a corollary, we obtain the following:

Corollary 3.2. *Let $\vec{G}$ have a 3-folding to $\vec{K}_N$. Then, for any K_k-free subgraph H in G geting p colors, we have $\chi(H) \leq \lfloor p\frac{k-1}{k} \rfloor$.*

Proof. Partition the p colors in $\lfloor \frac{p}{k} \rfloor$ classes of size p and one with the remaining $(p - k.\lfloor \frac{p}{k} \rfloor)$ elements (if any). Then, any of the classes of size p induces a graph of chromatic number at most $p-1$. The chromatic number of the whole graph has then the requested bound. □

With help of a deeper folding, we may improve this result a bit:

Theorem 3.3. *Let $\vec{G}$ have a $\chi(G)$-folding to $\vec{K}_N$. Then, for any subgraph H in G geting p colors, we have $\chi(H) \leq \frac{p+\omega(H)}{2}$.*

Proof. Compute a new proper coloring of H according to the acyclic orientation: each source of H is affected the color 1 and, following a topological sort order, each vertex v is colored by the minimum color which is not used by one of its incoming neighbours. According to this coloring, each vertex with color i has an incoming edge from a vertex with color j, for all $1 \leq j < i$. As this coloring is proper, at least one vertex z of H gets color $\chi(H)$. Consider the $\chi(G)$-downset D_z of z. Let n_i denote the number of vertices of D_z colored i. Then, $p \geq \sum_i n_i$, as D_z is a downset of depth at most $\chi(G)$. Moreover, as each n_i is at least 1, and as the set of the colors with $n_i = 1$ obviously forms a clique, $\sum_{i=1}^{\chi(H)} n_i \geq 2\chi(H) - |\{i, n_i = 1\}| \geq 2\chi(H) - \omega(H)$. Thus, $p \geq 2\chi(H) - \omega(H)$. □

4. K_k-free Bounds

It follows from the main result (Theorem 5) of [8] that our Theorem 3.1 implies the existence of a K_k-free bound. Note that Theorem 5 of [8] is more general as it deals with *locally U-colorable graphs* (for a fixed graph U); but presently we do not have analogy of Theorem 3.1 in this more general case.

For the sake of completeness of this paper we include a direct proof of our particular case.

Let $N, k, N > k > 1$ be positive integers. We denote by $[N]$ the set $\{1, 2, \ldots, N\}$ and by $[N]^k$ the set of all k-element subsets of $[N]$. We define the graph H_k^N as follows: The vertices $V(H_k^N)$ are all pairs (i, c) where $i \in [N]$ and c is a coloring of $[N]^k$ by $k-1$ colors (i.e. $c : [N]^k \to [k-1]$). The edges of H_k^N are pairs $\{(i, c), (i', c')\}$ where $i \neq i'$ and every set $S \in [N]^k$ which contains both i and i' gets different colors in c and c' (i.e. $c(S) \neq c'(S)$).

It is clear that for any choice of N and k H_k^N does not contain K_k (for if we would denote its vertices by $(i_1, c_1), (i_2, c_2), \ldots, (i_k, c_k)$ then the k-set $S = \{i_1, i_2, \ldots, i_k\}$ would have to get k distinct colors contrary to our assumption.) On the other hand given a K_k-free graph G with a coloring $f : G \to K_N$ as in Theorem 3.1 we can define a homomorphism $g : G \to H_k^N$ by $g(v) = (f(v), c)$ where for any k-tuple S of colors we consider a $k-1$-coloring c_S of the subgraph G_S onduced by the vertices of colors from S and we define $c(S) = c_S(v)$ (with $c(S)$ beeing arbitrary if $f(v) \notin S$). One can check easily that this mapping g is indeed a homomorphism $G \to H_k^N$.

This together with Theorem 3.1 finishes the proof of Theorem 1.2.

5. Remarks

1. The acyclic chromatic number $\chi_a(G)$ is a minimal number of colors needed to color vertices of G such that the subgraph induced by any two colors is a forest. d-folding is a structural strengthening of this notion. It has been proved in [10] that there is a function f such that any graph G with $\chi_a(G) \leq k$ has a 1-folding which uses at most $f(k)$ colors. This result is best possible as for every k there exists a graph G_k with acyclic chromatic number 3 for which any 2-folding uses at least k colors. One such example we can obtain as follows:

Denote by K the complete graph K_N of order N, with all the edges subdivided twice. Such a graph has acyclic chromatic number 3 (all the original vertices are colored 1 and the two vertices on an edge are colored 2 and 3). Consider any orientation of K and assume there exists a 2-folding with x colors. Let A be the original set of vertices K_N. If there is a sink in the subdivision vertices between two vertices a, b in A, then a and b have different colors. Otherwise, at least one of a and b gets an incoming edge from the subdivided vertices. As any vertex has at most $x-1$ incoming edges (because of the folding), there are at most $(x-1)N$ subdivided edges having a source. Hence, the number of colors in A is at least equal to the minimum chromatic number of a graph on N vertices with $((N-1)/2-(x-1)N$ edges. But, by Turan's theorem, N is at most equal to $2x(x-1)+1$.

Thus, for any x, the clique of order $2x(x-1)+1$ with edges subdivided twice has acyclic chromatic number 3 and 2-folding "number" is at least x.

2. Our Conjecture 1 may be seen as a finitary approximation to Hadwiger conjecture, see e.g. [7]. In our language Hadwiger conjecture may be expressed as follows:

Conjecture 2. *(Hadwiger)*
Any minor closed class $\mathcal{K}$ with bounded chromatic number has greatest element which is a complete graph.

In the other words, if a minor closed class $\mathcal{K}$ is bounded (by a finite graph, for example by a large complete graph) then it has the greatest element which is a complete graph. In this context one may see our Conjecture 1 as an approximation to Hadwiger conjecture: instead of asking for the greatest element of class $\mathcal{K}$ we ask for a bound with local properties similar to those in $\mathcal{K}$ (such as not containing a given complete graph). On the other hand the following naturally arises as a weaker form of Hadwiger conjecture:

Conjecture 3. *Any minor closed class $\mathcal{K}$ with bounded chromatic number has greatest element.*

In our setting the Hadwiger conjecture is not an isolated statement. For example the following problem is a weakening of it in another direction. We know that the coloring (or homomorphism) order $\leq$ is a quasiorder (which may be turned into a partial order by a suitable canonical factorization). Given a class $\mathcal{K}$ we say that a graph H is *supremum of* $\mathcal{K}$ if

i. $G \leq H$ for any $G \in \mathcal{K}$ (i.e. H is a bound for $\mathcal{K}$);

ii. If $G < H$ then there exists $G' \in \mathcal{K}$ such that $G \leq G' \leq H$ (i.e. H is the smallest bound of $\mathcal{K}$).

It follows from the *density theorem* [13] that for any positive integer k the class of all K_k-free k-colorable graphs has supremum K_k (and this class fails to contain a graph which is its greatest element). The following conjectures that this density phenomena cannot happen for geometrically restricted classes of graphs:

Conjecture 4. *If H is a supremum of proper minor closed class $\mathcal{K}$ then $H \in \mathcal{K}$ (i.e. the supremum of $\mathcal{K}$, if it exists, is also the greatest element of $\mathcal{K}$).*

3. The following is a consequence of various results of this paper and of [10]. It shows the diversity and fine interplay of very different and seemingly unrelated coloring characteristics when restricted to minor closed classes.

Remark 5.1. (Characterization of proper minor closed classes)

Let $\mathcal{C}$ be a a minor closed class of graphs. The following statements are equivalent

i. the acyclic chromatic number $\chi_a(G)$ is bounded for $G \in \mathcal{C}$;

ii. the oriented chromatic number $\vec{\chi}(G)$ is bounded for $G \in \mathcal{C}$, see [11];

iii. the star chromatic number $\chi_{st}(G)$ is bounded for $G \in \mathcal{C}$, see [10];

iv. colored mixed graphs in $\mathcal{C}$ may be colored by a fixed number of colors (in the sense of [1, 12]);

v. the chromatic number $\chi(G)$ is bounded for all $G \in \mathcal{C}$;

vi. the clique number $\omega(G)$ is bounded for all $G \in \mathcal{C}$;

vii. the edge density of all graphs G is bounded for all $G \in \mathcal{C}$.

vii. $\mathcal{C}$ is *proper* minor closed class of graphs (i.e. $\mathcal{C}$ is not the class of all graphs).

viii. For any positive integer k the class $\mathcal{K} \cap \mathrm{Forb_h}(\mathcal{K}_k)$ is bounded by a K_k-free graph;

ix. There exists $k \geq 3$ such that the class $\mathcal{K} \cap \mathrm{Forb_h}(\mathcal{F})$ is bounded by a K_k-free graph.

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