Flows and Colorings of Graphs

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Abstract

We present an overview of the theory of nowhere zero flows, in particular the duality of flows and colorings, and the extension to antiflows and strong oriented colorings. As the main result, we find the asymptotic relation between oriented and strong oriented chromatic number.

A preliminary version of this paper appears as [8]. The paper is organized as follows.

1. Introduction — contains necessary definitions
2. Overview of the theory of nowhere zero flows
3. Upper bound — we find an upper bound on $\vec{\chi}_s$ by improving the estimates of the order of a group which contains a Sidon set of a given size.
4. Tightness of the upper bound — we use the probabilistic method to find graphs with a large $\vec{\chi}_s$, $\vec{\chi}$ being fixed.

1 Introduction

The vertex set of a graph (or oriented graph) G is denoted by $V(G)$, its edge set by $E(G)$. What we call here a graph is really a multigraph, i.e. there can be several edges between two vertices, also loops are allowed. By

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edge uv (or (u,v) in the oriented case) we then mean some of the edges connecting u to v.

First we present the notion of coloring, which is of crucial importance in graph theory and also its various modifications. By *coloring* of an (unoriented) graph G we mean a mapping $c : V(G) \to S$ which satisfies

- for no edge uv of G there is $c(u) = c(v)$

Here S is an arbitrary set (of "colors"), for convenience we will often consider S to be an abelian group. However, in this definition the only important property of S is its size; the smallest size of a set S for which there is some coloring is called the *chromatic number* of G, denoted by $\chi(G)$. If G has no coloring (which happens if it has a loop, an edge (u,u)), we let $\chi(G) = \infty$.

Next we present a modification of coloring which is natural when considering oriented graphs, i.e. graphs where edges are ordered pairs. An *oriented coloring* of an oriented graph G is a mapping $c : V(G) \to S$ such that

- for no edge (u,v) of G there is $c(u) = c(v)$; and
- for no edges (u,v), (x,y) of G there is $c(u) = c(y)$ and $c(v) = c(x)$.

The *oriented chromatic number* of G (denoted by $\vec{\chi}(G)$) is the minimal $|S|$ for which the oriented coloring exists. If G has no oriented coloring (which happens if it has a loop or an oriented 2-cycle, i.e. edges (u,v) and (v,u)), we let $\vec{\chi}(G) = \infty$.

There is another approach to define coloring of oriented graphs. Let S be an abelian group. By a *strong oriented coloring* of an oriented graph G we mean a mapping $c : V(G) \to S$ such that

- for no edge (u,v) of G there is $c(u) = c(v)$; and
- for no edges (u,v), (x,y) of G there is $c(v) - c(u) = c(x) - c(y)$.

The *strong oriented chromatic number* of G (denoted by $\vec{\chi}_s(G)$) is the minimal size of a group S, for which there is a strong oriented coloring with values in S or, again, it is ∞ if there is no strong oriented coloring. This notion was first defined in [4] (where it is called just strong coloring).

To show better the relationship of these three variants of coloring, we present alternative definitions by means of homomorphism. Let G, H be two (oriented) graphs. We say that a mapping $f : V(G) \to V(H)$ is a *homomorphism* iff for every edge uv of G, graph h contains an edge $f(u)f(v)$; in the case of oriented graphs, the orientations has to be preserved as well.

It is easy to see that coloring of a graph G by k colors is the same as a homomorphism from G to K_k. Similarly, oriented coloring of G by k colors is equivalent to a homomorphism from G to some tournament with k vertices, i.e. to some orientation of K_k. Finally, strong oriented coloring of G by k colors is equivalent to a homomorphism from G to some oriented Cayley graph with k vertices; where Cayley graph is defined as follows: Let M be an abelian group and B its subset such that $B \cap -B$ is empty. Than the *oriented Cayley graph* $\mathrm{Cay}(M, B)$ has M as its vertex set, and (m, m') is its edge iff $m' - m \in B$.

The other important notion of this paper is a flow on a graph. As we will see in the next section, flows and colorings are deeply interconnected by duality of plane graphs. Let S be an abelian group, G an oriented graph. We say that $\phi : E(G) \to S$ is a *flow* iff for every vertex v of G

$$\sum_{(u,v) \in E(G)} \phi\big((u,v)\big) = \sum_{(v,u) \in E(G)} \phi\big((v,u)\big)$$

(we sum over all edges going out of u on the left hand side, over all edges going into u on the right hand side) i.e. it satisfies a condition similar to Kirchhoff's current law from elementary physics. If ϕ satisfies an additional condition

- for no edge $e \in E(G)$ we have $\phi(e) = 0$;

we call ϕ a *nowhere zero flow.* If ϕ satisfies also the third condition

- for no edges $e, f \in E(G)$ we have $\phi(e) = -\phi(f)$,

we call ϕ an *antiflow.* The smallest size of a group S for which there is a nowhere zero flow on G is called a *NZF number* of G and denoted by $\mathrm{NZF}(G)$. The smallest size of a group S for which there is a antiflow on G is called an *antiflow number* of G and denoted by $\mathrm{AF}(G)$. If G admits no nowhere zero flow (no antiflow) we define $\mathrm{NZF}(G) = \infty$ ($\mathrm{AF}(G) = \infty$).

An edge e of G is called a *bridge* iff a removal of e increases the number of components of G. Edges e, f of (an oriented graph) G form its *oriented 2-cut* iff there is a partition of $V(G)$ into two sets A, and B such that e, f are the only edges connecting a vertex of A to a vertex of B and both are oriented from A to B.

If we have an oriented plane graph G, we define its (oriented) *dual* to be $G^* = (V^*, E^*)$: V^* is the set of faces of G, and $E^* = \{e^* \mid e \in E(G)\}$, where e^* denotes an edge connecting the face to the left of e to the face to the right of e.

As an auxiliary notion we will need yet another variant of coloring, the *acyclic coloring*. It is an assignment of colors to the vertices, for which each cycle of the graph contains vertices of at least three colors. The minimal number of colors needed for an acyclic coloring of a graph G is called the *acyclic chromatic number* of G and denoted by $a(G)$.

We construct strong oriented coloring using the following notion, which comes from combinatorial number theory. Let S be a group and M be its subset. We say that M is a *Sidon subset* of S, iff the differences $m - n$ (for $m, n \in M$ and $m \neq n$) are pairwise different. (Note that equivalent definition is to ask all of the sums $m + n$ to be different.) For example the set $\{0, 1, 4, 14, 16\}$ is a Sidon subset of $\mathbb{Z}_{21}$ (by $\mathbb{Z}_n$ we denote the additive group of integers modulo n). By a theorem of [6], if m is a power of a prime, then there is a Sidon subset of $\mathbb{Z}_{m^2+m+1}$ with $m + 1$ elements. By s_k we will denote the size of the smallest group, which contains a Sidon subset with k elements.

2 Overview of the theory of nowhere zero flows

In this section we present various results about colorings and flows. We also point out which results we are going to improve in this paper. The interested reader may find more about the topic and also the proofs of the stated results in [2] or in [3].

Perhaps the most important (although rather simple) theorem about flows claims that nowhere zero flow is a dual notion to coloring. Let again S be an abelian group. For a mapping $c : V(G) \to S$ it is natural to define the mapping $\partial c : E(G) \to S$ by $\partial c((u, v)) = c(v) - c(u)$ and the mapping $(\partial c)^* : E(G^*) \to S$ by $(\partial c)^*(e^*) = \partial c(e)$. Then it holds that

c is a coloring iff $(\partial c)^*$ is a nowhere zero flow.

By the same token we have the duality for antiflows:

c is a strong oriented coloring iff $(\partial c)^*$ is an antiflow.

Now it easily follows that

$$\chi(G) = \mathrm{NZF}(G^*)\,, \qquad \vec{\chi}_s(G) = \mathrm{AF}(G^*)\,.$$

Now let's consider the question of when does a graph admit a nowhere zero flow (antiflow). Clearly if G has a bridge e, then every flow acquires value 0 at e. Hence a graph with a bridge does not admit a nowhere zero flow, neither an antiflow. Similarly, if G has an oriented 2-cut, then the edges of the 2-cut receive opposite values. Hence graphs with an oriented 2-cut have no antiflow. Quite surprisingly, these are the only obstacles, hence every graph without a bridge has a nowhere zero flow, and every graph without a bridge and an oriented 2-cut has antiflow in some group. (The proof of the first statement is standard, the latter one is shown in [4].) Note that for planar graphs these statements are equivalent to the (rather obvious) facts, that a graph has a coloring, whenever it has no loop, and has an oriented coloring, whenever it has no loop and no oriented 2-cycle.

Obviously every antiflow is also a nowhere zero flow, every strong oriented coloring is an oriented coloring, and every oriented coloring is a coloring. Hence we have the following easy inequalities for every oriented graph G

$$\mathrm{NZF}(G) \le \mathrm{AF}(G)\,, \qquad \chi(G) \le \vec{\chi}(G) \le \vec{\chi}_s(G)\,.$$

The two variants of oriented coloring are, however, more closely connected. In [7] it is shown (see Theorem 2) that

$$\vec{\chi}_s(G) \le 4\vec{\chi}(G)^2\,.$$

We will improve this result in the next section. In the last section we will find the precise relationship between these two numbers as Theorem 5.

By the Four color theorem and the duality of coloring and nowhere zero flow we get that every planar graph G has $\mathrm{NZF}(G) \le 4$. For non-planar graphs we have no duality and, indeed, the situation is rather more complicated. On contrary to the chromatic number, the NZF number is bounded, Seymour proved that for every bridgeless graph G we have $\mathrm{NZF}(G) \le 6$. The famous Tutte 5-flow conjecture claims that this can be improved to $\mathrm{NZF}(G) \le 5$.

In the case of the antiflow, the situation is even less understood. For general graphs, we have the estimate by [1]: for every graph G with no bridge and no 2-cut $\mathrm{AF}(G) \le 1\,259\,712$. In ([7]) is shown that the antiflow number of planar graphs is bounded by 672. This bound is obtained as a corollary of a more general bound

$$\vec{\chi}_s(G) \le s_{a(G)} 2^{a(G)}\,.$$

In the next section we are going to improve the bound for s_k, thus getting a better estimate for $\vec{\chi}_s$ by means of a. (Unfortunately this yields no better estimate for planar graphs.)

3 The upper bound

In [7] it is shown (using a theorem of Singer and the Bertrand's postulate):

Lemma 1 *For every $k \geq 1$,*

$$k(k-1)+1 \leq s_k \leq 4k^2 .$$

If $k-1$ is a power of a prime, then the lower bound is tight.

In this paper we improve this lemma by using a strengthening of the Bertrand's postulate, given by the following theorem (the first part of it is a well known consequence of the Prime number theorem, the second one appears in [5]).

Theorem 1 *(i) For every $\varepsilon > 0$ exists a k_0 such that for ever integer $k > k_0$ there is a prime p such that*

$$k < p < (1+\varepsilon)k .$$

(ii) For every integer $k \geq 647$ there is a prime p such that

$$k < p < 14/13k .$$

Lemma 2 *(i) $s_k = (1+o(1)) \cdot k^2$*

(ii) For every $k \geq 1$,

$$s_k \leq 64/49 \cdot k^2 \doteq 1.31 \cdot k^2 .$$

Proof: (i) Fix an $\varepsilon > 0$. We use Theorem 1 to find a k_0. For any $k > k_0$ we use the prime p between k and $(1+\varepsilon)k$ for the following estimate

$$s_k \leq s_{p+1} = (p+1)p+1$$

(here we use Lemma 1), further

$$(p+1)p+1 \leq (p+1)^2 \leq ((1+\varepsilon)k+1)^2 \leq (1+\varepsilon+\frac{1}{k_0})^2 \cdot k^2 .$$

As this can be done for any $\varepsilon > 0$ and we can choose larger k_0 if we need, the result follows.

(ii) The proof follows the same idea, but in this case we have to treat the numbers below 647 specially. In the proof we will use an increasing sequence of integers q_i, such that each $q_i - 1$ is a power of a prime (i.e. it is p^t for some prime p and some positive integer t), $q_1 = 3$ and for every i we have

$$q_{i+1} \leq 8/7 \cdot (q_i + 1)\,.$$

Before constructing the sequence let us use it to prove the statement of our lemma. Let $k \geq 1$. The cases $k < 4$ are handled separately ($\{0\}$ is a Sidon subset of $\mathbb{Z}_1$, hence $s_1 = 1$, $\{0, 1\}$ is a Sidon subset of $\mathbb{Z}_2$, hence $s_2 \leq 3$, $\{0, 1, 3\}$ is a Sidon subset of $\mathbb{Z}_7$, hence $s_3 \leq 7$), so suppose $k \geq 4$. Let q_j be the last of the q_i's less then k. Then

$$q_j < k \leq q_{j+1} \leq 8/7 \cdot (q_j + 1) \leq 8/7 \cdot k\,.$$

We denote $q = q_{j+1}$ and use Lemma 1 to find the exact value of s_q. So we obtain

$$s_k \leq s_q = q(q-1) + 1 \leq q^2 \leq (8/7)^2 \cdot k^2$$

which we wanted to prove.

It remains to find the q_i's. The first elements (for $i = 1, \ldots, 38$) are given explicitly, it is straightforward to verify, that the following numbers (there is 38 of them) have the desired property 3, 4, 5, 6, 8, 10, 12, 14, 17, 20, 24, 28, 33, 38, 44, 50, 54, 62, 72, 82, 90, 104, 114, 129, 140, 158, 180, 200, 228, 258, 294, 332, 380, 434, 492, 558, 632, 648, (note that the largest "gap" is between the numbers 8 and 6). For the next members of the sequence we just take "primes plus one", that is we let (for $i \geq 38$) $q_i = p_{i+c} + 1$, where p_n denotes the n-th prime number and c is such constant that for $i = 38$ the two definitions coincide. By Theorem 1 for $m = q_i + 1 = p_{i+c} + 2$ we have that there is a prime p such that

$$m = p_{i+c} + 2 < p < 14/13 \cdot m\,,$$

hence

$$q_{i+1} \leq p + 1 < 14/13 \cdot m + 1 = m(14/13 + 1/m) \leq 8/7 \cdot (q_i + 1)\,.$$

□

In [7] the following two estimates of strong oriented chromatic number by means of s_k are given.

Theorem 2 *Let G be an oriented graph with $\vec{\chi}(G) \le k$. Then G has a strong oriented coloring in a group with s_k elements (this coloring uses just k elements of the group), hence*

$$\vec{\chi}_s(G) \le s_k \le 4k^2 .$$

Theorem 3 *Let G be an oriented graph with $a(G) \le k$. Then G has a strong coloring in a group with $s_k \cdot 2^k$ elements, hence*

$$\vec{\chi}_s(G) \le s_k \cdot 2^k \le 4k^2 \cdot 2^k .$$

The mentioned coloring uses just $k \cdot 2^{k-1}$ elements of the group.

Using Lemma 2 to estimate s_k we have the following corollaries:

Corollary 1 *Let G be an oriented graph with $\vec{\chi}(G) \le k$. Then*

$$\vec{\chi}_s(G) \le (1 + o(1)) \cdot k^2$$

and particularly for every k

$$\vec{\chi}_s(G) \le 64/49 \cdot k^2 .$$

Corollary 2 *Let G be an oriented graph with $a(G) \le k$. Then*

$$\vec{\chi}_s(G) \le (1 + o(1)) \cdot k^2 \cdot 2^k$$

and particularly for every k

$$\vec{\chi}_s(G) \le 64/49 \cdot k^2 \cdot 2^k .$$

4 Tightness of the upper bound

In this section we will prove that the second part of Corollary 1 gives the correct asymptotic behaviour of $\vec{\chi}_s$ in terms of $\vec{\chi}$. To do so we use a simple probabilistic argument to find tournaments with large $\vec{\chi}_s$. Note that the proof also shows that for a random tournament, $\vec{\chi}_s \doteq \vec{\chi}^2$ asymptotically almost surely.

Theorem 4 *Let k, n be sufficiently large integers such that $n < k^2 - 5k \log k$. Then there is a tournament T_k with k vertices such that $\vec{\chi}_s(T_k) > n$. Consequently, we have*

$$\vec{\chi}_s(T_k) \ge (1 - o(1)) \cdot \vec{\chi}(T_k)^2 .$$

As an immediate corollary we get our main theorem, which strenghtens Corollary 1.

Theorem 5 *Let G be an oriented graph with $\vec{\chi}(G) \leq k$. Then*

$$\vec{\chi}_s(G) \leq (1 + o(1)) \cdot k^2$$

and this bound is tight.

For the proof of Theorem 4, we need a few results from group theory (see, e.g. [9]).

Theorem 6 *Let M be an abelian group of order m, let the prime factorization of m be $\prod_{i=1}^{t} p_i^{\alpha_i}$. Let*

$$M_{(p)} = \{a \in M; \textit{for some positive integer } n \textit{ we have } p^n a = 0\}.$$

Then

- *$M_{(p_i)}$ is a nontrivial group for each $i = 1, \ldots, t$.*
- *M is isomorphic to the product $\prod_{i=1}^{t} M_{(p_i)}$.*
- *$M_{(p_i)}$ is isomorphic to a product of cyclic groups $\prod_{j=1}^{t_i} \mathbb{Z}_{p_i^{\beta_j}}$, such that $\alpha_i = \sum_{j=1}^{t} \beta_j$.*

Proof of Theorem 4: Fix k, n as stated. We choose a random tournament T on k vertices: for each of the edges we randomly independently choose one of the two orientations, each with probability $1/2$. We will prove that with positive probability T has no homomorphism to a Cayley graph with at most n vertices thus showing that there is a tournament, name it T_k, for which the same holds, therefore $\vec{\chi}_s(T_k) > n$.

Let $\mathrm{Cay}(M, B)$ be an oriented Cayley graph with m vertices ($m \leq n$), without loss of generality we may suppose it is a maximal oriented Cayley graph. That is, let M be an abelian group with at most n elements and B be its subset such that for any $a \in M$ either $a + a = 0$ and a is not in B or $a + a \neq 0$ and exactly one of a, $-a$ is an element of B. Further let $f : V(T) \to M$ be an injective mapping. If for some $u, v \in V(T)$, $f(u)$ and $f(v)$ are not connected by an edge of $\mathrm{Cay}(M, B)$ in either direction, then surely f is not a homomorphism. Otherwise, the probability that f is a homomorphism is $2^{-\binom{k}{2}}$ as each edge has probability $1/2$ to be in the "right" direction. The number of possible mappings f is $k!\binom{|M|}{k} < n^k$. The number of possible sets B is at most $2^{m/2} \leq 2^{n/2}$ (it is exactly $2^{(m-z)/2}$, where z is the number of $a \in M$ such that $a + a = 0$).

Finally, the number of abelian groups with at most n elements is at most n^2: let $m \leq n$ be an integer with prime factorization $\prod_{i=1}^{t} p_i^{\alpha_i}$, Let N_m be the number of abelian groups of order m. By Theorem 6

$$\begin{aligned} N_m &= \prod_{i=1}^{t} \Big(\text{number of groups } \textstyle\prod_{j=1}^{t_i} \mathbb{Z}_{p_i^{\beta_j}} \text{ of order } p_i^{\alpha_i}\Big) \\ &= \prod_{i=1}^{t} \Big(\text{number of tuples of integers } t_i, (\beta_j) \text{ s.t. } \textstyle\sum_{j=1}^{t_i} \beta_j = \alpha_i\Big) \end{aligned}$$

If we distinguish the tuples of b_j's which differ only in the order, the i-th factor in the last expression is equal to $2^{\alpha_i - 1}$ (this well-known fact may be proved by a bijection to the subsets of $\{1, \dots, \alpha_i - 1\}$, which maps (β_j) to $\{\sum_{j=1}^{s} \beta_j; 1 \leq s < t_i\}$). As we do not distinguish such tuples, this factor is possibly smaller, but certainly less than $2^{\alpha_i} \leq p_i^{\alpha_i}$. Hence $N_m \leq \prod_{i=1}^{t} p_i^{\alpha_i} = m$. So the number of abelian groups of order at most n is the sum of integers upto n, which is bounded by n^2.

Hence the probability that T has a homomorphism to a Cayley graph of size at most n is bounded by

$$2^{-\binom{k}{2}} n^k 2^{n/2} n^2 .$$

This expression is $o(1)$, as the following routine calculation shows. Take logarithm with base 2. We get

$$\begin{aligned} -\frac{k^2}{2} + \frac{k}{2} + k \log n + \frac{n}{2} + 2 \log n &< -\frac{k^2}{2} + \frac{k}{2} + k \log k^2 + \frac{k^2}{2} - \frac{5k \log k}{2} + 2 \log k^2 \\ &\leq -\frac{1}{2} k \log k + O(k) , \end{aligned}$$

which tends to $-\infty$. Hence for k large enough the probability is less then 1, and we are done. □

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