Geometric Separation and Exact Solutions for the Parameterized Independent Set Problem on Disk Graphs

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Abstract

We consider the parameterized problem, whether for a given set $\mathcal{D}$ of n disks (of bounded radius ratio) in the Euclidean plane there exists a set of k non-intersecting disks. We expose an algorithm running in time $n^{O(\sqrt{k})}$, that is—to our knowledge—the first algorithm for this problem with running time bounded by an exponential with a sublinear exponent. For λ-precision disk graphs of bounded radius ratio, we show that the problem is fixed parameter tractable with respect to parameter k.

The results are based on a new "geometric $\sqrt{\cdot}$-separator theorem" which holds for all disk graphs of bounded radius ratio. The presented algorithm then performs, in a first step, a "geometric problem kernelization" and, in a second step, uses divide-and-conquer based on our geometric separator theorem.

Our techniques can be extended to various other graph problems, such as DOMINATING SET, to obtain similar results for disk graphs of bounded radius ratio.

1 Introduction

The problem and its motivation. In this paper, we study the parameterized INDEPENDENT SET problem on disk graphs, which takes as an input a set $\mathcal{D}$ of disks in the plane and an integer k and the task is to determine whether there are k mutually disjoint disks in $\mathcal{D}$. The problem is motivated by numerous applications, among which we want to highlight the area of frequency assignment problems in cellular networks [22]. Here, one considers a set of antennas which transmit data on a given frequency to their local environment. Assuming that this environment can be modeled by a disk centered at the position of the antenna, the task to determine the maximum number of antennas which can operate simultaneously without any conflict using the same frequency becomes a maximum INDEPENDENT SET problem on a disk graph.[1]

Previous work. It is known [7] that the problem is NP-hard even for unit disk graphs. A way to cope with this hardness was proposed by approximation theory [15, 17]. Very recently, Erlebach *et. al.* [15] gave a PTAS for INDEPENDENT SET on disk graphs, which is based on a sophisticated use of so-called shifting techniques as they were introduced in [4, 16]. Note that the running time of the given PTAS is far from being practical, since the degree of the polynomial running time for obtaining an approximation ratio $(1 + \frac{1}{\ell-1})^2$ grows as $\ell^2 n^{O(\ell^4)}$.

In this paper, however, we are interested in *exact* solutions for the given problem. Moreover, we take the viewpoint of parameterized complexity (see [12]). Formally, a parameterized problem is a two-dimensional language $\mathcal{L} \subseteq \Sigma^* \times \mathbb{N}$, where Σ is some alphabet. The input instances are objects of the form (I, k) and the integer k is called the *parameter*. We ask for an algorithm which decides if $(I, k) \in \mathcal{L}$ running in time $2^{g(n,k)}$, where $n = |I|$. A parameterized problem is called *fixed parameter tractable*, if there is an algorithm with $g(n, k) = f(k) + h(n)$, where f is some arbitrary function and $h(n) \in O(\log n)$. The class of fixed parameterized tractable problems is denoted by FPT.

We want to briefly summarize various results on the parameterized and classical complexity of the INDEPENDENT SET problem for general graphs and for planar graphs (the latter are equivalent to the class of coin

[1] Note that we always consider disk graphs with given representation, i.e., with given set of disks in the plane. This makes sense, since most applications which are modeled by disk graphs already provide this representation in a very natural way.

graph class	(classical) complexity	parameterized complexity
general graphs	$2^{0.276\,n}$ [25] rel. lower bound: $2^{\Omega(n)}$ [18]	W[1]-complete [11]
disk graphs DG_σ	$2^{O(\sqrt{n}\log(n))}$ [Rem. 17]	*open problem* $2^{O(\sqrt{k}\log(n))}$ [Thm. 16]
disk graphs $DG_{\sigma,\lambda}$ (with λ-precision)	$2^{O(\sqrt{n})}$ [Rem. 13]	FPT [Cor. 7] $2^{O(\sqrt{k}\log(k))+\log(n)}$ [Cor. 7]
planar graphs	$2^{O(\sqrt{n})}$ [21][Rem. 11] rel. lower bound: $2^{\Omega(\sqrt{n})}$ [6]	FPT $2^{O(\sqrt{k})+\log(n)}$ [2, 3]

Table 1: Relating our results on INDEPENDENT SET for disk graphs to known results for general graphs and for planar graphs.

graphs [19], i.e., disk graphs where disks are not allowed to overlap); see Table 1 for an overview.

In parameterized complexity study, it is known [11] that INDEPENDENT SET is complete for the class $W[1]$, which captures parameterized *intractable* problems (see [12] for details). However, restricted to planar graphs, INDEPENDENT SET is in FPT, and for the (asymptotically) best known algorithm we get $h(n) = \log(n)$ and $f(k) = O(\sqrt{k})$ being sublinear in k (see [2, 3]). Moreover, very recently, Cai and Juedes [6] showed that there is no fixed parameter algorithm with $f(k) = o(\sqrt{k})$ unless 3SAT$\in DTIME(2^{o(n)})$, which is generally considered to be very unlikely.

In the classical (one-dimensional) complexity study, the best known algorithm running in time $2^{e(n)}$ with $e(n) = 0.275n$ is due to Robson. Moreover, $e(n) \in o(n)$ is impossible unless 3SAT$\in DTIME(2^{o(n)})$ (see [18]). If restricted to planar graphs, Lipton and Tarjan applied their well-known planar separator theorem [20] to get an algorithm with $e(n) = O(\sqrt{n})$. This is the best possible asymptotic behavior for e (unless 3SAT$\in DTIME(2^{o(n)})$), since otherwise an algorithm with $e(n) \in o(\sqrt{n})$ in combination with a known linear problem kernel would lead to an algorithm for the parameterized problem better than the relative lower bound shown by Cai and Juedes.

Main results and methods used. It is an interesting phenomenon that, for planar graphs, in both cases (parameterized and classical complexity) the asymptotically best algorithms were derived based on separator theorems: in the case of classical complexity, by a direct divide-and-conquer approach using the planar separator theorem [21], and in the case of parameterized complexity, by a combination of so-called "reduction to a linear problem kernel" (see Section 3 for details) and divide-and-conquer [3]. Due to the large constants involved, the algorithms are considered to be impractical, but relevant from a theoretical point of view, since they match the corresponding relative lower bounds.

In this paper, for the case of disk graphs, we pursue a similar strategy of combining a geometric version of reduction to problem kernel with a divide-and-conquer approach based on an appropriate separator theorem. However, for disk graphs, so far such separator theorems are known only for so-called intersection graphs of τ-neighborhood systems [14, 23, 24], which are closely related to (unit) disk graphs with λ-precision, where all centers are at mutual distance of at least $\lambda > 0$. With respect to general disk graphs, we quote from the introduction of Hunt *et. al.* [17]:

> *"The [...] drawback is that problems such as maximum independent set and minimum dominating set [...]* **cannot** *be solved at all by the separator approach. This is because an arbitrary (unit) disk graph of n nodes can have a clique of size n."*

The key result in this paper is to show a way out of this dilemma by proving a new type of "geometric separator theorem" which holds for disk graphs with bounded radius ratio. Our geometric separator theorem can be seen as a generalization of (classical) separator theorems, where the guarantee is not on the size of the separator in terms of its number of vertices, but in terms of the space occupied by its disks.

This result is used to optimally solve the parameterized INDEPENDENT SET problem on disk graphs of bounded radius ratio in time $2^{g(n,k)}$ with $g(n,k) = O(\sqrt{k}\log(n))$, which is — to our knowledge — the first algorithm for this problem with running time bounded by a function with an exponent sublinear in k. In the worst case (i.e., when $k = n$) this turns into an algorithm of running time $2^{e(n)}$ with the sublinear term $e(n) = \sqrt{n}\log(n)$; a running time which cannot be achieved for general graphs (unless 3SAT $\in DTIME(2^o(n))$).

In addition, in the case of disk graphs with λ-precision, we can show that the INDEPENDENT SET problem is in FPT with $g(n,k) = f(k) + \log(n)$ and $f(k) = O(\sqrt{k}\log(k))$. The results are summarized in Table 1.

2 Preliminaries and Notation

Our subject to explore are intersection graphs of geometric objects in the plane, namely of disks. We assume that each point z of the plane is determined by its x any y coordinates and the plane is equipped by the standard distance $d(z, z') = \sqrt{(x - x')^2 + (y - y')^2}$.

If $\mathcal{S} = \{S_1, \ldots, S_n\}$, $S_i \subseteq \mathbb{R}^2$ is a collection of geometric objects, we denote by $\bigcup \mathcal{S} = \bigcup_{i=1}^{n} S_i$ the union of $\mathcal{S}$. For a collection $\mathcal{S}$, let $G_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$ denote the *intersection graph* of $\mathcal{S}$, i.e., $V_{\mathcal{S}} = \{v_1, \ldots, v_n\}$ and $E_{\mathcal{S}} = \{(v_i, v_j) \mid S_i \cap S_j \neq \emptyset\}$. The collection $\mathcal{S}$ is called the *representation* of $G_{\mathcal{S}}$. Moreover, for a subset $\mathcal{S}' \subseteq \mathcal{S}$, we denote by $V_{\mathcal{S}'} \subseteq V_{\mathcal{S}}$ the subset of vertices induced by $\mathcal{S}'$, i.e., $V_{\mathcal{S}'} = \{v_i \mid S_i \in \mathcal{S}'\}$. In this setting $G_{\mathcal{S}'} = G_{\mathcal{S}}[V_{\mathcal{S}'}]$ is the subgraph of $G_{\mathcal{S}}$ induced by the set of vertices $V_{\mathcal{S}'}$.

Disk graphs. A disk $D \subseteq \mathbb{R}^2$ is specified by a triple $(r, x, y) \in \mathbb{R}^3$, where (x, y) are coordinates of the center of the disk in the Euclidean plane and r is its radius. The graph class of *disk graphs*, denoted by DG is the set of all graphs G, for which we find a collection of disks $\mathcal{D} = \{D_1, \ldots, D_n\}$ such that $G = G_{\mathcal{D}}$. Note that for given collection $\mathcal{D}$, the graph $G_{\mathcal{D}}$ is given with a natural embedding in the plane, where v_i sits in the position of the center of D_i.

The class of disk graphs of *bounded radius ratio* σ is the subclass $DG_\sigma \subset DG$ of all graphs $G \in DG$ which admit a representation $\mathcal{D} = \{D_1, \ldots, D_n\}$, such that $(\max_{i=1,\ldots n} r_i)/(\min_{i=1,\ldots n} r_i) \leq \sigma$, where r_i denotes the radius of disk D_i. The parameter σ is called *radius ratio*. By a rescaling argument, for a graph $G \in DG_\sigma$ with representation $\mathcal{D}$, we can always achieve, that the smallest disk in $\mathcal{D}$ has radius one and, hence, all radii being upper bounded by σ.

Finally, a collection $\mathcal{D}$ is said to have λ-precision if all centers of disks are pairwise at least λ apart (see [17, Definition 3.2]). Again, by a rescaling argument, all disk graphs have a representation with λ-precision, however only some graphs of DG_σ, allow a representation with radii in $[1, \sigma]$ and precision λ. We denote this class of graphs by $DG_{\sigma,\lambda}$.

Throughout the paper, we assume that a disk graph G is given together with its representation witnessing its membership in DG, DG_σ or $DG_{\sigma,\lambda}$, respectively. This makes sense from an application point of view, since the graph is usually derived from the placement or real objects in the space, and it is natural that these objects have bounded size as well as distinguishable placement.

Grid graphs. Fix an arbitrary constant $\delta > 0$, and consider the infinite grid of span δ as the planar graph $H^\delta = (W^\delta, E^\delta)$ with vertices $W^\delta = \{w_{i,j} \mid i,j \in \mathbb{Z}\}$ and edges $E^\delta = \{(w_{i,j}, w_{k,l}) \mid |i-j| + |k-l| = 1\}$. The canonical (straight-line) embedding of H^δ is given by putting vertex $w_{i,j}$ at the coordinates $(i\delta, j\delta)$. The set of faces $\mathcal{F}^\delta$ of H^δ contains all closed squares $F^\delta_{i,j} = [i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta] \subseteq \mathbb{R}^2$. For a grid vertex $w \in W^\delta$, we define the *face neighborhood* $\hat{N}(w) := \{F \in \mathcal{F}^\delta \mid w \in F\}$. Similarly for $W' \subseteq W^\delta$, $\hat{N}(W') = \bigcup_{w \in W'} \hat{N}(w)$.

Definition 1 For a collection of disks $\mathcal{D} = \{D_1, \ldots, D_n\}$, we define $H^\delta_\mathcal{D}$ to be the smallest subgraph of the infinite grid H^δ induced by a set of grid points which completely covers all disks in $\mathcal{D}$. We call H^δ the *covering grid (of span δ) for $\mathcal{D}$.* In other words, if we define the set of faces hit by $\mathcal{D}$ as $\mathcal{F}^\delta_\mathcal{D} = \{F \in \mathcal{F}^\delta \mid F \cap \bigcup \mathcal{D} \neq \emptyset\}$, and if $W^\delta_\mathcal{D}$ is the set of all grid points of $\mathcal{F}^\delta_\mathcal{D}$, then the covering grid $H^\delta_\mathcal{D}$ is the subgraph $H^\delta_\mathcal{D}$ which is induced by $W^\delta_\mathcal{D}$.

An example which illustrates the construction of $H^\delta_\mathcal{D}$ is given in Fig. 4 in the Appendix.

Finally, for a collection $\mathcal{D}$ of disks and a set $S \subseteq \mathbb{R}^2$ (e.g., a set of grid vertices or a set of faces), we call $\mathcal{D}[S] := \{D \in \mathcal{D} \mid D \cap S \neq \emptyset\}$ the set of disks induced by S.

Measures. We use the standard Lebesgue measure μ in $\mathbb{R}^2$ as follows: For a Lebesgue measurable set $S \subseteq \mathbb{R}^2$, $\mu(S)$ denotes the *size* of S, i.e. the space in $\mathbb{R}^2$ occupied by S. In particular, for a collection of disks $\mathcal{D} = \{D_1, \ldots, D_n\}$ let $\mu(\mathcal{D}) = \mu(\bigcup \mathcal{D})$ be the space covered by the union of disks $D_1, \ldots, D_n$.

Definition 2 Let $\mathbb{G}$ be a graph class closed on taking subgraphs. A function $\xi : \mathbb{G} \to \mathbb{R}^+$, that is monotonous with respect to the subgraph ordering, i.e., $\xi(G) \leq \xi(G')$ if $G \subseteq G'$, and for which $\xi((\emptyset, \emptyset)) = 0$, is called a *graph measure.*

Example 3 We specify two graph measures which play a decisive role throughout the paper.

1. The usual *counting measure* $|\cdot|$ which assigns to any graph G the size of its vertex set $|V_G|$ clearly is a graph measure.
2. The *Lebesgue measure* $\mu(\cdot)$ which assigns to a disk graph $G_\mathcal{D}$ with representation $\mathcal{D}$ the value $\mu(G_\mathcal{D}) = \mu(\mathcal{D})$ is a graph measure for DG, when we restrict the subgraph ordering to $G_\mathcal{D} \subseteq G_{\mathcal{D}'} \Leftrightarrow \mathcal{D} \subseteq \mathcal{D}'$.

3 A Geometric Problem Kernelization

A well known tool for the design of fixed parameter algorithms is *reduction to problem kernel*.

Definition 4 Let $\mathcal{L}$ be a parameterized problem, i.e., $\mathcal{L}$ consists of pairs (I, k), where problem instance I has a solution of size k (the parameter). *Reduction to problem kernel*, then, means to replace problem (I, k) by a "reduced" problem (I', k') (called *problem kernel*) such that

$$k' \leq c \cdot k, \quad |I'| \leq p(k), \quad \text{and} \quad (I, k) \in \mathcal{L} \text{ iff } (I', k') \in \mathcal{L}, \tag{1}$$

where c is a constant,[2] and the function p, called the *size of the problem kernel*, depends only on k. Furthermore, we require that the reduction from (I, k) to (I', k') is computable in time $T_K(|I|, k)$, which is a polynomial.

It is well-known that a parameterized problem is fixed parameter tractable if and only if it admits a reduction to a problem kernel (see [13]).

As an example, we mention that, due to the 4-color theorem it is easy to derive a problem kernel of size $4k$ for INDEPENDENT SET on planar graphs. For disk graphs, we can prove a geometric version of a problem kernel. By this, we mean that the size of the reduced instance is upper bounded by $O(k)$, when measured by the (Lebesgue) measure $\mu(\cdot)$ instead of the counting measure $|\cdot|$ (see Example 3).

Proposition 5 *For the parameterized* INDEPENDENT SET *(IS) problem on* DG_σ *there exists a "geometric" problem kernel, i.e., there is a procedure (which is computable in linear time), that transforms an instance* $(G_\mathcal{D}, k)$ *to an instance* $(G_{\mathcal{D}'}, k)$*, such that* $(G_\mathcal{D}, k) \in IS$ *iff* $(G_{\mathcal{D}'}, k) \in IS$ *and*

$$\pi k \;\leq\; \mu(G_{\mathcal{D}'}) \;\leq\; 9\pi\sigma^2 k.$$

Proof. Recall that for a given instance $(G_\mathcal{D}, k)$ with the representation $\mathcal{D}$—due to our assumptions—all disks have radius in the range $[1, \sigma]$ and $G_\mathcal{D}$ is naturally embedded in the plane with respect to $\mathcal{D}$.

[2] Usually, $c \leq 1$. In general, however, it would even be allowed that $k' = g(k)$ for some function g. For our purposes, however, we need that k and k' are linearly related. We are not aware of a concrete, natural parameterized problem with problem kernel where this is not the case.

```
kernelize(disk graph G_D, integer k)
// Answers either that G_D has an independent set of size k,
// or that no such set exists, or returns "geometric problem kernel" (G_D', k).

  1. scale D such that the smallest disk has unit radius.
     // Since G_D ∈ DG_σ, the largest radius of a disk in D is σ.

  2. set F_• = F^• = ∅, δ = 1/20
     // to be compatible with Theorem 14, however any fixed δ > 0 would suffice

  3. for each D ∈ D do
     F_•  :=  F_• ∪ {F ∈ F^δ | F ⊆ D}
     F^•  :=  F^• ∪ {F ∈ F^δ | F ∩ D ≠ ∅}

  4. if |F^•|δ^2 > 9πσ^2 k then return "YES"
     // i.e., find greedily a set of k independent disks D_I and return (G_{D_I}, k),
     respectively.
     else if |F_•|δ^2 < πk then return "NO"
     // i.e., return (∅, k).
     else return (G_D, k))
```

Figure 1: Geometric problem kernel reduction.

Observe first, that $\mu(G_{\mathcal{D}}) > 9\pi\sigma^2 k$ implies that $(G_{\mathcal{D}}, k) \in \mathit{IS}$. We use the fact that $\mu(\mathcal{D}[N(v)]) \leq (3\sigma)^2\pi$ for any vertex $v \in V$, i.e., that the neighborhood of any vertex may occupy the space at most $9\pi\sigma^2$.

And, secondly, if $\mu(G_{\mathcal{D}}) < \pi k$, then $(G_{\mathcal{D}}, k) \notin \mathit{IS}$, since the representation of any independent set of k vertices needs space at least πk. □

The procedure which in linear time transforms $(G_{\mathcal{D}}, k)$ to $(G_{\mathcal{D}'}, k)$ with $\mu(G_{\mathcal{D}'}) \leq c\sigma^2 k$ is given in Fig. 1. (Observe that $c \searrow 9\pi$ as $\delta \to 0$.)

Note that this is not a problem kernel according to Definition 4, since the size of $G_{\mathcal{D}}$ is measured by the (Lebesgue) measure $\mu(\cdot)$, which, in general, is not related to the (input) size of G. For disk graphs with λ-precision, however, we can upper bound the counting measure by the Lebesgue measure.

Lemma 6 *Let $G_{\mathcal{D}} = (V, E) \in DG_{\sigma,\lambda}$ be a graph with and representation $\mathcal{D}$. Then,*

$$|V| \leq \frac{4}{\pi}\lambda^{-2}\mu(G_{\mathcal{D}})$$

Proof. The proof is deferred to the Appendix.

Corollary 7 *The parameterized* INDEPENDENT SET *problem on disk graphs $DG_{\sigma,\lambda}$ (with λ-precision) admits a problem kernel of size ck with constant $c = 36(\frac{\sigma}{\lambda})^2$, which can be computed in linear time,*

Moreover, the problem can be solved in time $O(k^{O(k)} + n)$, hence, it is fixed parameter tractable.

Proof. The claimed problem kernel reduction follows immediately by Proposition 5 and Lemma 6.

In order to solve the problem, we perform the problem kernel reduction in time $O(n)$ and then, on the reduced instance of size ck, we may check all $\binom{ck}{c} = O(k^k)$ subsets of k disks for their independence. This results in the claimed total running time. □

4 A Geometric Separator Theorem

In the following, we prove our key result—a new geometric $\sqrt{\cdot}$-separator theorem—that makes our divide-and-conquer strategy work. For this purpose, in a first subsection, we briefly introduce the notion of vertex separators and give a short overview on known $\sqrt{\cdot}$-separator theorems for planar graphs and unit disk graphs with λ-precision. In a second subsection, we prove our key result, a geometric $\sqrt{\cdot}$-separator theorem for general disk graphs of bounded radius ratio.

4.1 Classical $\sqrt{\cdot}$-separator theorems

We start with a somewhat generalized notion of separator theorems.

Definition 8 Let $G = (V, E)$ be an undirected graph. A *separator* $V_S \subseteq V$ of G partitions V into two *parts* V_A and V_B such that

- $V_A \dot\cup V_S \dot\cup V_B = V$, and
- no edge joins a vertex of V_A to V_B.

The triple (V_A, V_S, V_B) is also called a *separation* of G.

In order to provide a quantitative approach to separators, we need the notion of "measure" as introduced in Section 2.

Definition 9 Let ξ be a graph measure. An *$f(\cdot)$-separator theorem for the measure ξ (and constants $\alpha < 1$, $\beta > 0$)* on a class of graphs $\mathbb{G}$ which is closed under taking subgraphs is a theorem of the following form:
For any $G \in \mathbb{G}$ there exists a separation (V_A, V_S, V_B) of G such that

1. $\xi(G[V_S]) \leq \beta\, f(\xi(G))$
2. $\xi(G[V_A]), \xi(G[V_B]) \leq \alpha\, \xi(G)$

$\sqrt{\cdot}$-separator theorems on planar graphs. Stated in this framework, the planar separator theorem due to Lipton and Tarjan [20] can be formulated as follows.

Theorem 10 *On the class of planar graphs, there exists a $\sqrt{\cdot}$-separator theorem for the counting measure $|\cdot|$ with constants $\alpha = 2/3$ and $\beta = 2\sqrt{2}$. Moreover, the corresponding separation can be found in linear time.*

Later, Djidjev [8] improved the constants to $\alpha = 2/3$ and $\beta = \sqrt{6}$. The current record for $\alpha = 2/3$ is $\beta \approx 1.97$ [10]. Djidjev has also shown a lower bound of $\beta \approx 1.55$ for $\alpha = 2/3$ [8]. Similar $\sqrt{\cdot}$-separator theorems are also known for other graph classes, e.g., for the class of graphs of bounded genus, see [9].

Remark 11 Note that Theorem 10 can directly be used to obtain a $2^{O(\sqrt{n})}$-algorithm for INDEPENDENT SET on planar graphs (see [21], or [3, conference version, Prop.1]).

$\sqrt{\cdot}$-separator theorems on disk graphs with λ-precision. In terms of geometric graphs, a $\sqrt{\cdot}$-separator theorem for the counting measure was proven on the class of intersection graphs of so-called τ-neighborhood systems (see [17]). Here, a τ-neighborhood system is a collection $\mathcal{B} = \{B_1, \ldots, B_n\}$ of balls in a space of arbitrary fixed dimension, such that the intersection of any $(\tau + 1)$ distinct balls in $\mathcal{B}$ is empty. It can be verified that every unit disk graph with λ-precision is an intersection graph of a τ-neighborhood system (τ depending on λ) and, vice versa, every intersection graph of a τ-neighborhood system in $\mathbb{R}^2$ is λ-precision disk graph (λ being the minimum distance between the centers of any two disks), see [17]. In the two-dimensional case the corresponding separator theorem reads as follows (see [23, Theorem 2.5] and [14, Theorem 5.1]):

Theorem 12 *On the class of intersection graphs of τ-neighborhood systems, there exists a $\sqrt{\cdot}$-separator theorem for the measure $|\cdot|$ with constants $\alpha = 3/4$ and $\beta = O(\sqrt{\lambda})$. Moreover, the corresponding separation can be found in linear time.*

Remark 13 As exhibited, e.g., in [3, conference version, Section 4.1], a divide-and-conquer approach yields that INDEPENDENT SET on intersection graphs of τ-neighborhood systems, and hence, on unit disk graphs with λ-precision as well as for graphs from $DG_{\sigma,\lambda}$, can be solved in time $2^{O(\sqrt{n})}$.

4.2 A new geometric $\sqrt{\cdot}$-separator theorem

In this subsection, we prove an analogue to the classical $\sqrt{\cdot}$-separator theorems for the class DG_σ of disk graphs with bounded radius ratio. Note that graphs in DG_σ may contain arbitrary large cliques, which means that a $\sqrt{\cdot}$-separator theorem does not hold for the counting measure $|\cdot|$. However, if we use the (Lebesgue) measure $\mu(\cdot)$ (see Example 3), we obtain the following geometric $\sqrt{\cdot}$-separator theorem. Recall the notion from Section 2.

Theorem 14 *On the class DG_σ of disk graphs with bounded radius ratio, there exists a $\sqrt{\cdot}$-separator theorem for the Lebesgue graph measure $\mu(\cdot)$.*

More precisely there exist constants $\alpha < 1$ and β such that, for every graph $G_{\mathcal{D}} \in DG_\sigma$ with representation $\mathcal{D}$, we find three sets $\mathcal{D}_A, \mathcal{D}_S, \mathcal{D}_B \subseteq \mathcal{D}$, such that $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ is a separation for $G_{\mathcal{D}}$, satisfying

1. $\mu(\mathcal{D}_S) \leq \sigma^2 \beta \sqrt{\mu(\mathcal{D})}$,
2. $\mu(\mathcal{D}_A), \mu(\mathcal{D}_B) \leq \alpha \mu(\mathcal{D})$.

Moreover, this separation can be found in time linear in $|\mathcal{D}|$.

The idea for the proof of Theorem 14 is to construct the covering grid $H^\delta_{\mathcal{D}}$ (of suitable span) for a given collection $\mathcal{D}$ of disks (see Definition 2). Then, in a second step, one applies a planar $\sqrt{\cdot}$-separator theorem on $H^\delta_{\mathcal{D}}$ from which, in a suitable manner, the three sets $\mathcal{D}_A, \mathcal{D}_S, \mathcal{D}_B \subseteq \mathcal{D}$ will be constructed. In order to prove the Theorem we need the following key result which interrelates the space covered by $\mathcal{D}$ with the size of the covering grid.

Proposition 15 *For any ε there exists a δ such that for any set $\mathcal{D}$ of disks of radius at least one:*

$$|W_{\mathcal{D}}| \leq \frac{1+\varepsilon}{\delta^2} \mu(\mathcal{D}).$$

```
geometric_separator(disk graph G_D)
// Returns sets D_S, D_A and D_B corresponding to a separation (V_{D_A}, V_{D_S}, V_{D_B})
// of G_D with respect to the measure μ(G_D).

  1. scale D such that the smallest disk has unit radius.
  2. fix arbitrary ε < 1/2 and select δ according to Proposition 15
     // E.g. ε = 1/4, δ = 1/20
     and construct the graph H^δ_D
  3. run the algorithm of Lipton and Tarjan (see Theorem 10) on the
     covering grid H^δ_D for D to obtain a separation (W_A, W_S, W_B)
     with
      a)  |W_S| ≤ β' sqrt(|W^δ_D|), and
      b)  |W_A|, |W_B| ≤ α'|W^δ_D|,
     for the constants β' = sqrt(8) and α' = 2/3.
  4. return the three sets
            D_S  :=  D[N̂(W_S)],   // N̂(W_S) is defined in Section 2.
            D_A  :=  D[W_A] \ D_S,
            D_B  :=  D[W_B] \ D_S.
```

Figure 2: Separator algorithm corresponding to Theorem 14.

Proof. The proof is deferred to the Appendix. □

Proof. (of Theorem 14) The sets $\mathcal{D}_S$, $\mathcal{D}_A$ and $\mathcal{D}_B$ will be determined according to the procedure given in Fig. 2. We now prove that, indeed $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ is a separation of G and that properties (1.) and (2.) of the theorem hold for the computed sets $\mathcal{D}_S, \mathcal{D}_A$, and $\mathcal{D}_B$:

$(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ *is a separation of* G*:* Showing that $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ is a separation of $G_\mathcal{D}$ is equivalent to proving that $\bigcup \mathcal{D}_A \cap \bigcup \mathcal{D}_B = \emptyset$. Recall that (W_A, W_S, W_B) is the separation of the covering grid $H^\delta_\mathcal{D}$ obtained by the algorithm of Lipton and Tarjan. First of all we claim that

$$\bigcup \mathcal{D}_A \cap W^\delta_\mathcal{D} \subseteq W_A, \quad \text{and} \quad \bigcup \mathcal{D}_B \cap W^\delta_\mathcal{D} \subseteq W_B. \tag{2}$$

To see this, note that $\bigcup \mathcal{D}_A \cap W_S = \emptyset$, since if there is a disk $D \in \mathcal{D}_A$ containing a point $w \in W_S$—by construction of $\mathcal{D}_S$—we had $D \in \mathcal{D}_S$. Suppose now that there is a vertex $w_B \in W_B$ which lies in a disk $D \in \mathcal{D}_A$. By definition of the set $\mathcal{D}_A$, we find a vertex $w_A \in W_A$ inside D as well. Then, there exists a path P in $H^\delta_\mathcal{D}$ which connects w_A and w_B and is completely placed inside the disk D. Since (W_A, W_S, W_B) is a separation of $H^\delta_{\mathcal{D}}$ then there exists a vertex of W_S on P contradicting $\bigcup \mathcal{D}_A \cap W_S = \emptyset$. This also implies that $\bigcup \mathcal{D}_A \cap W_B = \emptyset$. Since $W^\delta_{\mathcal{D}} = W_A \cup W_S \cup W_B$, we get $\bigcup \mathcal{D}_A \cap W^\delta_{\mathcal{D}} \subseteq W_A$. The property $\bigcup \mathcal{D}_B \cap W^\delta_{\mathcal{D}} \subseteq W_B$ follows similarly.

Assume now, for contradiction that vertices $v_A \in V_{\mathcal{D}_A}$ and $v_B \in V_{\mathcal{D}_B}$ form an edge of $E_{\mathcal{D}}$, that means there exists some point z in $D_A \cap D_B$. Let $F_z \in \mathcal{F}^\delta$ be any of the (at most four) squares containing z, and let W_z be the four grid vertices adjacent to F_z in $H^\delta_{\mathcal{D}}$.

We first consider the case when D_B does not intersect W_z. Then it intersects one side of F_z, and since $\delta \ll 1$ it contains two grid points of the square sharing this side (see the last case depicted in Fig. 5 in the Appendix). Note that we use equation (2) in the sense that all grid points intersected by D_B belong to W_B. Then as D_A must contain at least one point of W_z (if not, either there is no possible place for z or $W_A \cap W_B \neq \emptyset$).

In this case, or when we symmetrically exchange subscripts A and B, and also when both D_A and D_B intersect W_z, there are two grid points $w_A \in D_A$, $w_B \in D_B$, that are in $H^\delta_{\mathcal{D}}$ at distance at most two. (All essential constellations are schematically depicted in Fig. 5.) Since w_A and w_B must be separated by W_S, and at the same time they belong to $\hat{N}(W_S)$, that is in contradiction with the definition of sets $\mathcal{D}_A$ and $\mathcal{D}_B$.

ad property (1.): Consider a vertex $w \in W_S$ and the neighborhood $\hat{N}(w)$. As shown in Fig. 6 in the Appendix, all disks in $\mathcal{D}$ which intersect $\hat{N}(w)$ must lie inside a cycle of radius $(2\sigma + \sqrt{2}\delta)$ centered at w. This is clear, since disks in $\mathcal{D}$ have radius bounded by σ and since the grid has span δ. More formally, we get

$$\mu(\mathcal{D}[\hat{N}(w)]) \leq (2\sigma + \sqrt{2}\delta)^2\pi.$$

Moreover, this implies that

$$\begin{aligned} \mu(\mathcal{D}_S) &= \mu(\mathcal{D}[\hat{N}(W_S)]) = \mu\Big(\bigcup_{w \in W_S} \mathcal{D}[\hat{N}(w)]\Big) \\ &\leq \sum_{w \in W_S} \mu(\mathcal{D}[\hat{N}(w)]) \leq \Big((2\sigma + \sqrt{2}\delta)^2\pi\Big)\, |W_S| \leq 5\sigma^2\pi|W_S| \qquad (3) \end{aligned}$$

Since, by our choice, $\varepsilon < \frac{1}{2}$ we may use in the last step $\sqrt{2}\delta < \frac{1}{6} \leq \frac{\sigma}{6}$. Using property (a) of step (3.) of the algorithm in Fig. 2 and Proposition 15, we have $|W_S| \leq \beta' \sqrt{|W_{\mathcal{D}}|} \leq \beta' \sqrt{\frac{1+\varepsilon}{\delta^2}\mu(\mathcal{D})}$ which together with the estimate (3) establishes

$$\mu(\mathcal{D}_S) \le \sigma^2 \beta \sqrt{\mu(\mathcal{D})} \quad \text{for} \quad \beta = \frac{5\pi\beta'}{\delta}\sqrt{(1+\varepsilon)}. \tag{4}$$

ad property (2.): First of all, observe that the set $\bigcup \mathcal{D}_A$ is completely covered by the square faces of the subgraph $H^\delta_{\mathcal{D}}[W_A]$, induced by the vertices of W_A. To see this, suppose there is a point $z \in D$ (for some $D \in \mathcal{D}_A$) which lies in some square $F_z \in \mathcal{F}^\delta$ of $H^\delta_{\mathcal{D}}$ but not of $H^\delta_{\mathcal{D}}[W_A]$. As above, if the four vertices adjacent to F_z host a vertex of W_B or W_S, we get $D \cap \hat{N}(W_S) \neq \emptyset$, a contradiction.

By this observation and by the fact that $|W_A| \le \alpha'|W_{\mathcal{D}}|$, we get

$$\mu(\mathcal{D}_A) \;\le\; \mu(H^\delta_{\mathcal{D}}[W_A]) \;\le\; \delta^2|W_A| \;\le\; \delta^2\alpha'|W^\delta_{\mathcal{D}}| \;\le\; \alpha'(1+\varepsilon)\,\mu(\mathcal{D}),$$

where Proposition 15 was used in the last step.

Similarly, one proves $\mu(\mathcal{D}_B) \;\le\; \alpha'(1+\varepsilon)\,\mu(\mathcal{D})$. □

We note here that by our choice of $\varepsilon = \frac{1}{4}$, we get $\alpha = \alpha'(1+\varepsilon) = \frac{5}{6}$; but α can be arbitrarily close to α' by a sufficiently small choice of ε. However then we have to consider the tradeoff of getting a small β according to Equation (4), on the one hand, and enlarging $|W^\delta_{\mathcal{D}}|$, on the other hand.

5 The Algorithm and its Analysis

We use the geometric kernelization of Section 3 and a divide-and-conquer approach based on the new geometric separator theorem from Section 4.2 to derive an algorithm for the INDEPENDENT SET problem on disk graphs of bounded radius ratio.

Theorem 16 *Let $\mathcal{D}$ be a collection of n disks with $G_{\mathcal{D}} \in DG_\sigma$. Then, there is an algorithm running in time $n^{O(\sqrt{k})}$ which decides if $G_{\mathcal{D}}$ admits an independent set of size at least k, and if the answer is "YES" it constructs one.*

Proof. On input instance $(G_{\mathcal{D}}, k)$, in a first step, perform the geometric kernelization explained in Section 3. After this step, without loss of generality, we may assume that $\mu(\mathcal{D}) \le ck$ for the constant c given in Proposition 5.

In a second step, the divide-and-conquer procedure `indep_set` shown in Fig. 3 is applied to the instance $(\mathcal{D}, ck)$.

```
disks indep_set(disks 𝒟, space s)
// Returns an optimal independent set of G_𝒟,
// where s gives an upper bound on the space μ(𝒟) occupied by 𝒟.
  • if (𝒟 = ∅) then return indep_set(disks 𝒟, space s)=∅; else
    • compute 𝒟_S, 𝒟_A, and 𝒟_B
      // According to the geometric separator theorem (Theorem 14).
    • for all independent sets V_IS of G_{𝒟_S} do
      • construct the sets
        𝒟'_A := 𝒟_A \ {D ∈ 𝒟_A | ∃D' ∈ 𝒟[V_IS] : D ∩ D' ≠ ∅}
        𝒟'_B := 𝒟_B \ {D ∈ 𝒟_B | ∃D' ∈ 𝒟[V_IS] : D ∩ D' ≠ ∅}
      • compute result_{V_IS} := V_IS ∪ indep_set(𝒟'_A,αs) ∪
        indep_set(𝒟'_B,αs)
    • return result_{V_IS_0}, for the independent set V_IS_0 with
      | result_{V_IS_0} | = min{| result_{V_IS} | where V_IS is independent set of G_{𝒟_S}}
```

Figure 3: Divide-and-conquer algorithm for INDEPENDENT SET for disk graphs of bounded radius ratio based on the new geometric separator theorem.

Denote by $T(n,s)$ the time needed to execute `indep_set(`$\mathcal{D}$`,`s`)` on a collection of n disks $\mathcal{D}$ with $\mu(\mathcal{D}) \leq s$. Let $p(|\mathcal{D}|)$ be the polynomial time needed to compute the sets $\mathcal{D}_S, \mathcal{D}_A$, and $\mathcal{D}_B$ according to Theorem 14, and $q(|\mathcal{D}|)$ be the polynomial time needed to perform constructions of $\mathcal{D}'_A$ and $\mathcal{D}'_B$. Note that in $\mathcal{D}_S$ at most $\lfloor \frac{\beta\sqrt{s}}{\pi} \rfloor$ many disks can be independent, since $\mu(\mathcal{D}_S) \leq \beta\sqrt{s}$ and every disk has radius at least one. Hence, the total number of independent sets in $G_{\mathcal{D}_S}$ is upper bounded by

$$\sum_{i=0}^{\lfloor \frac{\beta\sqrt{s}}{\pi} \rfloor} \binom{n}{i} \;\leq\; n^{\widehat{\beta}\sqrt{s}},$$

where $\widehat{\beta}$ is some constant. Then, the recursion we have to solve in order to compute an upper bound on $T(n,s)$ reads as follows:

$$T(n,s) \leq p(n) \;+\; n^{\widehat{\beta}\sqrt{s}} \cdot q(n) \cdot 2\,T(n, \alpha s).$$

Hence, for n large enough, and a suitable constant $\widetilde{\beta}$ we have

$$\begin{aligned} T(n,s) \quad &\leq \quad n^{\widetilde{\beta}\sqrt{s}} \cdot T(n,\alpha s) \quad \leq \quad \prod_{i=0}^{\log_{\frac{1}{\alpha}}(s)} n^{\widetilde{\beta}\sqrt{\alpha^i s}} \cdot T(n,1) \\ &\leq \quad n^{\widetilde{\beta}\sqrt{s}(\sum_{i=0}^{\infty}(\sqrt{\alpha})^i)} \cdot T(n,1) \quad = \quad n^{\frac{\widetilde{\beta}\sqrt{s}}{1-\sqrt{\alpha}}} \cdot T(n,1). \end{aligned}$$

Note that $T(n,1)$ is constant, since $\mu(\mathcal{D}) \leq 1$ implies $\mathcal{D} = \emptyset$, because for every disk D we have $\mu(D) \geq \pi$. By plugging in the values $n = |\mathcal{D}|$ and $s = ck$, we obtain the running time as we have claimed. $\square$

Remark 17 Note that in the worst case we have $k = n$, which means that in the sense of classical complexity theory we have an algorithm running in time $O(2^{\sqrt{n}\log(n)})$. As already mentioned in the introduction, such a running time with a sublinear exponent cannot be achieved for general graphs (unless 3SAT$\in DTIME(2^{o(n)})$).

Remark 18 We want to mention that the methods which yield the $n^{O(\sqrt{k})}$ time algorithm can be carried over to all problems which, on the one hand, admit a geometric problem kernel and which, on the other hand, can be solved by a divide-and-conquer approach based on a separator theorem. This latter property was characterized in [3] using the notion of "weak glueability." However, things get more involved here and, due to the lack of space, we want to refer to [3] and the long version for details. As an example, we mention the weakly glueable DOMINATING SET problem, for which a (linear) geometric problem kernel can be proven similarly to Proposition 5.

6 Conclusion and Open Problems

In this paper, we prove a geometric separator theorem which in some sense extends the idea of common separator theorems for planar graphs and (unit) disk graphs of λ-precision to arbitrary disk graphs of bounded radius ratio.

The geometric separator theorem together with a geometric problem kernelization is applied to design an algorithm for the INDEPENDENT SET problem for disk graphs of bounded radius ratio. The running time of this algorithm is $n^{O(\sqrt{k})}$.

In terms of the methods used in this paper, it is an interesting question, whether a similar geometric separation theorem also holds for for disk graphs with arbitrary radius ratio or for intersection graphs of other geometric objects.

As to parameterized complexity, we leave it as an open problem, whether INDEPENDENT SET and DOMINATING SET, respectively, on disk graphs (of bounded radius ratio) are in FPT or complete for the classes $W[1]$ and $W[2]$, respectively. We want to emphasize, that we are not aware of a (non-artificial) $W[1]$-complete problem which allows for an algorithm of running time $n^{o(k)}$, i.e., with running time bounded by an exponential with a sublinear exponent.

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Appendix

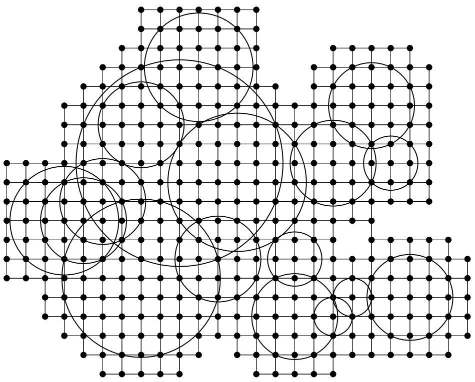

Figure 4: According to Definition 1: Constructing the covering grid $H_{\mathcal{D}}^{\delta}$ for a collection of disks $\mathcal{D}$.

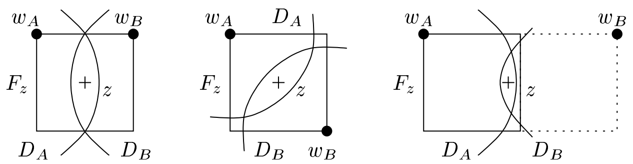

Figure 5: For the proof of Theorem 14: The intersection of disks selects two vertices w_A and w_B from the grid which are at distance at most two.

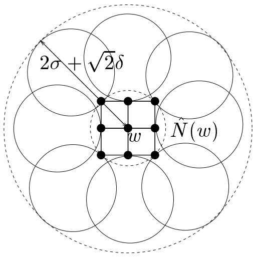

Figure 6: For the proof of Theorem 14: $\mu(\mathcal{D}_S) \leq (2\sigma + \sqrt{2}\delta)^2\pi|W_S|$.

Proof of Lemma 6:
Observe that, for given set of n disks $\mathcal{D}$ with radius at least one and centers of mutual distance at least λ, the smallest value $\mu(\mathcal{D})$ is obtained by optimally placing all centers in the interior of a disk. However, such a disk must have radius at least $\frac{\lambda\sqrt{n}}{2}$. Since every disk has diameter at least two, we have

$$\mu(\mathcal{D}) \geq \left(\frac{\lambda\sqrt{n}}{2} + 2\right)^2 \pi \geq \frac{\pi}{4}\lambda^2 n.$$

□

Proof of Proposition 15:
We use the theorem of Bern and Sahai [5] stating that if any set of disks is shifted in the plane by a continuous motion, such that the center-to-center distance does not increase at any time, then the total area of the union of disks is also non-increasing.

In particular, if we multiply all radii of disks in $\mathcal{D}$ by an arbitrary factor $\eta \geq 1$, the total area of the new set is at most $\eta^2\mu(\mathcal{D})$. To see this, first multiply the coordinates of centers as well as the radii by η. Consider this transformation of disks as a continuous mapping ϕ. These disks now cover the space $\eta^2\mu(\mathcal{D})$. We shift all disk centers to the original position by the continuous mapping ϕ^{-1}. Applying the Bern and Sahai's theorem to ϕ^{-1} yields the claim.[3]

Fix arbitrary positive $\delta \leq \frac{1}{3}(\sqrt{2(1+\varepsilon)} - \sqrt{2})$ and consider the set $W_{\mathcal{D}}$. Each point w of $W_{\mathcal{D}}$, could be represented as a unique square of side length δ with w placed at the center. (These squares correspond to the grid squares

[3] We thank Jiří Matoušek for pointing us to the result of Bern and Sahai.

shifted by $\frac{\delta}{2}$ in both coordinates.) All these new squares could be covered by the original disks if we enlarge all radii by the factor $(1 + \frac{3\sqrt{2}}{2}\delta)$. Then, due to the choice of δ,

$$|W_{\mathcal{D}}| \leq \left(1 + \frac{3\sqrt{2}}{2}\delta\right)^2 \frac{\mu(\mathcal{D})}{\delta^2} \leq \frac{1+\varepsilon}{\delta^2}\mu(\mathcal{D}).$$

$\square$