The Compactness Lower Bound of Shortest-path Interval Routing on $n \times n$ Tori with Random Faulty Links

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Abstract. In this paper we consider that a communication network is given by a 2-dimensional square torus with independently faulty links that is modelled by a random graph. We prove the lower bound on compactness of shortest-path interval routing for a random 2-dimensional torus of the order N in the form $\Omega(\sqrt{\log N})$ with probability of at least $1 - o(1)$. This is the first non-constant compactness lower bound for random tori. The trivial upper bound is $O(N)$.

Keywords: interval routing scheme, torus, grid, random graph

1 Introduction

In recent years, massive exploition of information technologies based on communication networks and multiprocessor architectures has caused the area of distributed (computing) systems to became of great interest by many researchers. One of the primary objects of study is the problem of reliable function of distributed systems. Many unreliable and fault models of distributed systems were examined: there are various types of faults in combination with several different models of their distribution. Many of them are discussed e.g. in [1], [4], [5], [7], [13], [15] and [23]. In this paper we adopt the model of interconnection networks with crash-fault communication links satisfying probabilistic distribution. This model is more realistic in practice than a so called *worst-case* one in which a bounded number of

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adverse distributed faults occur. The model used in our paper was formally described in [15] by a generalization of the standard Erdös-Rényi random graph model. More precisely, if H be an undirected graph then a random graph of type-H is obtained by selecting edges of H independently and with probability p. This way a communication network H, in which the link fails independently and with the probability $f = 1 - p$, can be represented.

In this paper, we focus on the shortest-path interval routing on a random graph of type-H, where H is a torus $n \times n$. Interval routing is a popular space-efficient routing method for point-to-point communication networks. It was introduced in [22] and generalized in [24]. Interval routing is attractive because it needs to keep only a small amount of information in each node to route massages correctly through the network. That is why it has been used in industrial applications in the INMOS T9000 Transputer design such as the C 104 Router Chip [14]. It was also exploited in fault-tolerant parallel networks [25].

Interval routing is based on a suitable labeling scheme for the vertices and edges in a given graph. A vertex label is an integer and arc label consists of at most k consecutive cyclic intervals. If the routing strategy guarantees that the messages always arrive at their destination and always via the shortest path, then the scheme is called shortest path k-interval routing scheme and denoted as k-IRS. To measure the space efficiency of the given k-IRS, we use the *compactness* measure defined as the smallest k such that a given graph supports a shortest path k-IRS.

The compactness of many graph classes has been studied intensively during recent years. In particular, its value is 1 for complete graphs, rings [22], trees [22], hypercubes [3], and d-dimensional grids [3], and it is 2 for tori [24] (see also surveys in [9], [21] and [23], Chapter 4). The problem of compactness lower bounds for IRS and initial proving techniques were opened in [19] and [20]. Matching upper and lower bounds on the compactness of general graphs of the order N have been proved in [11] in the following form:

$$N/4 - o(N) < \mathit{compactness}(G) < N/4 + o(N) \ .$$

Also, other types of networks require more than a constant number of intervals for shortest path k-IRS; namely they are shuffle-exchanges, De Bruijn graphs, cube connected cycles, butterflies, star graphs, cf. [12], [21]. Proving related nontrivial upper bounds for these graphs is still open. The lower bound proving technique, which is used in our paper, was developed in [12]. The interval routing on random graphs (in a sense of the standard Erdös-Rényi model) was considered in [8] and [10]. In these papers, the random

graph can be viewed as a model of "average-case" network topology and such an approach avoids the use of proving techniques based on the probability theory. The following results were shown:
(1) There exist classes of random graphs $\mathcal{G}_{N,p}$ for all N sufficiently large such that with high probability a shortest-path IRS for a graph $G \in \mathcal{G}_{N,p}$ requires the compactness $\Omega(N^{1-\varepsilon})$, $0 < \varepsilon < 1$, if $p = N^{-1+1/s}$ for an integer $s > 0$, [8].
(2) Almost all graphs support a shortest-path interval routing with at most 3 intervals per outgoing edge [10]. More precisely, there were distinguished three models of node labelling in $\mathcal{G}_{N,p}$ [10]: adversary (A), random (R) and designer (D) and accordingly three models of interval routing were defined, namely IRS_A, IRS_R and IRS. It was shown, in particular, that for a fraction at least $1-o(1)$ of all graphs $G \in \mathcal{G}_{N,p}$ it holds: $1 \leq \mathrm{IRS}(G) \leq \mathrm{IRS}_R(G) = 2$ and $\mathrm{IRS}_A(G) \leq 3$. The question of whether $\mathrm{IRS}(G) = 1$ or $\mathrm{IRS}(G) = 2$ remains open.
According to [6] and [15], the probability space $\mathcal{G}_{N,p}$ is defined as a complete graph K_N with randomly deleted edges. Note that it is a random graph of type-K_N and it can be also written as $\mathcal{G}(H,p)$, where $H = K_N$.

As mentioned above, interval routing is popular for its good space-efficient properties for several types of graphs. The main disadvantage of this routing strategy arises in its poor robustness. In order to preserve a good compactness measurement of a network, it may be necessary to recompute arc interval labels completely, including the assignment of new labels to each vertex, in the case of removing/adding only one communication link in the network. (In contrast, universal routing strategy based on Tajibnapis' Netchange algorithm does not require the rearrangement of vertex labels in such a case, cf. [23], Chapter 4.) According to weak fault-tolerant properties of interval routing, it is of a great interest to measure the compactness also for other random graph of type-H, where H represents a communication network with randomly failed/overloaded links. In this paper we focus on the probability space $\mathcal{G}(T_{n\times n},p)$, which is a random graph of type-H, where H is a 2-dimensional square torus $n \times n$. We show that the lower bound on the compactness of shortest-path interval routing for a random graph of type-H, where H is a 2-dimensional torus of the order $N = n^2$, is $\Omega(\sqrt{\log N})$ with the probability at least $1 - o(1)$. This result is shown in the following table with respect to the previous works.

Random graph p - constant	*compactness*
$G \in \mathcal{G}_{N,p}$	≤ 2 [10]
$T \in \mathcal{G}(T_{n\times n}, p)$	$\Omega(\sqrt{\log n})$ **[Thm. 1]**

Tab. 1. Results for random graphs

This table can be understand in such a way that if the compactness of random graphs from $\mathcal{G}_{N,p}$ is at most 2 with high probability, then the compactness of random tori $T \in \mathcal{G}(T_{n\times n}, p)$ is $\Omega(\sqrt{\log n})$ with high probability for constant p.
Other fault-tolerant properties of the random graph of type-$T_{n\times n}$ have been studied e.g. in [16].

The organization of the paper is as follows. We will introduce useful notions in Section 2. The lower bound proving technique based on the results from [12] is described in Section 3. Our main result is proved in Section 4.

2 Preliminaries and Terminology

2.1 Graph Theory

For graphs $G = (V(G), E(G))$ and $H = (V(H), E(G))$ we will denote by $H \subseteq G$ if H is a *subgraph of* G. A subgraph $H \subseteq G$ is itself a graph whose vertex set $V(H)$ is a subset of $V(G)$ and $E(H)$ is subset of $E(G)$. If every pair of vertices $u, v \in V(H)$ are adjacent in H iff they are adjacent in G, then H is said to be an *induced subgraph of* G.
Let n be positive integer, such that $n > 2$. Let us denote the set of n integers $\{0, 1, \ldots, n-1\}$ by $[n]$. A *two-dimensional vector over* $[n]$ is the pair $\overline{x} = (x_1, x_2)$ such that $x_1, x_2 \in [n]$. We can also write $\overline{x} \in [n] \times [n]$. For two vectors $\overline{x}, \overline{y} \in [n] \times [n]$, their *Hamming distance* is defined as $\rho(x, y) = \sum_{i=1}^{2} |x_i - y_i|$.

For n as above, a *two-dimensional torus* is the undirected graph $T_{n\times n} = (V, E)$, where $V(T_{n\times n}) = [n]\times[n]$ (each vertex is labelled by a two-dimensional vector $\overline{x} \in [n] \times [n]$) and $E(T_{n\times n})$ is defined as follows. Vertices $\overline{x}, \overline{y} \in V(T_{n\times n})$ are adjacent iff

(i) $\rho(\overline{x}, \overline{y}) = 1$, or

(ii) $\rho(\overline{x}, \overline{y}) = n - 1$ and $\overline{x}, \overline{y}$ differ only in one number.

The degree of each vertex of $T_{n\times n}$ is equal to 4. The order (number of vertices) of the torus $T_{n\times n}$ is given by $|V(T_{n\times n})| = n^2$ and its number of edges is $|E(T_{n\times n})| = 2n^2$.

For two integers $r, s \geq 1$, a *two-dimensional rectangular grid* is the undirected graph $G_{r\times s} = (V, E)$, where $V(G_{r\times s}) = [r] \times [s]$ and $E(G_{r\times s}) = \{(\overline{x}, \overline{y}) \mid \overline{x}, \overline{y} \in [r] \times [s],\ \rho(\overline{x}, \overline{y}) = 1\}$. If $r = s$, then the parameter r is called the *size of the grid* $G_{r\times r}$.
The order of the grid $G_{r\times s}$ is $|V(G_{r\times s})| = r.s$ and its number of edges is given by $|E(G_{r\times s})| = s(r-1) + r(s-1)$.

For integers $r, s \geq 2$ let us define graph $RG_{r\times s}$ as a grid $G_{r\times s}$ in which all internal and horizontal edges have been removed. Especially, if $r = s$, then $|V(RG_{r\times r})| = r^2$ and $|E(RG_{r\times r})| = (r+2)(r-1)$. Analogously, let $RG^T_{r\times s}$ be a graph obtained from a grid $G_{r\times s}$ by removing of all internal vertical edges. It is easy to see that the graph $RG^T_{s\times r}$ is the graph $RG_{r\times s}$ orthogonally rotated. Thus, it holds: $|V(RG_{r\times s})| = |V(RG^T_{s\times r})|$ and $|E(RG_{r\times s})| = |E(RG^T_{s\times r})|$.

2.2 Random Graph Model

Let $p \in \mathbf{R}$ be a constant such that $0 < p < 1$ and let H be a graph with N vertices and M edges. Graph H is said to be *the sample graph.* Let us define *random graph* G obtained from graph H by randomly removed edges as follows. Put $V(G) = V(H)$. Consider that for a random graph G each edge exists independently and with the probability p. It means that $Pr[e \in E(G)] = p$ for all edges $e \in E(H)$. The constant p is called the *probability of an edge.* We introduce the corresponding probability space.

Definition 1. *Let $(\Omega, \mathbf{F}, Pr)$ be a probability space, where the class Ω consists of all (labelled) graphs G on N vertices such that $V(G) = V(H)$ and $E(G) \subseteq E(H)$. If G has q edges, $0 \leq q \leq M$, then the probability of obtaining G as a result of random edge generation is given by:*

$$Pr[G] = p^q(1-p)^{M-q}. \tag{1}$$

The graph G will be called random graph induced by graph H. The probability space $(\Omega, \mathbf{F}, Pr)$ will be denoted by $\mathcal{G}(H, p)$ and it is called probability space induced by the graph H.

Equality $Pr[\Omega] = 1$ in the probability space $\mathcal{G}(H, p)$ follows directly from Binomial equation.

We will use the probability space $\mathcal{G}(T_{n\times n}, p)$ - *probability space of random tori*. A graph $T \in \mathcal{G}(T_{n\times n}, p)$ is said to be *random torus*.

In order to describe a property of random graphs we will use the notions from probability theory, e. g. random variables, expectations, variances, etc. (Cf. [2].) We will use only discrete random variables in this paper. We will also use the following bounds of the random variables.

Proposition 1. (Markov's inequality) *Let X be a nonnegative random variable with expectation $E(X)$ and let $\lambda > 0$. Then the following inequality holds:*

$$Pr[X \geq \lambda] \leq E(X) \cdot \lambda^{-1} \ . \tag{2}$$

Proposition 2. (Chebyshev's inequality) *Let X be a nonnegative (discrete) random variable with expectation $E(X)$ and variance $Var(X)$. Then the following inequality holds:*

$$Pr[X > 0] \geq 1 - \frac{Var(X)}{E^2(X)} \ . \tag{3}$$

The proof technique based on the Chebyshev's inequality is said to be a *second moment method*. For more details see [2].

2.3 Interval Routing Scheme

We assume a point-to-point asynchronous communication network. The network topology is modelled by a simple graph $G = (V, E)$, where V is a set of vertices (or processors) and E is a set od edges (or bidirectional communication links) in G. Assume $|V| = N$.

A *routing in* G is a set $\mathcal{R} = \{P_{uv} \mid (u, v) \in V \times V, u \neq v\}$ of $|V|(|V| - 1)$ paths in G, where each individual path P_{uv} has initial vertex u and terminal vertex v. (Note that the paths P_{uv} and P_{vu} may be different.) If G' is an induced subgraph of G, then we say that routing $\mathcal{R}' \subseteq \mathcal{R}$ is the *routing in* G' if all $P \in \mathcal{R}'$ are paths in G'.

Interval routing is based on a suitable labeling scheme for the vertices and edges in G. A vertex label is an element of the set $\{1, \ldots, N\}$ and arc label is a cyclic interval $[a, b]$ with $a, b \in \{1, \ldots, N\}$. (Note that $[a, b] = \{a, a+1, \ldots, N, 1, \ldots, b\}$ for $a > b$.) Given a vertex $v \in V$, by $I(v)$ we denote the set of arcs outgoing from v. An *interval labeling scheme* (for short denoted by ILS) of G is a scheme, where

- a vertex labeling is an assignment of unique labels to vertices of V and
- for each vertex $v \in V$, an edge labeling is an assignment of disjoint intervals to arcs $e \in I(v)$.

Given an ILS on G, messages to a destination vertex having a label w are routed via the arc labeled by the interval $[a, b]$ such that $w \in [a, b]$.

If the edge labeling assigns at most k intervals per arc, the scheme is called k *interval labeling scheme* (shortly k-ILS). If the routing strategy guarantees that the messages always arrive at their destination and always via the shortest path, then k-ILS is said to be *shortest path* (or *optimal*) k *interval routing scheme* (shortly optimal k-IRS). (For more precise formulations and other details see [9] and [12].) In this paper we consider shortest path interval routing schemes only.
The *compactness* of a graph G, denoted as *compactness*(G), is the smallest integer k such that G supports a k-IRS that provides only one shortest path between any pairs of nodes.

3 Lower Bound on Compactness

In order to prove our lower bound we use the combination of two proving techniques. The first one is the second moment method, cf. [2]. The second one is based on so called "wq-property" and it is taken from [12], derived from [8]. We will describe it in this section.
The following notation is proposed. Let an arbitrary optimal k-IRS of the graph G be given. For a vertex $v \in V$ and arc $e \in I(v)$ let us denote $S(v, e)$ be the subset of vertices $w \in V$ which can be reached optimally from v over its outgoing arc e and $Z(v, e)$ be the subset of vertices $w \in V$ such that every optimal path from v to w follows the outgoing arc e.

Proposition 3. *([12]) Let G be a graph with maximum degree Δ and ρ be an optimal k-IRS of G. Let Q and W be disjoint vertex subsets of G satisfying that for $w_i, w_j \in W$, $w_i \neq w_j$, there is $v \in Q$ such that for each $e \in I(v)$ it holds $w_i \notin S(v, e)$ or $w_j \notin S(v, e)$. Then it holds*

$$k \geq \frac{|W|}{\Delta \cdot |Q|} \ . \tag{4}$$

Using this proposition we obtain the lower bound on the compactness of the graphs $RG_{r \times s}$. (The compactness of $RG_{(s+3) \times s}$ was proved in [12].)

Lemma 1. *([12]) It holds that compactness*$(RG_{r\times s}) \geq \frac{(r-1)(s-1)}{6(r+s-1)}$. *Moreover, the asymptotic maximum is acquired for* $r = s$ *and it holds:*

$$compactness(RG_{r\times r}) = \Omega(r) \ .$$

Proof. Our idea of the proof is based on the previous proposition but it is a little bit different from the one in [12]. It is sufficient to choose the sets W and Q as it is shown on the following figure. The set Q contains all the vertices which lie on the leftmost vertical path and on the top horizontal path of $RG_{r\times s}$. The set W contains all the vertices which lie on the each horizontal row except the vertices from Q with the odd vertical index from the top. (Recall that the labeling starts from 0.) Note that sets Q and W are disjoint. It can be easy to check that the property of the proposition 3 holds.
(1) If vertices $w_i, w_j \in W$, $w_i \neq w_j$, are from the same vertical path of $RG_{r\times s}$, then the corresponding vertex $q \in Q$ lies on the leftmost vertical path of $RG_{r\times s}$. (For example, for w_1 and w_4 it is q_3.)
(2) If vertices $w_i, w_j \in W$ are from the different vertical paths, then the corresponding vertex $q \in Q$ lies on the top horizontal paths of $RG_{r\times s}$.
Thus, for the sets Q and W as above, for all the pairs $w_i, w_j \in W$, $w_i \neq w_j$, there is $q \in Q$ such that for each arc e outgoing from q, both w_i, w_j cannot be reached optimally from q over its outgoing arc e.
It holds that $|Q| = r + s - 1$ and $|W| = \lceil (r-1)/2 \rceil (s-1) \geq (r-1)(s-1)/2$. Putting $r = s$ we obtain *compactness*$(RG_{r\times r}) \geq (r-1)^2/6(2r-1)$. It yields that the asymptotic maximum for the compactness is acquired for $r = s$, and then *compactness*$(RG_{r\times r}) = \Omega(r)$. □

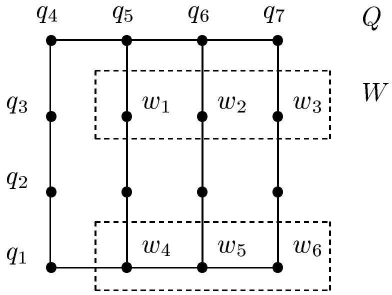

Fig. 1. The sets W and Q in the graph $RG_{4\times 4}$

For the reason presented above, we consider only the square (sub)graphs $RG_{r\times r}$.

Lemma 2. *Let $T \in \mathcal{G}(T_{n\times n}, p)$ be a random torus. If $RG_{r\times r}$ is an induced subgraph of the random torus T and $r < n/2$ then it holds:*

$$compactness(T) \geq compactness(RG_{r\times r}) \ . \tag{5}$$

Proof. If the set of all shortest paths in T between all pairs of vertices $u, v \in V(RG_{r\times r})$ is entirely contained in the induced subgraph $RG_{r\times r}$, then the inequality (5) follows directly from the Lemma 1.
A graph $RG_{r\times r}$ consists of two horizontal paths - *rows* and r vertical paths - *columns*. Let $u, v \in V(RG_{r\times r})$ be arbitrary different vertices. We can distinguish three cases.
(1) If u and v lie in the same columns, then the length of the shortest u-v path in the graph $RG_{r\times r}$ is equal to their Hamming distance. Hence there exists exactly one optimal routing path $P_{uv} = P_{vu}$ in the graph $RG_{r\times r}$.
(2) If at least one vertex of u and v lies in a row, then this situation is the same as the previous one. Also in this case there exists exactly one optimal routing path $P_{uv} = P_{vu}$ in the graph $RG_{r\times r}$.
(3) If both vertices u and v lie in different columns as internal vertices, then the length of the shortest u-v path in $RG_{r\times r}$ is greater then their Hamming distance. We construct a routing path P_{uv} in such a way that from u it tends towards the nearest row along the u-column and it continues to the v-column (along the chosen row) and to the vertex v. The optimal routing

path P_{vu} is constructed in the same manner, however P_{uv} and P_{vu} may be different.

Since $RG_{r\times r}$ is the induced subgraph of T, then each optimal routing scheme for $RG_{r\times r}$ is also optimal in T for all $v \in V(RG_{r\times r})$. Moreover, there is no $u, v \in V(RG_{r\times r})$ such that any optimal routing path P_{uv} or P_{vu} in T is not also an optimal routing path in $RG_{r\times r}$. It implies that all optimal interval routing schemes that are constructed for the graph $RG_{r\times r}$ are also exactly the same optimal IRS for the induced subgraph $RG_{r\times r} \subseteq T$ and vice versa. It follows that the inequality (5) holds. □

Remark 1. The same property holds also for the graph $RG^T_{r\times r}$.

4 Result

The main result of the paper is the following.

Theorem 1. *Let* p *be a fixed constant such that* $0 < p < 1$ *and let* $0 < \gamma < 1$. *Then for a random torus* $T \in \mathcal{G}(T_{n\times n}, p)$ *it holds*

$$compactness(T) = \Omega(\sqrt{\log n})$$

with probability at least $1 - O(n^{-2\gamma})$.

Our proof of this theorem is based on the following idea. We show that "almost all" random torus contains a graph $RG_{r\times r}$ as an induced subgraph. By probabilistic arguments we calculate the size r of such subgraphs.
For more precise calculation, it is necessary to consider that random torus contains not only the induced subgraphs $RG_{r\times r}$ but also the induced subgraphs $RG^T_{r\times r}$. As we will see later (remark 2) this assumption is not important and therefore it is sufficient to focus only on one type of induced subgraphs.

For $r \in \mathbf{N}$ let X_r be an indicator random variable defined on $\mathcal{G}(T_{n\times n}, p)$ as follows:

- $X_r(T) = 1$, if T contains an induced subgraph $RG_{r\times r}$ and
- $X_r(T) = 0$, otherwise

for all $T \in \mathcal{G}(T_{n\times n}, p)$.
The expectation of the random variable X_r is expressed in the following lemma.

Lemma 3. *Let $r \geq 2$. For the random variable X_r it holds*

$$E(X_r) = n^2 p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} \ ,$$

and

$$n^2 \cdot g(p,r) \leq E(X_r) \leq n^2 \cdot f(p,r) \ , \tag{6}$$

where

$$g(p,r) = p^{(r+2)^2}(1-p)^{(r+2)^2} \ ,$$
$$f(p,r) = p^{(r-2)^2}(1-p)^{(r-2)^2} \ .$$

Proof. If $RG_{r\times r}$ is an induced subgraph of a graph $T \in \mathcal{G}(T_{n\times n}, p)$, then there are n^2 possibilities how to choose the placement of $RG_{r\times r}$ in T. Graph $RG_{r\times r}$ has $(r+2)(r-1)$ edges and $(r-1)(r-2)$ free slots. Thus, the probability of a placement is $p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)}$.

The following inequality holds:

$$p^{(r-2)^2}(1-p)^{(r-2)^2} \leq p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)},$$

since $0 < p < 1$. This gives the lower bound of $E(X_r)$.
The upper bound follows from the fact that for $0 < p < 1$ it holds:

$$p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} \leq p^{(r-2)^2}(1-p)^{(r-2)^2}.$$

□

As a consequence we obtain the following upper bound. The proof follows from the Markov's inequality, for its details see [16].

Lemma 4. *Let $T \in \mathcal{G}(T_{n\times n}, p)$ be a random torus. Then for the size of the largest induced subgraph $RG_{r\times r} \subseteq T$ holds that $r \leq r_0$ with the probability tending to 1 as $n \to \infty$, where*

$$r_0 = \sqrt{2\log_{1/p(1-p)} n} + 2 \ . \tag{7}$$

Proof. By contradiction. By substituting $r = \sqrt{2\log_{1/p(1-p)} n} + 2$ into the upper bound of $E(X_r)$ (6) we obtain

$$E(X_r) \leq n^2[p(1-p)]^{2\log_{1/p(1-p)} n} = n^2 \cdot n^{-2} = 1 \ . \tag{8}$$

¿From the Markov's inequality (2) for $\lambda = 1$ and from (8) it follows that

$$Pr[X_t > 1] \leq E(X_t) < 1 \tag{9}$$

for arbitrary $t > \sqrt{2\log_{1/p(1-p)} n} + 2$. The random variable X_r counts the number of induced subgraphs $RG_{r\times r}$ in the random torus T. The inequality (9) yields that there is no such a graph $RG_{r\times r} \subseteq T$ with the size $r > \sqrt{2\log_{1/p(1-p)} n} + 2$ with the probability tending to 1 as $n \to \infty$. Hence a contradiction. □

Remark 2. If we consider that a random torus $T \in \mathcal{G}(T_{n\times n}, p)$ may contain induced subgraphs $RG_{r\times r}$ or $RG^T_{r\times r}$, then this value is $r'_0 = \sqrt{1/2 + 2\log_4 n} + 2$ for an arbitrary constant p. It is very close to r_0 and therefore it is not necessary to assume the case when also $RG^T_{r\times r}$ is an induced subgraph of T. (For other details see the appendix.)

We will express the variance of the random variable X_r.

Lemma 5. *Let $r \geq 2$. For the random variable X_r it holds the following equation:*

$$Var(X_r) = E(X_r)[1 + 2c_1 + 4c_2 + 8c_3 - \Psi(r,p)],$$

where

$$c_1 = \frac{p^{r+1}(1-p)^{r-2} - p^{(r+1)(r-1)}(1-p)^{(r-1)(r-2)}}{1 - p^{r+1}(1-p)^{r-2}} \ ,$$

$$c_2 = p^{(r+1)(r-1)}(1-p)^{(r-1)(r-2)},$$
$$c_3 = p^{r^2}(1-p)^{(r-1)(r-2)}[1 - p^{r-2}]/(1-p),$$
$$\Psi(r,p) = (2r-3)(2r+1)p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)}.$$

Proof. The variance of X_r we will enumerate by the following equation, [18]:

$$Var(X_r) = E(X_r^2) - E^2(X_r) \ . \tag{10}$$

In order to enumerate the value $E(X_r^2)$ we will use indicator method, cf. [17].
Let A and B are two graphs $RG_{r\times r}$ and let us define the indicator random variable $\eta_{A,B}$ on $\mathcal{G}(T_{n\times n}, p)$ as follows:

- $\eta_{A,B}(T) = 1$, if A and B are formed in such placement that T contains at least one induced subgraph $RG_{r\times r}$ and
- $\eta_{A,B}(T) = 0$, if T contains no $RG_{r\times r}$

for all $T \in \mathcal{G}(T_{n\times n}, p)$.
It is easy to see that $X_r^2 = \sum \eta_{A,B}$, where the summation ranges over all ordered pairs A and B. The sum $\sum \eta_{A,B}$ can be divided into three parts according to the following three conditions:

1. graphs A and B are disjoint,
2. graphs A and B are the same,
3. graphs A and B are different and have a nonempty intersection.

Thus, by the linearity of expectation it holds:

$$E(X_r^2) = \sum_{A,B} E(\eta_{A,B}) =$$

$$= \sum_{P(A,B)} E(\eta_{A,B}) + \sum_{A=B} E(\eta_{A,B}) + \sum_{Q(A,B)} E(\eta_{A,B}) \ , \tag{11}$$

where $P(A,B) \equiv (A \cap B = \emptyset)$ and $Q(A,B) \equiv (A \cap B \neq \emptyset) \wedge (A \neq B)$.
We express these three sums separately.

$$\sum_{P(A,B)} E(\eta_{A,B}) = n^2[n^2 - (2r-1)^2]\ p^{2(r+2)(r-1)}(1-p)^{2(r-1)(r-2)} \tag{12}$$

$$\sum_{A=B} E(\eta_{A,B}) = n^2 p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} \tag{13}$$

$$\sum_{Q(A,B)} E(\eta_{A,B}) = 2n^2 \sum_{i=1}^{r-2} p^{(r+2)(r-1)+i(r+1)}(1-p)^{(r-1)(r-2)+i(r-2)} +$$
$$+8n^2 \sum_{i=1}^{r-2} p^{2(r+2)(r-1)-i}(1-p)^{2(r-1)(r-2)} +$$
$$+4n^2 p^{2(r+2)(r-1)}(1-p)^{2(r-1)(r-2)} +$$
$$+4n^2 p^{2(r+2)(r-1)-r+1}(1-p)^{2(r-1)(r-2)} \ . \tag{14}$$

We enumerate the summation expressions as follows.

$$\sum_{i=1}^{r-2} p^{(r+2)(r-1)+i(r+1)}(1-p)^{(r-1)(r-2)+i(r-2)} =$$

$$= p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} \times$$

$$\times \frac{p^{r+1}(1-p)^{r-2} - p^{(r+1)(r-1)}(1-p)^{(r-1)(r-2)}}{1-p^{r+1}(1-p)^{r-2}} \tag{15}$$

$$\sum_{i=1}^{r-2} p^{2(r+2)(r-1)-i}(1-p)^{2(r-1)(r-2)} =$$

$$= p^{(r+2)(r-1)+r^2}(1-p)^{2(r-1)(r-2)} \times \frac{1-p^{r-2}}{1-p} \tag{16}$$

After the substitution of equations (11), (12), (13), (14), (15) and (16) into (10) we obtain the resulting equation. □

Our lower bound proof is based on the second moment method, cf. [2], [6]: If $Var(X) = o(E^2(X))$, then $X \sim E(X)$. Therefore, we will consider parameters r and r_0 as functions of n and we will write $r = r(n)$ and $r_0 = r_0(n)$.

Lemma 6. *Let p, $0 < p < 1$ be a constant. Let r be a function given by $r(n) = \varepsilon \cdot r_0(n) + 2(1-\varepsilon)$ for a fixed constant ε such that $0 < \varepsilon < 1$. Then for n sufficiently large, a random torus $T \in \mathcal{G}(T_{n\times n}, p)$ contains at least one graph $RG_{r\times r}$ as an induced subgraph with the probability at least $1 - O(n^{2(\delta^2-1)})$, where $\delta = \varepsilon + 4/[r_0(n)-2]$ and $0 < \varepsilon < 1 - 4/[r_0(n)-2]$.*

Proof. For the function $r(n)$ it holds

$$r(n) = \varepsilon \cdot r_0(n) + 2(1-\varepsilon) = \varepsilon \cdot \sqrt{2\log_{1/p(1-p)} n} + 2 < r_0(n) \tag{17}$$

for all $\varepsilon < 1$.

We estimate the variance of the random variable X_r for $r(n)$ given as above and relative to n. We use the equality from the Lemma 5. It is easy to see that parameters c_1, c_2 and c_3 are at most positive constants and it is sufficient to estimate the expression $(2r-3)(2r+1)p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)}$. The inequality (17) yields that for ε as above it holds:

$$(2r-3)(2r+1) = O(\log n) \ . \tag{18}$$

On the other hand, from the lemma 3 it holds that $p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} \leq f(p,r)$ and

$$f(p, r(n)) = n^{-2\varepsilon^2} \ . \tag{19}$$

¿From (18) and (19) it follows that:

$$(2r-3)(2r+1)p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} \to 0 \quad as \quad n \to \infty$$

for fixed $\varepsilon > 0$. It means that for all $c > 0$ there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$ it holds $(2r-3)(2r+1)p^{(r+2)(r-1)}(1-p)^{(r-1)(r-2)} < c$. In particular, this expression is less than 1 for a sufficiently large n. It yields that there exists a constant $t > 0$ such that

$$Var(X_r) = t \cdot E(X_r)$$

for sufficiently large n.
Let us estimate the fraction $Var(X_r)/E^2(X_r)$ according to the previous conditions and Lemma 3.

$$\frac{Var(X_r)}{E^2(X_r)} = \frac{t}{E(X_r)} \le \frac{t}{n^2 g(p,r)} \tag{20}$$

Let n_1 be a constant such that $4/[r_0(n_1)-2] < 1$. Let us fix ε such that $0 < \varepsilon < 1 - 4/[r_0(n)-2]$ for n sufficiently large. (It means that it holds for all $n > n_1$.) Putting $\delta = \varepsilon + 4/[r_0(n)-2]$ it holds:

$$\begin{aligned} r(n) &= \varepsilon \cdot r_0(n) + 2(1-\varepsilon) = \\ &= \left(\delta - \frac{4}{r_0(n)-2}\right) \cdot r_0(n) + 2\left(1 - \delta + \frac{4}{r_0(n)-2}\right) = \\ &= \delta \cdot \sqrt{2\log_{1/p(1-p)} n} - 2 \ . \end{aligned} \tag{21}$$

Note that $\delta \to \varepsilon$ as $n \to \infty$. By substituting of (21) to (20) we have:

$$\frac{Var(X_r)}{E^2(X_r)} \le \frac{t}{n^2 \cdot n^{-\delta^2}} = t \cdot n^{2(\delta^2-1)} \ .$$

Now we use the Chebyshev's inequality.

$$Pr[X_r > 0] \ge 1 - \frac{Var(X)}{E^2(X)} \ge 1 - O\left(n^{2(\delta^2-1)}\right) \ ,$$

since the previous conditions hold.
It means that a random torus $T \in \mathcal{G}(T_{n\times n}, p)$ contains at least one graph $RG_{r\times r}$ as an induced subgraph with the probability at least $1 - O(n^{2(\delta^2-1)})$.
□

Thus, the Theorem 1 follows directly from the Lemmas 1, 2 and 6 for $\gamma = 1 - \delta^2$.

5 Concluding Remarks

We have proved the lower bound on compactness of shortest-path interval routing for a random 2-dimensional torus representing a communication network in which the link fails independently and with the constant probability $f = 1 - p$. This is the first non-constant compactness lower bound for random tori. The determination of related nontrivial upper bound is still open. A future research may also concern other topologies (as hypercubes). Also it is not clear whether our technique can be extend to random d-dimensional torus for $d > 2$.
Our proving technique is based on the combination of two methods: the compactness lower bound property for the graph $RG_{r \times r}$ (cf. [12]) and the second moment method. The idea of using the second moment method in this paper is similar to the famous and celebrated proof of the clique number estimation on the random graphs from $\mathcal{G}_{N,p}$, cf. [6], [18]. However, the main difference is as follows: The probability distribution of the random variable for cliques attains a binomial-like distribution, but the distribution of the random variable X_r (according to variable r) in case of random tori is close to the geometric distribution. In spite of this reason it was necessary to use different combinatorial and asymptotical approximations.

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Appendix

The following lemma is the formal reasoning of the property in the remark 2.

Lemma 7. *Let $T \in \mathcal{G}(T_{n\times n}, p)$ be a random torus. Then for the size of the largest induced subgraphs $RG_{r\times r} \subseteq T$ or $RG^T_{r\times r} \subseteq T$ the following inequality holds*

$$r \leq \sqrt{\frac{1}{2} + 2\log_4 n} + 2 \ . \tag{22}$$

with the probability tending to 1 as $n \to \infty$.

Proof. It is analogous as in the previous lemma. Let us define the indicator random variable X^T_r on the probability space $\mathcal{G}(T_{n\times n}, p)$ as follows:

- $X^T_r(T) = 1$, if T contains an induced subgraph $RG^T_{r\times r}$ and
- $X^T_r(T) = 0$ otherwise,

for all $T \in \mathcal{G}(T_{n\times n}, p)$.
It is easy to see that $E(X^T_r) = E(X_r)$ for all r. Next we define random variable Z_r on $\mathcal{G}(T_{n\times n}, p)$ such that:

$$Z_r = X_r + X^T_r \ .$$

This random variable counts the number of induced subgraphs $RG_{k\times k}$ or $RG^T_{r\times r}$; it is $Z_r(T) = 1$ iff $T \in \mathcal{G}(T_{n\times n}, p)$ contains the induced subgraph $RG_{r\times r}$ or $RG^T_{r\times r}$. Let us express the expectation of the random variable Z_r. By linearity of expectation it holds:

$$E(Z_r) = E(X_r + X^T_r) = E(X_r) + E(X^T_r) = 2E(X_r) \ .$$

Thus, by the inequality (6), it holds:

$$E(Z_r) \leq 2n^2[p(1-p)]^{(r-2)^2} \leq 2n^2 4^{-(r-2)^2} \ .$$

By the substitution $r = \sqrt{1/2 + 2\log_4 n} + 2$ we have

$$E(Z_r) \leq 2n^2 4^{-1/2 - 2\log_4 n} = 2n^2 \cdot 4^{-1/2} n^{-2} = 1 \ .$$

The rest follows from the Markov's inequality. $\square$