# A MONAD FOR DOMAINS AND OTHER CATEGORIES

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*Dedicated to the memory of Ivan Rival*

ABSTRACT. A small modification of Vickers' definition of continuous information systems allows for a representation of the category of continuous domains (continuous DCPOs) and several other categories (Scott domains, continuous Scott domains, continuous lattices, algebraic lattices, and others) as Kleisli categories of suitable monads.

In his article [11], Vickers presented a very general approach to information systems (following the definition of Scott, [10]). The (continuous) information systems were defined as sets with transitive interpolative relations (here we should also mention the paper of Hoofman [6] in which the same approach had been chosen, and the papers od Erné – e.g. [4] – in which the importance of the interpolativity had been obseved); the approximable maps were defined as in the original paper of Scott [10], just modified for not necessarily reflexive relations. In among many other results, Vickers shows that the obtained category is equivalent to that of continuous domains (in his terminology, continuous posets).

In this paper we work with the objects slightly restricted. Basically, the requirement of interpolativity ($a \prec b \;\Rightarrow\; \exists a', a \prec a' \prec b$) is replaced by a simultaneous one for couples ($a_i \prec b, i = 1, 2 \;\Rightarrow\; \exists a', a_i \prec a' \prec b$); the approximable maps are the same as before. With this modification, and a very natural definition of morphisms (ld-maps, see 2.3 below) we obtain a category **LDir** such that

- the category **CDom** of continuous DCPOs is its full subcategory, and

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- the ideal functor yields a monad $\mathbb{I}$ on **LDir** such that the original approximable maps coincide with the Kleisli morphisms.

Thus, the category of information systems becomes the Kleisli category $\mathbf{LDir}_{\mathbb{I}}$ and the ideal functor yields a reflection of **LDir** onto **CDom** that, lifted to $\mathbf{LDir}_{\mathbb{I}}$ becomes an equivalence of categories.

Further, we show that $\mathbf{LDir}_{\mathbb{I}} \cong \mathbf{LDir}^{\mathbb{I}}$ (thus, our representation is unrelated to another Kleisli representation, of a smaller category of domains, in [2]), and that in many cases the monad can be restricted to naturally characterized subcategories of **LDir** to obtain Kleisli representations of several important subcategories of **CDom** (algebraic domains, Scott domains, continuous Scott domains, continuous lattices, algebraic lattices, and others).

## 1. Preliminaries

**1.1.** A binary relation $R$ is *interpolative* if for any $aRb$ there is a $c$ such that $aRcRb$.

If $(X, R)$ is a set with a transitive relation (not necessarily a partial order), a subset $D \subseteq X$ is said to be *directed* if it is non-void and if for any $a, b \in D$ there is a $c \in D$ such that $aRc$ and $bRc$.

**1.2.** A supremum of a subset $M$, if it exists, will be denoted by $\bigvee M$ and if the set $M$ is directed the fact will be indicated by writing $\bigvee^{\uparrow} M$. In a poset $(X, \leq)$ we write $a \ll b$ and say that $a$ is *way below* $b$ if, for every directed $D \subseteq X$, $b \leq \bigvee^{\uparrow} D$ implies that $a \leq d$ for some $d \in D$.

**1.3.** A poset $(X, \leq)$ is said to be a *continuous domain* (often refferred to as *continuous DCPO*, or *continuous poset* in [11]) if each directed subset has a supremum and if for each $a \in X$, $a = \bigvee^{\uparrow}\{x \mid x \ll a\}$. An important fact is that

**1.3.1.** *In a continuous domain, each $\{x \mid x \ll a\}$ is directed also in $\ll$ (that is, if $x_1, x_2 \ll a$ then there is an $x$ such that $x_1, x_2 \ll x \ll a$.) In particular, $\ll$ is interpolative.*

(Indeed, using the property of continuous domains for the $x \ll a$ we see that $a = \bigvee^{\uparrow}\{y \mid \exists x, y \ll x \ll a\}$. Thus, if $x_i \ll a$ we have $x_i \ll x'_i \ll a$ and, since $\{x \mid x \ll a\}$ is directed in $\leq$, $x_i \ll x'_i \leq x \ll a$ for some $x$.)

Using the interpolativity one immediately deduces that

**1.3.2.** *In a continuous domain, if $a \ll \bigvee^{\uparrow} D$ then there is a $d \in D$ such that $a \ll d$.*

**1.4.** Let $(X,\leq),(Y,\leq)$ be continuous domains. A *domain map* $f:(X,\leq)\to(Y,\leq)$ is a map $f:X\to Y$ preserving the suprema of directed sets. Note that

**1.4.1.** *$f:X\to Y$ is a domain map iff for each $a\in X$, $f(a)=\bigvee^{\uparrow}\{f(x)\mid x\ll a\}$.*

Indeed, let $D$ be directed in $X$. We have $\bigvee^{\uparrow} D=\bigvee^{\uparrow}\{x\mid x\ll\bigvee^{\uparrow} D\}$ and hence $f(\bigvee^{\uparrow} D)=\bigvee^{\uparrow}\{f(x)\mid x\leq d\in D\}\leq\bigvee^{\uparrow}\{f(d)\mid d\in D\}$.)

The category of continuous domains and domain maps will be denoted by

**CDom**.

**1.5.** The following definition is a slight modification of the Vickers' definition of a *continuous information system* (or *infosys* – see [11]).

A *locally directed set* (briefly, *ld-set*) $(X,\prec)$ is a set with a transitive relation such that

(ld1) for every $a\in X$ there is a $b$ such that $b\prec a$, and
(ld2) if $b_1,b_2\prec a$ then there is a $b$ such that $b_1,b_2\prec b\prec a$.

(In other words, *each* ${\prec}a=\{x\mid x\prec a\}$ *is directed.*)

An *approximable map* $f:(X,\prec)\multimap(Y,\prec)$ is a relation $f\subseteq Y\times X$ such that

(am1) for every $a\in X$ there is a $b\in Y$ such that $bfa$,
(am2) if $bfa'\prec a$ then $bfa$,
(am3) if $bfa$ then $bfa'\prec a$ for some $a'$,
(am4) if $b\prec b'fa$ then $bfa$, and
(am5) if $b_1,b_2fa$ then $b_1,b_2\prec bfa$ for some $b$

(if the objects are ld-sets, this is equivalent with the definition in [11].) The resulting category (in which the relations are composed in the order usual for maps, and the identities are the relations $\prec$) will be denoted by

**Infos**.

**1.6. Notes.** The category **Infosys** in [11] is larger: the objects are all the $(X,\prec)$ with transitive interpolative $\prec$.

The main motivation for the restriction has been to get closer to the Scott's *information systems* ([10] – the generalization of which the continuous information systems in [11] are) by introducing a counterpart of the Scott's axiom

"if $u\vdash v_1$ and $u\vdash v_2$ then $u\vdash(v_1\cup v_2)$"

which the (ld2), in the absence of unions and reflexivity, does. Other facts that speak for the modification are, e.g., that

(a) the way-below relation in continuous domains has this stronger property,
(b) the ld-sets are exaxtly the infosyses for which the pre-topology from [11], 2.2, is a topology,
(c) in the resulting representation there is in fact no restriction (we will show that **Infos** $\cong$ **CDom**, and hence, after all, **Infos** $\cong$ **Infosys**),
(d) and, last but not least, it makes everything technically easier.

2. Our notation is reversed as compared to that of [11] or [10]. This is done just for convenience: when representing an approximable map $f$ as a multimap, as we will do later, we have the optically comfortable transit from $bfa$ to $b \in f(a)$. Also, this way we are seldom forced to reverse the symbol of the basic relation ($\prec$ to $\succ$, $<$ to $>$, etc.).

## 2. The category **LDir** and continuous domains

**2.1.** Let $(X, \prec)$ be an ld-set. On $X$ define a relation $\precsim$ by setting

$$a \precsim b \quad \text{iff} \quad (x \prec a \;\Rightarrow\; x \prec b).$$

Then obviously

**2.1.1.** *$\precsim$ is a preorder and it is a partial order iff* ${\prec}a = {\prec}b \;\Rightarrow\; a = b$.

**2.1.2.** $a \prec b \quad \Rightarrow \quad a \precsim b$.

**2.1.3.** $a \prec b \precsim c \quad \Rightarrow \quad a \prec b$.

Also the following is easy (interpolate $x \prec y \prec a$).

**2.1.4.** *if* $(x \prec a \;\Rightarrow\; x \precsim b)$ *then* $a \precsim b$.

**2.2. Proposition.** *Let $(X, \leq)$ be a continuous domain. Then $(X, \ll)$ is an ld-set and the resulting $\precsim$ coincides with the original $\leq$.*

*Proof.* For the first statement recall 1.3.1. Now if $a \leq b$ then trivially $x \ll a$ implies $x \ll b$, and if the implication holds we have $a = \bigvee^{\uparrow}\{x \mid x \ll a\} \leq \bigvee^{\uparrow}\{x \mid x \ll b\} = b$ □

**Convention.** From now on, we will view a continuous domain $(X, \leq)$ as the ld-set $(X, \ll)$. By the Proposition, this causes no loss of information.

**2.3.** An *ld-map* $f : (X, \prec) \to (Y, \prec)$ is a mapping $f : X \to Y$ such that

(a) $a \precsim b \quad \Rightarrow \quad f(a) \precsim f(b)$, and

(b) $y \prec f(a) \quad \Rightarrow \quad \exists a' \prec a$ such that $y \prec f(a')$.
The category of ld-sets and ld-maps will be denoted by

$$\mathbf{LDir}.$$

**Note.** The ld-maps are easily shown to be exactly the continuous maps in the Vickers' topology from [11], 2.2 (also recall (b) in 1.6.1 above).

**2.3.1. Proposition.** *1. The condition* (a) *can be replaced by the formally weaker*

(a') $a \prec b \quad \Rightarrow \quad f(a) \precsim f(b)$.

*2. If* $(Y, \ll)$ *is a continuous domain then the condition* (b) *can be replaced by*

(b') *for every* $a \in X$, $f(a) = \bigvee^{\uparrow}\{f(a') \mid a' \prec a\}$.

*Consequently (recall 1.4.1), if both* $(X, \leq)$ *and* $(Y, \leq)$ *are continuous domains then the domain maps* $(X, \leq) \to (Y, \leq)$ *are exactly the ld-maps* $(X, \ll) \to (Y, \ll)$. *Thus, in the sense of the Convention in 2.2,* **CDom** *is a full subcategory of* **LDir**.

*Proof.* 1. Let $a \precsim b$ and $y \prec f(a)$. We have an $a' \prec a$ with $y \prec f(a')$. Now $a' \prec b$ and hence $y \prec f(a') \precsim f(b)$.

2. If we have (b) then $f(a) = \bigvee^{\uparrow}\{y \mid y \ll f(a)\} = \bigvee^{\uparrow}\{y \mid \exists a' \prec a, y \ll f(a')\} = \bigvee^{\uparrow}\{f(a') \mid a' \prec a\}$. If (b') holds and $y \ll f(a) = \bigvee^{\uparrow}\{f(a') \mid a' \prec a\}$ then $y \ll f(a')$ for an $a' \prec a$. $\square$

**2.4.** An *ideal* in an ld-set $(X, \prec)$ is a subset $J \subseteq X$ such that

(J1) $a \in J \ \&\ b \prec a \ \Rightarrow \ b \in J$, and

(J2) $J$ is directed in $\prec$

(cf. [11]). Thus in particular,

$$\textit{each } {\prec} a = \{x \mid x \prec a\} \textit{ is an ideal in } (X, \prec).$$

The set of all ideals in $(X, \prec)$, ordered by inclusion, will be denoted by

$$\mathfrak{J}(X, \prec).$$

**2.4.1. Lemma.** *1. A union of a directed system of ideals is an ideal.*

*2. Each* $J \in \mathfrak{J}(X, \prec)$ *is the directed union*

$$\bigcup^{\uparrow}\{{\prec} a \mid a \in J\}.$$

*3. In* $\mathfrak{J}(X, \prec)$, $K \ll J$ *iff there is an* $a \in J$ *such that* $K \subseteq {\prec} a$.

*Proof.* 1 is obvious. 2: If $x \in J$ take (by (J2)) an $a \in J$ such that $x \prec a$. The union is directed by (J2) again.

3. $\Rightarrow$ follows immediately from 2; on the other hand, if $K \subseteq {\prec}a$, $a \in J$ and $J \subseteq \bigcup^{\uparrow} J_i$ then $a \in J_i$ for some $i$, and $K \subseteq {\prec}a \subseteq J_i$. $\square$

**2.4.2. Corollary.** *$\mathfrak{I}(X, \prec)$ is a continuous domain.*

**2.4.3.** From 2 in 2.4.1 we immediately obtain
**Observation.** *Let $(Y, \prec)$ be an ld-set and let $f : \mathfrak{I}(X, \prec) \to (Y, \prec)$ be an ld-map. Then if $y \prec f(J)$ there is an $a \in J$ such that $y \prec f({\prec}a)$.*

## 3. The ideal monad $\mathbb{I}$

**3.1.** Recall that a *monad* $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathcal{A}$ consists of a functor $T : \mathcal{A} \to \mathcal{A}$ and transformations $\mu : T \cdot T \to T$ and $\eta : \mathrm{Id} \to T$ satisfying $\mu \cdot T\mu = \mu \cdot \mu T$ and $\mu \cdot T\eta = \mu \cdot \eta T = \mathrm{id}$, and that adjoint situations $F \dashv U : \mathcal{B} \to \mathcal{A}$ create monads with $T = UF$; in among those one has the extremal cases:

- the "largest one" with $\mathcal{B} = \mathcal{A}^{\mathbb{T}}$, the category of Eilenberg-Moore algebras; the objects are couples $(A, \alpha)$ where $\alpha : TA \to A$ satisfies $\alpha \cdot T\alpha = \alpha \cdot \mu_A$ and $\alpha \cdot \eta_A = \mathrm{id}_A$; the morphisms (homomorphisms of the algebras) $(A, \alpha) \to (B, \beta)$ are the $f : A \to B$ such that $f \cdot \alpha = \beta \cdot Tf$,
- and the "smallest one" with $\mathcal{B} = \mathcal{A}_{\mathbb{T}}$, the Kleisli category of $\mathbb{T}$, with objects same as in $\mathcal{A}$, morphisms $f : A \multimap B$ the original $f : A \to TB$, and the composition defined by $g \circ f = \mu_C \cdot Tg \cdot f$. Note that in $\mathcal{A}_{\mathbb{T}}$, $\eta_A$ is the identity $A \multimap A$.

For more details see, for instance, Mac Lane [7].

**3.2. The Manes description of a monad.** In [8], Manes presented a very useful equivalent description of a monad. For every $A \in \mathcal{A}$ one has an object $TA \in \mathcal{A}$ and a morphism $\eta_A : A \to TA$, and with every $f : A \to TB$ one has associated an $f^*; TA \to TB$; these data are subjected to the conditions

- $\eta_A^* = \mathrm{id}_{TA}$,
- $f^* \eta_A = f$, and
- $g^* f^* = (g^* f)^*$.

The monad in the standard sense is obtained by defining $T(f) = (\eta_B f)^*$ for $f : A \to B$, thus making $T$ to a functor and $\eta$ to a transformation $\eta : \mathrm{Id} \to T$, and defining $\mu_A = (\mathrm{id}_{TA})^*$. (This constitutes a one-one correspondence with the monads as above, the inverse given by setting $f^* = \mu_B \cdot Tf$.)

**3.3.** The construction $\mathfrak{I}$ from 2.4 can be extended to a monad $\mathbb{I} = (\mathfrak{I}, \eta, {}^*)$ by defining

$$\eta_{(X,\prec)} = (a \mapsto {\prec} a) : (X, \prec) \to \mathfrak{I}(X, \prec)$$

and setting

$$f^*(J) = \bigcup\!\!\!\uparrow \{f(a) \mid a \in J\} \quad \text{for} \quad f : (X, \prec) \to \mathfrak{I}(Y, \prec).$$

Indeed we have

**Proposition.** *1. $\eta$ and $f^*$ are morphisms in* **LDir**.

*2. $\eta^*_{(X,\prec)} = \mathrm{id}_{\mathfrak{I}(X,\prec)}$, and $f^* \cdot \eta = f$.*

*3. $g^* f^* = (g^* f)^*$.*

*Proof.* 1 follows immediately from 2.3.1 and 2.4.1.

2: $\eta^* = \mathrm{id}$ by 2.4.1.2, and $f^*(\eta(a)) = \bigcup\!\!\!\uparrow \{f(a) \mid a' \prec a\} = f(a)$ by 2.3.1 (b').

3: $(g^* f)^*(J) = \bigcup\!\!\!\uparrow \{g^* f(a) \mid a \in J\} = \bigcup\!\!\!\uparrow \{\bigcup\!\!\!\uparrow \{g(b) \mid b \in f(a)\} \mid a \in J\} = \bigcup\!\!\!\uparrow \{g(b) \mid \exists a \in J, b \in f(a)\} = g^*(\bigcup\!\!\!\uparrow \{f(a) \mid a \in J\}) = g^* f^*(J)$. $\square$

**3.3.1. Lemma.** *If $(X, \leq)$ is a continuous domain then $\eta : (X, \ll) \to \mathfrak{I}(X, \ll)$ is an isomorphism.*

*Proof.* Set $\varphi = (J \mapsto \bigvee\!\!\!\uparrow J) : \mathfrak{I}(X, \ll) \to X$. Obviously, $\varphi$ is monotone, $\varphi(\eta(a)) = a$ and, by (J2), $J \subseteq \eta\varphi(J)$. Now let $x \in \eta(\varphi(J)) = \{x \mid x \ll \bigvee\!\!\!\uparrow J\}$. Then $x \ll a$ for some $a \in J$ and hence $x \in J$, and we see that $\eta(\varphi(J)) = J$. Thus, $\varphi$ is the inverse of $\eta$. $\square$

**3.3.2.** Recall 1.3. Let us view the approximable map $f$, instead of as a relation, as a map associating with the $a \in X$ the sets $f(a) = \{b \mid bfa\}$. Doing this we obtain

**Proposition.** *The category* **Infos** *coincides with the Kleisli category* $\mathbf{LDir}_{\mathbb{I}}$.

*Proof.* Recall 1.5. By (1), (4) and (5), each $f(a) = \{b \mid bfa\}$ is an ideal. By (2), $a' \prec a$ implies $f(a') \subseteq f(a)$ and by (3), $f(a) = \bigcup\!\!\!\uparrow \{f(a') \mid a' \prec a\}$. Thus, by 2.3.1, $f$ is an ld-map $(X, \prec) \to \mathfrak{I}(Y, \prec)$ and hence a Kleisli morphism $(X, \prec) \multimap (Y, \prec)$. On the other hand, obviously, an ld-map $\varphi : (X, \prec) \to \mathfrak{I}(Y, \prec)$ viewed as the relation $b\varphi a \equiv b \in \varphi(a)$ is an approximable map. It remains to be shown that the Kleisli composition coincides with the composition of relations. Indeed, we have $g \circ f = \mu \cdot Tg \cdot f = (\mathrm{id})^*(\eta g)^* f =$

$(\mathrm{id}^*\eta g)^* f = g^* f$ and hence $c \in (g \circ f)(a) = g^*(f(a)) = \bigcup\!\!\!\uparrow\{g(b) \mid b \in f(a)\}$ iff there is a $b$ such that $cgbfa$. $\square$

**3.4.** Define a functor $\mathbf{LDir}_{\mathbb{I}} \to \mathbf{CDom}$ by setting $F(X, \prec) = \mathfrak{I}(X, \prec)$, $F(f) = f^*$ (since $g \circ f = \mu \cdot Tg \cdot f = g^* f$ we have $(g \circ f)^* = g^* \cdot f^*$; also recall 2.3.1).

**Proposition.** *The functor $F$ constitutes an equivalence of the categories* **Infos** *($\equiv$* $\mathbf{LDir}_{\mathbb{I}}$*) and* **CDom***.*

*Proof.* There is only one domain map $g : \mathfrak{I}(X, \prec) \to \mathfrak{I}(Y, \prec)$ such that $g \cdot \eta = f$, namely $f^*$ (indeed, $g(J) = g(\bigcup\!\!\!\uparrow\{\prec a \mid a \in J\}) = \bigcup\!\!\!\uparrow\{g(\prec a) = f(a) \mid a \in J\})$ and hence $F$ is a full embedding. By 3.2.1, it constitutes an equivalence if categories. $\square$

**3.4.1. Note.** In the standard adjuncion $L \dashv R : \mathbf{LDir}_{\mathbb{I}} \to \mathbf{LDir}$ associated with the monad, the functor $F$ is the right adjoint $R$ with the range restricted to **CDom**. Also note that **CDom** is a reflective subcategory of **LDir** with the reflection $\eta : \mathrm{Id} \to FL$ ($FL$ is, of course, $\mathfrak{I}$ with the range restricted – also recall 3.3.1).

**3.5. Proposition.** *The Eilenberg-Moore category* $\mathbf{LDir}^{\mathbb{I}}$ *coincides with* **CDom***. More precisely, for an Eilenberg-Moore algebra $((X, \prec), \alpha)$, $(X, \precsim)$ is a continuous domain, $\alpha$ is uniquely determined, and the corresponding $(X, \ll)$ is isomorphic to $(X, \prec)$ in* **LDir***; on the other hand, each continuous domain admits (exactly one) operation.*

*Proof.* An operation $\alpha : \mathfrak{I}(X, \prec) \to (X, \prec)$ has to satisfy $\alpha\eta = \mathrm{id}$ and $\alpha\mathfrak{I}(\alpha) = \alpha\mu$, and has to be an ld-map. That is, we should have

(a) $\alpha(\prec a) = a$,

(b) $\alpha(\bigcup\!\!\!\uparrow\{\prec\alpha(J) \mid j \in \mathcal{J}\}) = \alpha(\bigcup\!\!\!\uparrow\{J \mid J \in \mathcal{J}\})$ for ideals $\mathcal{J}$ in $\mathfrak{I}(X, \prec)$, and

(c) $\alpha$ monotone, and for each $x \prec \alpha(J)$ an $a \in J$ such that $x \prec a$ (use (a) and 2.3.4).

First, from (a) and 2.1.1 we immediately see that $(X, \precsim)$ is a poset. Let $D$ be directed in $(X, \precsim)$. Set $J = \{a \mid a \prec d \in D\}$. If $d \in D$ and $b \prec d$ we have a $b \in J$, $\prec b \subseteq J$ and $b = \alpha(\prec b) \subseteq \alpha(J)$. Thus, by 2.1.4, $d \precsim \alpha(J)$. If for each $d \in D$, $d \precsim c$ and if $b \prec \alpha(J)$ then by (c) $b \in J$ and, for some $d \in D$, $x \prec d \precsim c$; hence $\alpha(J) \precsim c$. Thus, $\alpha(J)$ is the supremum of $D$ in $(X, \precsim)$.

Denote by $\ll$ the way below relation associated with $\precsim$. We will show that the identity is an isomorphism $(X, \prec) \to (X, \ll)$. First, we see that

$$x \prec a \quad \Rightarrow \quad x \ll a.$$

(Indeed, let $x \prec a \precsim \bigvee^{\uparrow} D$. Take the $J = \{c \mid c \prec d \in D\}$; thus $x \prec \alpha(J)$ and hence, since $\alpha$ is an ld-map, there is a $c \prec d \in D$ such that $x \prec \alpha(\prec c) = c$, and hence $x \precsim d$.) Thus, if $x \prec a$, interpolate $x \prec a' \prec a$, and we have $a' \ll a$. For the inverse, if $x \ll a = \bigvee^{\uparrow}(\prec a)$, then $x \ll a'$ for some $a' \prec a$.

If $(X, \precsim)$, $(Y, \precsim)$ are continuous domains, any ld-map $f : (X, \prec) \to (Y, \prec)$ preserves directed joins and we do not have to check the homomorphism property.

Now if $(X, \leq)$ is a continuous domain set $\alpha(J) = \bigvee^{\uparrow} J$. We have $\bigvee^{\uparrow}(\ll a) = a$ and if $b \ll \bigvee^{\uparrow} J$ there is an $a \in J$ such that $b \ll a$ (recall 1.3.2). The condition (c) holds since obviously $\bigvee^{\uparrow}(\bigcup^{\uparrow}\{\ll \bigvee^{\uparrow} J \mid J \in \mathcal{J}\}) = \bigvee^{\uparrow}\{\bigvee^{\uparrow} J \mid J \in \mathcal{J}\} = \bigvee^{\uparrow}(\bigcup^{\uparrow}\{J \mid J \in \mathcal{J}\})$. Finally, if $f : (X, \leq) \to (Y, \leq)$ is a domain map we have $\alpha(\mathfrak{I}(f)(J)) = \bigvee^{\uparrow}((\eta f)^*(J)) = \bigvee^{\uparrow}(\bigcup^{\uparrow}\{\ll f(a) \mid a \in J\}) = \bigvee^{\uparrow}\{f(a) \mid a \in J\} = f(\bigvee^{\uparrow}\{a \mid a \in J\}) = f\alpha(J)$. $\square$

**3.5.1. Note.** In the general monad setting, the couples $(TA, \mu_A)$ are obviously objects of $\mathcal{A}^{\mathbb{T}}$. They play the role of free algebras, and the correspondence $A \mapsto (TA, \mu_A)$, $f \mapsto f^* = \mu_B \cdot Tf$ represents $\mathcal{A}_{\mathbb{T}}$ as the free part of $\mathcal{A}^{\mathbb{T}}$.

In our particular case, this correspondence coincides with the functor from 3.4. Thus, the equivalence **Infos** $\cong$ **CDom** from 3.4 yields, by 3.5, an equivalence $\mathbf{LDir}_{\mathbb{I}} \cong \mathbf{LDir}^{\mathbb{I}}$. Thus in particular

- all the algebras in $\mathbf{LDir}^{\mathbb{I}}$ are free, and
- there is, up to equivalence, only one adjunction $L \dashv R : \mathcal{B} \to \mathbf{LDir}$ such that $RL \cong \mathfrak{I}$.

## 4. Some categories of special objects

**4.1.** In this section we will consider Kleisli representations of several (full) subcategories of **CDom**.

Recall that a domain is said to be *algebraic* if for each $a \in L$, $a = \bigvee^{\uparrow}\{x \mid x \ll x \leq a\}$. The category of algebraic domains will be denoted by

$$\mathbf{ADom}.$$

Further,

$$\mathbf{BCDom} \quad \text{resp.} \quad \mathbf{BADom}$$

will be the subcategory of **CDom** resp. **ADom** generated by the domains in which any bounded couple of elements has a supremum, and

$$\mathbf{CLat} \quad \text{resp.} \quad \mathbf{ALat}$$

will be the categories of continuous resp. algebraic lattices. In any of the categories, if it is not automatic, the subscript 0 or 1 will indicate the existence of a least or greatest element. Thus e.g. $\mathbf{BADom}_0$ is the category of Scott domains ([10], [1]).

**4.2.** An ld-set $(X,\prec)$ is *strongly interpolative* if for every $a \prec b$ we can interpolate $a \prec c \prec c \prec b$.

**4.2.1. Lemma.** *1. In $\Im(X,\prec)$, $J \ll J$ iff $J = \prec a$ for some $a \prec a$.*
*2. If $K \ll J$ in $\Im(X,\prec)$ then we can choose $a \prec b \in J$ such that $K \ll \prec a \ll \prec b \ll J$.*

*Proof.* 1. If $a \in J \subseteq \prec a$ then $a \prec a$. 2: $K \subseteq \prec c$ for some $c \in J$. Choose $a, b$ such that $c \prec a \prec b \in J$. □

**4.2.2. Proposition.** *The following statements are equivalent.*

(1) *$(X,\prec)$ is strongly interpolative.*
(2) *$\Im(X,\prec)$ is strongly interpolative.*
(3) *$\Im(X,\prec)$ is algebraic.*

*Proof.* (1)⇒(2): Take the $a, b$ from 4.2.1.2 and interpolate $a \prec c \prec c \prec b$.
(2)⇒(3) is trivial.
(3)⇒(1): Let $a \prec b$. By 4.2.1, $a \in \prec b = \bigsqcup^{\uparrow}\{\prec c \mid \ \prec c \subseteq \prec b,\ c \prec c\}$. Thus, $a \prec c \prec c \prec b$ for some $c$. □

**4.2.3. Proposition.** *An ld-set $(X,\prec)$ is strongly interpolative iff it is, in* **Infos***, isomorphic to a reflexive one.*

*Proof.* Let $(X,\prec)$ be strongly interpolative. Set $Y = \{x \mid x \prec x\}$ and define $f : (X,\prec) \multimap (Y,\prec)$, $g : (Y,\prec) \multimap (X,\prec)$ ($\prec$ in $Y$ as in $X$) by seting $yfx$ iff $y \prec x$, and $xgy$ iff $x \prec y$. It is easy to check that $f$ and $g$ are approximable maps and, interpolating, that $g \circ f = \prec$ and $f \circ g = \prec$.

Now let $(Y,\leq)$ be reflexive and $f : (X,\prec) \multimap (Y,\leq)$, $g : (Y,\leq) \multimap (X,\prec)$ mutually inverse. Let $x \prec x'$. then $xgyfx'$ for some $y$ and as $y \leq y$ we have a $z$ such that $yfzgy$. Then $xgyfzgyfzgyfx'$, that is, $x \prec z \prec z \prec x'$. □

**4.3. Lemma.** *$(X,\prec)$ is isomorphic to $\Im(X,\prec)$ in* **Infos***.*

*Proof.* Define $f : (X,\prec) \multimap \Im(X,\prec)$ and $g : \Im(X,\prec) \multimap (X,\prec)$ by setting

$$JfX \equiv J \ll \prec x \quad \text{and} \quad xgJ \equiv x \in J.$$

Checking that $f$ and $g$ are approximable maps is straightforward. Obviously $g \circ f \subseteq \prec$ and $f \circ g \subseteq \ll$. Now if $x \prec y$ interpolate $x \prec z \prec y$ to obtain $xg(\prec z)fy$, and if $K \ll J$ choose $x, y \in J$ such that $K \subseteq \prec x$ and $y \prec x$ to obtain $KfygJ$. □

**4.4.** An ld-set is said to have *conditional joins* if there is a partial associative and commutative operation $x \sqcup y$ defined whenever there is a $z$ such that $x, y \prec z$, satisfying the condition

$$x_1 \sqcup x_2 \prec x \quad \text{iff} \quad x_i \prec x \text{ for both } i = 1, 2.$$

**Note.** If we assume the axiom of choice we can easily see that it suffices to assume that for any two $x_1, x_2$ such that there is a $z$ with $x_1, x_2 \prec z$ there is an $\overline{x}$ such that $\overline{x} \prec x$ iff $x_i \prec x$ for both $i = 1, 2$.

**4.4.1. Proposition.** *The following statements are equivalent.*

(1) $(X, \prec)$ *is isomorphic in* **Infos** *to an ld-set with conditional joins.*
(2) $\mathfrak{I}(X, \prec)$ *has conditional joins.*
(3) $\mathfrak{I}(X, \prec)$ *is in* **BCDom**.

*Proof.* (1)$\Rightarrow$(2). We can assume that $(X, \prec)$ itself has conditional joins. Obviously, if $J$ is an ideal and $x_i \in J$ then $x_i \sqcup x_2$ exists and is in $J$. Let $J_1, J_2 \ll K$. Set $J = \{x \mid x \prec x_1 \sqcup x_2 \text{ for some } x_i \in J_i\}$. Then $J$ is an ideal (if $x^i \prec x_1^i \sqcup x_2^i$, $x_j^i \in J_j$, we have $x^1, x^2 \prec (x_1^1 \sqcup x_1^2) \sqcup (x_2^1 \sqcup x_2^2)$). Obviously $J_1, J_2 \subseteq J$, and if $J_1, J_2 \subseteq J'$ then $J \subseteq J'$.
(2)$\Rightarrow$(3) is trivial.
(3)$\Rightarrow$(1) follows from 4.3. $\square$

**4.5.** For an ld-set $(X, \prec)$ set $X^{\prec} = \{x \in X \mid x \prec y \text{ for some } y \in X\}$. Then

$$(X^{\prec}, \prec) \textit{ is isomorphic in } \textbf{Infos} \textit{ with } (X, \prec).$$

(Indeed, the restrictions of $\prec$ to $X^{\prec} \times X$ and $X \times X^{\prec}$ provide mutually inverse approximable maps.)
An ld-set $(X, \prec)$ is said to be *directed* if $X^{\prec}$ is directed in $\prec$.

**Proposition.** *The following statements are equivalent.*

(1) $(X, \prec)$ *is isomorphic in* **Infos** *to a directed ld-set.*
(2) $\mathfrak{I}(X, \prec)$ *is directed.*
(3) $\mathfrak{I}(X, \prec)$ *has a largest element.*

*Proof.* (1)$\Rightarrow$(3). If $(X, \prec)$ is directed then $X^{\prec}$ is the largest element in $\mathfrak{I}(X, \prec)$.
(3)$\Rightarrow$(2). Let $J_1, J_2$ be in $\mathfrak{I}(X, \prec)^{\ll}$, $J_i \ll J_i'$. then we have $J_i' \subseteq M$, the largest ideal, hence $J_i \ll M$ and hence there is a $J$ such that $J_i \ll J \ll M$.
(2)$\Rightarrow$(1) follows from 4.3. $\square$

**4.6. Proposition.** *$\mathfrak{I}(X,\prec)$ has a least element iff $(X,\prec)$ is isomorphic in* **Infos** *to an ld-set with a least element.*

*Proof.* If $a$ is least in $(X,\prec)$ then $\prec a$ is least in $\mathfrak{I}(X,\prec)$. Use 4.3. $\square$

**4.7.** Full subcategories of **Infos** resp **LDir** determined by the conditions above will be indicated by putting corresponding symbols into brackets, as follows:

- $\sqcup$ for conditional joins,
- SI for strong interpolation,
- R for reflexivity,
- D for directedness, and
- O for the existence of the least element.

(Thus, for instance $\mathbf{LDir}(\sqcup,\mathrm{D})$ is generated by the directed ld-sets admitting conditional joins.)

Summarizing the results above we obtain the following identities and equivalences.

$$\mathbf{Infos}(\mathrm{O}) \;\equiv\; \mathbf{LDir}(\mathrm{O})_{\mathbb{I}} \;\cong\; \mathbf{CDom}_0,$$
$$\mathbf{Infos}(\mathrm{D}) \;\equiv\; \mathbf{LDir}(\mathrm{D})_{\mathbb{I}} \;\cong\; \mathbf{CDom}_1,$$
$$\mathbf{Infos}(\mathrm{O,D}) \;\equiv\; \mathbf{LDir}(\mathrm{O,D})_{\mathbb{I}} \;\cong\; \mathbf{CDom}_{01},$$
$$\mathbf{Infos}(\mathrm{SI}) \;\equiv\; \mathbf{LDir}(\mathrm{SI})_{\mathbb{I}} \;\cong\; \mathbf{ADom},$$
$$\mathbf{Infos}(\sqcup) \;\equiv\; \mathbf{LDir}(\sqcup)_{\mathbb{I}} \;\cong\; \mathbf{BCDom},$$
$$\mathbf{Infos}(\mathrm{SI},\sqcup) \;\equiv\; \mathbf{LDir}(\mathrm{SI},\sqcup)_{\mathbb{I}} \;\cong\; \mathbf{BADom},$$
$$\mathbf{Infos}(\mathrm{SI},\sqcup,\mathrm{O}) \;\equiv\; \mathbf{LDir}(\mathrm{SI},\sqcup,\mathrm{O})_{\mathbb{I}} \;\cong\; \mathbf{BADom}_0 \text{ (the Scott domains)},$$
$$\mathbf{Infos}(\sqcup,\mathrm{D},\mathrm{O}) \;\equiv\; \mathbf{LDir}(\sqcup,\mathrm{D},\mathrm{O})_{\mathbb{I}} \;\cong\; \mathbf{CLat} \text{ (cf. [3])},$$
$$\mathbf{Infos}(\sqcup,\mathrm{D},\mathrm{SI},\mathrm{O}) \;\equiv\; \mathbf{LDir}(\sqcup,\mathrm{D},\mathrm{SI},\mathrm{O})_{\mathbb{I}} \;\cong\; \mathbf{ALat},$$

and similarly for a few other combinations. Replacing SI by R we obtain $\mathbf{Infos}(\mathrm{R},\dots) \;\cong\; \mathbf{Infos}(\mathrm{SI},\dots)$ (recall 4.2.3) but $\mathbf{Infos}(\mathrm{R},\dots)$ is not any more a Kleisli category under a restriction of $\mathbb{I}$.

**4.7.1.** The proof of 3.5 works under any of the restrictions given by the properties $\sqcup$, SI, D or O. Consequently we have

**Observation.** *For any combination $\kappa$ of $\sqcup$, SI, D or O,*

$$\mathbf{LDir}(\kappa)_{\mathbb{I}} \;\cong\; \mathbf{LDir}(\kappa)^{\mathbb{I}}.$$

**4.7.2.** If $(X,\leq)$ is a continuous domain then

- $(X, \ll)$ has a least element iff $(X, \leq)$ has one,
- $(X, \ll)$ is directed iff $(X, \leq)$ has a greatest element,
- $(X, \ll)$ is strongly interpolative iff $(X, \leq)$ is algebraic, and
- $(X, \ll)$ has conditional joins iff $(X, \leq)$ has suprema of bounded couples.

All these statements are obvious (as for the last one, $\Leftarrow$ is trivial and for $\Rightarrow$ construct $a_1 \vee a_2$ as $\bigvee^{\uparrow}\{x_1 \sqcup x_2 \mid x_i \ll a_i\}$).

Thus, we can add to the summary above also the identities

$$\begin{aligned}
&\mathbf{CDom}_0 \equiv \mathbf{LDir}(\mathrm{O})^{\mathbb{I}}, && \mathbf{CDom}_1 \equiv \mathbf{LDir}(\mathrm{D})^{\mathbb{I}},\\
&\mathbf{ADom} \equiv \mathbf{LDir}(\mathrm{SI})^{\mathbb{I}}, && \mathbf{BCDom} \equiv \mathbf{LDir}(\sqcup)^{\mathbb{I}},\\
&\mathbf{BADom} \equiv \mathbf{LDir}(\mathrm{SI},\sqcup)^{\mathbb{I}}, && \mathbf{BADom}_0 \equiv \mathbf{LDir}(\mathrm{SI},\sqcup,\mathrm{O})^{\mathbb{I}},\\
&\mathbf{CLat} \equiv \mathbf{LDir}(\sqcup,\mathrm{D},\mathrm{O})^{\mathbb{I}}, && \mathbf{ALat} \equiv \mathbf{LDir}(\sqcup,\mathrm{D},\mathrm{SI},\mathrm{O})^{\mathbb{I}},
\end{aligned}$$

etc.

**4.7.3 Note.** In [2], the category of Scott domain $\mathbf{BADom}_0$ has been represented as another Kleisli category, namely that of the down-set monad $\mathbb{D}$ on the category **SLat** of bounded semilattices. This is, however, an entirely different situation; the corresponding Eilenberg-Moore category $\mathbf{SLat}^{\mathbb{D}}$ is the category of frames and not every frame is free.

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