On-line and off-line distance constrained labeling of disk graphs *

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Abstract

A disk graph is the intersection graph of a set of disks in the plane. We consider the problem of assigning labels to vertices of a disk graph satisfying a sequence of distance constrains. Our objective is to minimize the distance between the least and the largest labels. We propose an on-line labeling algorithm on disk graphs, if the maximum and minimum diameters are bounded. We give the upper and lower bounds on its competitive ratio, and show that the algorithm is

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*An extended abstract of this paper was presented at ESA'01. The work of JF is partially supported by EU ARACNE project HPRN-CT-1999-00112, by GAČR 201/99/0242 and by the Ministry of Education of the Czech Republic as project LN00A056. AVF acknowledges support by the DFG-Graduiertenkolleg "Effiziente Algorithmen und Mehrskalenmethoden". Part of FVF's work on this paper was done while he was a visiting postdoc at DIMATIA-ITI partially supported by GAČR 201/99/0242 and by the Ministry of Education of the Czech Republic as project LN00A056. FVF also acknowledges support by EC contract IST-1999-14186, Project ALCOM-FT (Algorithms and Complexity - Future Technologies)

asymptotically optimal. In more detail we explore the case of distance constraints $(2,1)$, and present two off-line approximation algorithms. The last one we call robust, i.e. it does not require the disks representation and either outputs a feasible labeling, or answers the input is not a unit disk graph.

1 Introduction

The *frequency assignment* problem is a general framework which focuses on the point-to-point communication, e.g. in *radio* or *mobile telephony* networks. One of its main threads asks for an assignment of frequencies to transmitters while avoiding interference. Such a situation may appear when signals of the same or similar frequency are used in the same location. From another side, due to very high cost of the frequency spectrum, the assignment should use as less frequencies as possible.

Another important aspect of frequency problem concerns the fact that modern communication networks tend to grow, reflecting fast increase in the number of transmitters. This means that the corresponding transmitter systems should be flexible to possible changes, e.g. installing or disinstalling of a group of transmitters. Hence, an assignment of frequencies to new transmitters should not lead to major changes in already existing transmitter system, and should not decrease quality of communication. Thus, an on-line algorithm would be a good approach.

The most common model for an instance of frequency assignment is the *interference graph*. Each vertex of the interference graph represents a transmitter. If simultaneous broadcasting of two transmitters may cause an interference, then they are connected by an edge in the interference graph. The level of interference depends on the assigned frequencies. In the majority of research works, interference graphs are assumed to have a special structure, e.g. planar graphs, or grids [25]. However, there exists a very natural way to model the situation. One can associate the coverage area of a transmitter with a disk of a particular diameter, and afterwards, to model the interference graph as a graph whose edges connect transmitters at "close" distance, e.g. when their scopes intersect. Then, the underlying interference graph is a *disk graph* , that is the intersection graph of disks in the plane, or, sometimes, a *unit disk graph*, when all disks are of the same diameter [15]. The class of disk graphs is more general than the class of planar graphs, since every planar graph is a *coin graph*, i.e. the intersection graph of interior-disjoint disks [17].

Regarding the assumption that a pair of "close" transmitters should be assigned different frequencies, the frequency assignment is equivalent to the problem of coloring the interference graph. However, in [15] it was observed that the signal propagation may affect the interference even in distant regions (but with decreasing intensity). Hence, not only "close" transmitters should get different frequencies, but also frequencies used at some distance should be appropriately separated. In this case, the frequency assignment can be modeled as the problem of *distance constrained labeling*, or so called $L_{(p_1,\ldots,p_k)}$-*labeling* of the interference graph [18]:

Definition *Let* $p_1, \ldots, p_k$ *be a sequence of positive integers called* distance constraints. *The* $L_{(p_1,\ldots,p_k)}$-*labeling of a graph* G *is a mapping* $c\colon V(G) \to \{1, \ldots, \lambda\}$ *such that the following holds:*

$$\forall i : 1 \leq i \leq k, \forall u, v \in V(G) : \mathrm{dist}_G(u, v) \leq i \Rightarrow |c(u) - c(v)| \geq p_i.$$

The minimum number λ for which an $L_{(p_1,\ldots,p_k)}$-labeling of G exists, is denoted by $\chi_{(p_1,\ldots,p_k)}(G)$. Notice that $\chi_{(1)}(G) = \chi(G)$, where $\chi(G)$ is the chromatic number of G. Also for $p_1 = p_2 = \cdots = p_k = 1$, $\chi_{(p_1,\ldots,p_k)}(G) = \chi(G^k)$, where G^k is the k-th power of G, i.e. a graph which arise from G by adding edges connecting vertices at distance at most k.

Related works on graph labelings. In the last few years, an important amount of work has been devoted to the study of distance labeling for general graphs and its relationship to graph coloring.

First observe [13, 9], that for any integer t

$$\chi_{(tp_1,\ldots,tp_k)}(G) = t \cdot \chi_{(p_1,\ldots,p_k)}(G) - t + 1.$$

Hence, it can be assumed that parameters $p_1, \ldots, p_k$ have no common divisor. Moreover we can bound

$$\chi_{(p_1,\ldots,p_k)}(G) \leq \chi_{(p_1,\ldots,p_1)}(G) = p_1\chi_{(1,\ldots,1)}(G) - p_1 + 1 = p_1\chi_{(1)}(G^k) - p_1 + 1,$$

where $\chi_{(1)}$ is the standard chromatic number and G^k is the k-th power of G.

Regarding the computational complexity, finding the value of $\chi_{(1)}(G^k)$ is an *NP*-hard problem as well as for the ordinary chromatic number [22]. In the same paper it has been shown that First-fit coloring algorithm is

$O(\log n)$-competitive for the second power of a general graph. If we restrict ourselves to the case of planar graphs, it is known that the coloring of the second power of planar graph is an *NP*-hard problem, but there exist approximation algorithms with a small multiplicative factor. The long standing conjecture due to Wegner [26] is is that for any planar graph G with maximum degre $\Delta \geq 8$, the chromatic number of the square G^2 is at least $\lceil \frac{3}{2}\Delta \rceil + 1$. There is a number of consecutive papers with results coming closer and closer to the conjectured bound. The current champion (to our knowledge) is the bound of Molloy & Salavatipour $\chi(G^2) \leq \frac{5}{3}\Delta + 78$ [21]. We refer the interested reader to the paper of Molloy & Salavatipour [21] for a historical overview.

The most intensively studied case of distance labeling is $k = 2$ and the distance constraints $(p_1, p_2) = (2, 1)$. The existence of an $L_{(2,1)}$-labeling was explored for different graph classes in [3, 5, 12, 13, 25]. The exact value of $\chi_{(2,1)}$ can be derived for cycles and paths, and there are polynomial algorithms which compute the value $\chi_{(2,1)}$ for trees and co-graphs [5]. The problem of recognizing graphs such that $\chi_{(2,1)} \leq \lambda$ is *NP*-complete for all fixed $\lambda \geq 4$ [10]. For planar graphs, the problem of deciding $\chi_{(2,1)} \leq 9$ was shown to be *NP*-complete in [3]. The best known so far approximation algorithm is the algorithm of Molloy & Salavatipour [21] producing $L_{(p_1,p_2)}$ labeling with the largest label at most $\frac{5}{3}(2p_2 - 1)\Delta + 12p_1 + 144p_2 - 78$.

It is expected that for every k-tuple of distance constraints $(p_1, \ldots, p_k)$ there exists a bound λ_0 such that for every $\lambda \geq \lambda_0$ the decision problem $\chi_{(p_1,\ldots,p_k)}(G) \leq \lambda$ is *NP*-complete for general graphs. So far this conjecture has been only proved for $k = 2$ and $p_1 \geq 2p_2$ [8]. We follow this expectation and show an approximate solution to the labeling problem. Our motivation is to provide a positive (partial) answer to a (possibly) *NP*-hard optimization problem.

Related works on disk graphs. The classes of disk graphs and unit disk graphs are interesting by their own (from the computational point of view), since already the disk graph and unit disk graph recognition problems are *NP*-hard [4, 16]. Hence, algorithms that require the corresponding disk representation are substantially weaker than those which work only with graphs. On the other hand, for real radio or mobile telephony networks, the interference graphs are built with respect to the geographical placement of transmitters and the disk representation can be easily derived.

It is shown in [6] that the 3-coloring of unit disk graph is *NP*-complete (even if the input is a set of unit disks). However it is well known that the chromatic number of general graphs cannot be approximated within a constant factor [2], for unit disk graph coloring problem there is a 5-competitive on-line algorithm [19, 23, 7], and in the case of given disks representation, there is a 3-approximation off-line algorithm [4, 23]. Furthermore, there is a 5-approximate off-line algorithm for the disk graph coloring problem, which uses the First-fit technique on disks that are ordered from those with the biggest diameter to the smallest one [19]. From another side, one cannot expect the existence of an on-line coloring algorithm of a constant competitive ratio in the case of general disk graphs, since there is no such one even for planar graphs [14], and every planar graph is also a disk graph [17].

Our results. For fixed distance constraints $(p_1, \ldots, p_k)$ and fixed diameter ratio σ we present a constant-competitive on-line labeling algorithm which require the disk representation. This is the first on-line labeling algorithm known to our knowledge. We prove that when the representation is not given or when the diameter ratio is not bounded then no on-line algorithm with fixed competitive ratio exists.

For our algorithm we obtain upper and lower bounds on its competitive ratio, and show that in the case of unit disk graphs the algorithm is asymptotically optimal.

Finally, we explore the case of distance constraints $(2, 1)$, and present two off-line approximation algorithms for unit disk graph labeling. One of these two labeling algorithms is robust, i.e. an algorithm that does not require the disk representation and either outputs a feasible labeling, or answers the input is not a unit disk graph.

The following table summarizes the known and new upper bounds on the competitive or performance ratio on coloring and labeling problems on unit disk graphs (UDG), on disk graphs with diameter ratio bounded by σ (DG$_\sigma$) and on general disk graphs (DG).

Execution	Off-line		On-line	
Repres.	+	−	+	−
Coloring:				
UDG	3 [23]	3^1	5 [19, 23]	5^1
DG$_\sigma$	5 (as DG)	5 (as DG)	Theorem 4 [∗]	YES [7]
DG	5 [19]	5^1	NO [7]	NO [14]
$L_{(2,1)}$-labeling:				
UDG ^{2}	12 → 9 [∗]	10.6 → 10 [∗]	25 → 12.5 [∗]	NO [∗]
$L_{(p_1,\ldots,p_k)}$-labeling:				
UDG	Th. 4^4, C. 11^3 [∗]	Th. 16, C. 14^3 [∗]	Theorem 4 [∗]	NO [∗]
DG$_\sigma$	Theorem 4^4 [∗]	Theorem 16 [∗]	Theorem 4 [∗]	NO [∗]
DG	?	?	NO [∗]	NO [∗]

Positive results are marked either by "YES" or by the appropriate statement. "NO" means that no algorithm with fixed performance/competitive ratio exists. The sign "?" marks an open problem. The results presented in this paper are highlighted by "[∗]". The list of further explanations follows:

[1] The algorithm working without the disk representation can be derived from those which uses the representation as shown in Section 4.2.

[2] This rows shows the values of the upper bound in the worst case. Since our results are better for graphs with large cliques, we give also the limit of the upper bound as the clique size grows to infinity.

[3] Only for $k = 2, (p_1, p_2) = (2, 1)$.

[4] Every on-line algorithm can be executed off-line.

This paper is organized as follows: Next section introduces notation used later in the paper. The third section presents the on-line labeling algorithm considering general parameters $p_1, \ldots, p_k$ together with the lower and upper bounds on the competitive ratio. In the fourth section we consider the off-line $L_{(2,1)}$-labeling problem. In the last section some open problems and possible directions for further research are discussed.

2 Preliminaries

A *graph* G consists of a finite set of vertices V and of a set E of unordered pairs of vertices, which are called edges E. By this definition, graphs are simple, undirected and finite.

A *clique* C of a graph $G = (V, E)$ is a subset of V such that all the vertices of C are pairwise adjacent. A subset of vertices $I \subseteq V$ is *independent* if no two of its elements are adjacent. We denote by $\omega(G)$ the maximum number of vertices in a clique of G.

For a set of geometric objects, the corresponding *intersection graph* is the undirected graph whose vertices are objects and two vertices are adjacent if the corresponding objects intersect. Let $\mathcal{E}$ be a 2-dimensional Euclidean plane with the coordinates x, y. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a set of n disks in $\mathcal{E}$, where each D_i is uniquely determined by its center in (x_i, y_i) and by the diameter $d_i \in \mathbb{R}_+$. The intersection graph $G_\mathcal{D}$ of the set $\mathcal{D}$ is called a *disk graph*, more formally $V(G_\mathcal{D}) = \mathcal{D}, E(G_\mathcal{D}) = \{(D_i, D_j) \in \binom{\mathcal{D}}{2} : D_i \cap D_j \neq \emptyset\}$. When all disks of $\mathcal{D}$ have unit diameter, i.e. $d_i = 1$ for all $D_i \in \mathcal{D}$, then $G_\mathcal{D}$

is called a *unit disk graph*. In both cases, $\mathcal{D}$ is called the *disk representation* of $G_{\mathcal{D}}$. The value $\sigma(\mathcal{D}) = \frac{\max d_i}{\min d_i}$ is called the *diameter ratio* of $\mathcal{D}$.

We say that an algorithm $\mathcal{A}$ is a *ρ-approximation off-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm* if for a given graph $G_{\mathcal{D}}$ it runs in polynomial time and outputs an $L_{(p_1,\ldots,p_k)}$-labeling of $G_{\mathcal{D}}$ such that the maximum label used by $\mathcal{A}$ is at most $\rho\chi_{(p_1,\ldots,p_k)}(G_{\mathcal{D}})$. The value ρ is called the *approximation ratio* of $\mathcal{A}$.

We say that an algorithm $\mathcal{B}$ is an *on-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm* if it labels the vertices of a graph $G_{\mathcal{D}}$ in an externally determined sequence $D_1 \prec \cdots \prec D_n$. At the time t the algorithm $\mathcal{B}$ has to irrevocably assign a label to D_t, while it has been given only the edges connecting vertices $D_1, \ldots, D_t$. Such an algorithm $\mathcal{B}$ is a *ρ-competitive on-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm* if for every graph $G_{\mathcal{D}}$ and any ordering $\prec$ on $V(G_{\mathcal{D}})$ it always outputs an $L_{(p_1,\ldots,p_k)}$-labeling with the maximum label at most $\rho\chi_{(p_1,\ldots,p_k)}(G_{\mathcal{D}})$. We call the constant ρ the *competitive ratio* of $\mathcal{B}$.

Our on-line algorithm is based on the following partition of the plane $\mathcal{E}$. We call a *hexagonal tiling* on $\mathcal{E}$ the partition of the plane into the set $\mathcal{C}$ of hexagonal cells of diameter one. Each cell $C_{i,j} \in \mathcal{C}$ is characterized by two integer coordinates i, j, and is a simplex delimited by the following lines:

$$\begin{array}{rcl} 2i - j - 1 < & \frac{4}{3}\sqrt{3}x & \leq 2i - j + 1 \\ i + j - 1 < & \frac{2}{3}(\sqrt{3}x + 3y) & \leq i + j + 1 \\ -i + 2j - 1 < & \frac{2}{3}(-\sqrt{3}x + 3y) & \leq -i + 2j + 1. \end{array}$$

Each point of the plane belongs to exactly one cell $C_{i,j}$. Observe also, that each cell contains exactly two adjacent corners of the bounding hexagon, and the distance between every two points inside the same cell is at most one (see Figure 2).

We denote the *plane distance* of two points $p, p' \in \mathcal{E}$ by $\operatorname{dist}_{\mathcal{E}}(p, p')$. Similarly the plane distance of cells $C_{i,j}$ and $C_{k,l}$ is defined as

$$\operatorname{dist}_{\mathcal{E}}(C_{i,j}, C_{k,l}) = \inf\{\operatorname{dist}_{\mathcal{E}}(p, p') : p \in C_{i,j}, p' \in C_{k,l}\}.$$

3 On-line distance constrained labeling

In this section we explore the on-line distance constrained labeling problem for disk graphs.

We start with the following observation. Let $(p_1, \ldots, p_k)$ be a sequence of distance constraints ($k \geq 2$ and $p_2 \geq 1$) and let $\mathcal{B}$ be an arbitrary on-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm for disk graphs. Let $\mathcal{D}$ be a set of mutually disjoint disks, i.e. $G_{\mathcal{D}}$ has no edges.

Consider the following two cases. First assume that the disk representation is not a part of the input. Secondly, $\mathcal{B}$ the disk representation is given as a part of the input, but the diameter ratio $\sigma(\mathcal{D})$ is not known during the execution of $\mathcal{B}$.

In both cases, the algorithm $\mathcal{B}$ labels all vertices of $G_{\mathcal{D}}$ by distinct labels. Otherwise any new vertex may create a path of length two between an arbitrary pair of already labeled vertices (in the second case by extending $\mathcal{D}$ by a disk of large diameter). The maximal label used by $\mathcal{B}$ on $G_{\mathcal{D}}$ is at least $|\mathcal{D}|$, although $\chi_{(p_1,\ldots,p_k)}(G_{\mathcal{D}}) = 1$.

Due to the above arguments, we consider the case when the disk representation $\mathcal{D}$ of $G_{\mathcal{D}}$ is a part of the input and when the corresponding diameter ratio $\sigma(\mathcal{D})$ is known, i.e. it is bounded by a constant.

First we introduce a special circular labeling of cells and show how our on-line labeling algorithm uses such labeling. Later we derive the upper bound on its competitive ratio and show that the algorithm is asymptotically optimal for the class of unit disk graphs with at least one edge. Finally, we illustrate the algorithm performance for distance constraints $(2, 1)$.

3.1 Circular labeling

Let $\mathcal{D}$ be a set of disks with the diameter ratio $\sigma(\mathcal{D})$ and $G_{\mathcal{D}}$ be the corresponding disk graph. We suppose that the coordinates of plane $\mathcal{E}$ are scaled such that the smallest disk has the unit diameter. Moreover, through this section we assume that the diameter ratio σ and distance constraints $(p_1, \ldots, p_k)$ are fixed, i.e. for every selection of these parameters we design a specific algorithm.

Let $\mathcal{C}$ be the set of hexagonal cells. We say that a mapping $\varphi : \mathcal{C} \to \{1, 2, \ldots, l\}$ is a *circular l-labeling of $\mathcal{C}$* (with respect to $(p_1, \ldots, p_k)$ and $\sigma \geq 1$) if

$$dist_{\mathcal{E}}(C, C') \leq i \cdot \sigma \Rightarrow \min\{|\varphi(C) - \varphi(C')|, l - |\varphi(C) - \varphi(C')|\} \geq p_i,$$

for all cells $C, C' \in \mathcal{C}$ and all $i \in \{1, \ldots, k\}$.

An example of a circular 25-labeling (with respect to $(p_1, p_2) = (2, 1), \sigma = \frac{\sqrt{7}}{2}$) is depicted in Figure 1. In fact, the cells with equal labels are at plane

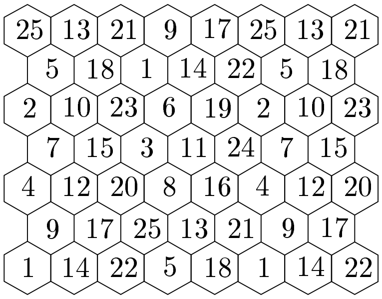
25 13 21 9 17 25 13 21
5 18 1 14 22 5 18
2 10 23 6 19 2 10 23
7 15 3 11 24 7 15
4 12 20 8 16 4 12 20
9 17 25 13 21 9 17
1 14 22 5 18 1 14 22

Figure 1: An example of a 25-circular labeling ($(p_1, p_2) = (2, 1), \sigma = \frac{\sqrt{7}}{2}$)

distance at least $2\sqrt{3}$ and those pairs labeled by consecutive labels have plane distance at least $\frac{\sqrt{7}}{2} \doteq 1.32$.

For general case, the existence of such labelings is guaranteed by the following theorem.

Theorem 1 *For every k-tuple of distance constraints $(p_1, \ldots, p_k)$, every $\sigma \geq 1$ and*

$$l^* = 1 + 6\Big(2p_1 - 1 + \sum_{i=2}^{\lfloor \frac{4k\sigma+4}{3} \rfloor} i \cdot (2p_{\lceil \frac{3i-4}{4\sigma} \rceil} - 1)\Big),$$

there exists a circular l^*-labeling of $\mathcal{C}$ and this labeling can be found in $O(l^* \sigma^4 k^4)$ time.

Proof: For the purposes of the proof we will represent cells as vertices of an infinite triangular mesh M shown in Figure 2, where edges connect adjacent cells. For any pair of cells C and C', $\text{dist}_M(C, C')$ is the *mesh distance* measured as the number of edges of the shortest path connecting the vertices corresponding to C and C' in M.

For $a = \lceil \frac{2k\sigma}{\sqrt{3}} \rceil$ the cells $C_{i,j}$, $C_{i+a+1,j}$, $C_{i,j+a+1}$ and $C_{i+a+1,j+a+1}$ have pairwise plane distance greater than $k\sigma$ and can be labeled by the same number. By this method, we have to define a suitable labeling of a^2 cells $C_{i,j}$ with coordinates $i, j \in \{1, .., a\}$.

We order these a^2 cells arbitrarily and label them by First-fit method, with respect to labels of adjoin cells shifted by a.

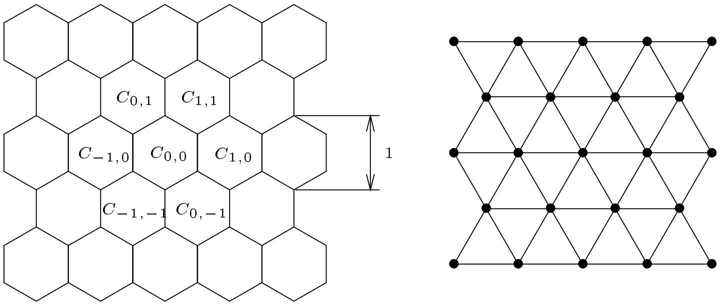

Figure 2: The hexagonal tiling and the corresponding mesh M

Consider that we have to select a label for a cell C. For any $i \geq 1$, there exist exactly $6i$ cells that are at mesh distance i from C, and their *plane distance* from C is at least $\frac{3i-4}{4}$ (it is nonnegative number for $i \geq 2$.) The maximum i we consider is $\lfloor \frac{4k\sigma+4}{3} \rfloor$, because for higher values of i the plane distance become greater than $k\sigma$ and even if some of these cells are already labeled, this cannot influent the First-fit assignment of the label for C.

If any cell at mesh distance i from C is already labeled, then by the definition of the circular l-labeling, at most $2p_{\lceil \frac{3i-4}{4\sigma} \rceil} - 1$ numbers are unavailable for C.

In the worst case at most

$$6\Big(2p_1 - 1 + \sum_{i=2}^{\lfloor \frac{4k\sigma+4}{3} \rfloor} i \cdot (2p_{\lceil \frac{3i-4}{4\sigma} \rceil} - 1)\Big) = l^* - 1$$

numbers are forbidden as a label for C. Hence, among l^* distinct labels there exists at least one feasible candidate.

The time complexity follows from the fact that for each of $a^2 = O(\sigma^2 k^2)$ cells we have to consider at most $a^2 = O(\sigma^2 k^2)$ labelled cells and test feasibility of at most $O(l^*)$ labels. $\square$

3.2 The algorithm

Assume now that the parameters of algorithm, i.e. distance constraints $(p_1, \ldots, p_k)$ and diameter ratio $\sigma \geq 1$ are fixed.

Algorithm On-line disk labeling (ODL)
Input: Set of disks $\mathcal{D}$, in an arbitrarily order $D_1 \prec \cdots \prec D_n$.
Output: An $L_{(p_1,\dots,p_k)}$-labeling $c : \mathcal{D} \to \mathbb{Z}^+$ of $G_{\mathcal{D}}$.

```
1. Find a circular $l$-labeling $\varphi : \mathcal{C} \to \{1, \dots, l\}$
(with respect to $(p_1, \dots, p_k)$ and $\sigma$).
2. For each cell $C_{i,j}$ put $\mathcal{D}_{i,j} = \emptyset$.
($\mathcal{D}_{i,j}$ is the set of already labeled disks of $\mathcal{D}$ with centers in a cell
$C_{i,j} \in \mathcal{C}$.)
3. For each $k$ from 1 to $n$:
    3a. Decide into which cell $C_{i,j}$ the disk $D_k$ belongs.
    3b. Define the label $c(D_k) = \varphi(C_{i,j}) + l \cdot |\mathcal{D}_{i,j}|$.
    3c. Set $\mathcal{D}_{i,j} = \mathcal{D}_{i,j} \cup \{D_k\}$.
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An implemention of the algorithm ODL in Java could be found at [20].

The next lemma follows directly from the properties of circular l-labeling.

Lemma 2 *Suppose that at the first step the algorithm ODL finds a circular l-labeling (with respect to $(p_1, \dots, p_k)$ and σ). Then for any set of disks $\mathcal{D}$ of diameter ratio $\sigma(\mathcal{D}) \leq \sigma$, the algorithm ODL produces a feasible $L_{(p_1,\dots,p_k)}$-labeling of $G_{\mathcal{D}}$ and the maximal label used by this algorithm is at most $l\omega(G_{\mathcal{D}})$.*

The analysis of our algorithms is based on the following easy fact: If a set of vertices X induces a complete subgraph of a graph G, then the maximal label used for vertices of X in any labeling is at least $p_1(|X| - 1) + 1$.

Lemma 3 *For any k-tuple $(p_1, \dots, p_k)$ of distance constraints and any set of disks $\mathcal{D}$,*

$$\chi_{(p_1,\dots,p_k)}(G_{\mathcal{D}}) \geq p_1(\omega(G_{\mathcal{D}}) - 1) + 1 \geq p_1(\max_{i,j}\{|V(G_{\mathcal{D}_{i,j}})|\} - 1) + 1,$$

where $\omega(G_{\mathcal{D}})$ is the size of the maximum clique in $G_{\mathcal{D}}$.

Proof: Observe that the subgraph $G_{\mathcal{D}_{i,j}}$ of $G_{\mathcal{D}}$ is isomorphic to a complete graph and the number of disks in $|\mathcal{D}_{i,j}|$ is at most $\omega(G_{\mathcal{D}})$. $\square$

Combining Theorem 1 with Lemmas 2 and 3, we get the main result of this section.

Theorem 4 *For every* $(p_1, \ldots, p_k)$ *and every* $\sigma \geq 1$ *the competitive ratio of algorithm ODL is bounded by*

$$\rho \leq \max_{\mathcal{D}} \frac{\omega(G_{\mathcal{D}}) \cdot l^*}{(\omega(G_{\mathcal{D}}) - 1) \cdot p_1 + 1},$$

where the maximum is taken over all sets of disks of diameter ratio at most σ.

Therefore, ODL is an l^*-competitive on-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm for the class of disks graphs $G_{\mathcal{D}}$ of the diameter ratio $\sigma(\mathcal{D}) \leq \sigma$. Moreover, if disk graphs have at least one edge then the competitive ratio $\rho \leq \frac{2l^*}{p_1+1}$, and if $\omega(G_{\mathcal{D}}) \to \infty$ then $\rho \to \frac{l^*}{p_1}$.

If we apply the above theorem on the labeling depicted in Fig. 1, we get:

Corollary 5 *For the distance constraints* $(2, 1)$, *the algorithm ODL is* $\frac{50}{3} \doteq 16.67$-*competitive on disk graphs with at least one edge and* $\sigma \leq 1.32$. *Moreover, the competitive ratio tends to* 12.5 *when* $\omega(G_{\mathcal{D}}) \to \infty$.

3.3 Lower bounds

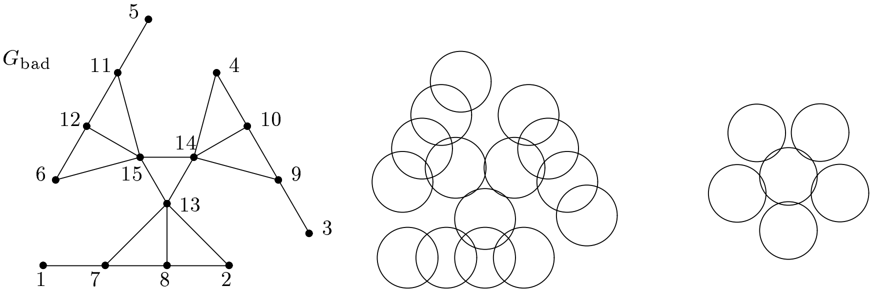
In this section we show several lower bounds of the for on-line coloring and labeling algorithms for unit disk graphs. We start with coloring algorithms.

Observation 6 *For any positive* ε, *there is no* $(2-\varepsilon)$-*competitive algorithm for the unit disk graph coloring problem.*

Proof: Suppose that such algorithm $\mathcal{B}$ exists. Consider the graph G_{bad} depicted in Fig. 3 a) and order the vertices as in the figure.

The vertices 1–6 form an independent set, hence, $\mathcal{B}$ colors them by the same color. Otherwise $\mathcal{B}$ is not $(2-\varepsilon)$-competitive even on graph without edges. Vertices 1–12 form a bipartite graph, and for their proper coloring $\mathcal{B}$ needs exactly two more additional colors. Then vertices 13, 14 and 15 require three extra new colors: They form a triangle, so they cannot share the same color and each of them is adjacent to three vertices among 1–12 that are colored by three distinct colors. In other words $\mathcal{B}$ is forced to use at least six colors on G_{bad}. But the graph is 3-colorable, hence $\mathcal{B}$ is not an $(2-\varepsilon)$-competitive algorithm. ☐

Now we focus our attention on distance labeling algorithms of disk graphs. Our method shows that even an independent set of disks should be



a) b)

Figure 3: Bad instances for online algorithms

colored by many different colors. For example, consider the five outer disks from the graph depicted in Fig. 3 b).

No online labeling algorithm can use the same label twice on these disks, because if the central disk is given later, then there exists no $L_{(2,1)}$-labeling of all six disks. Although all these five disks might be labeled the same, any on-line labeling algorithm needs at least five distinct labels. We conclude that there is no $(5-\varepsilon)$-competitive algorithm for the distance labeling problem for UDG.

Now we present a lower bound on the competitive ratio of any on-line algorithm solving the distance constrained labeling problem for disk graphs, which have a representation of bounded diameter ratio.

Theorem 7 *For any k-tuple $(p_1, \ldots, p_k)$, of distance constraints $k \geq 2$, any $\sigma \geq 1$ and any $\varepsilon > 0$, there is no $(\bar{\rho} - \varepsilon)$-competitive on-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm for the class of disk graphs with representation of diameter ratio at most σ, where*

$$\bar{\rho} = 1 + \frac{\sigma^2}{9} \max_{i=2,\ldots,k} \{i^2 p_i\}.$$

Proof:

Take any $t \in (1, \sqrt{2})$ and define an auxiliary sequence $a_i = \lfloor \frac{(i-1)\sigma+1}{t\sqrt{2}} + 1 \rfloor$ for $i = 2, \ldots, k$.

We define a set $\mathcal{D} = \{D_{1,1}, D_{1,2}, \ldots, D_{a_k,a_k}\}$ of a_k^2 unit disks, where disk $D_{j,l}$ has center (jt, lt) and j, l are integers from the set $\{1, 2, \ldots, a_k\}$. The graph $G_{\mathcal{D}}$ consist of a_k^2 independent vertices.

Now consider disks D and D' with coordinates j, l and j', l' and take the maximum i such that $|j - j'| \leq a_i$ and $|l - l'| \leq a_i$. It is easy to see that D and D' are at plane distance at most $(i-1)\sigma$, and therefore there exists a set $\overline{\mathcal{D}(D, D')}$ of $(i-1)$ disks of diameter σ such that in $G_{\mathcal{D} \cup \overline{\mathcal{D}(D,D')}}$ the vertices $D_{j,l}$ and $D_{j',l'}$ are at distance i.

Let $\overline{\mathcal{D}}$ be the union of $\overline{\mathcal{D}(D, D')}$ over all $D, D' \in \mathcal{D}$. We have showed that for every $i : 2 \leq i \leq k$ the a_i^2 vertices from $\{D_{j,l} : 1 \leq j, l \leq a_i\}$ are in $G_{\mathcal{D} \cup \overline{\mathcal{D}}}$ mutually at distance at most i. Hence any on-line distance labeling algorithm $\mathcal{B}$ needs at least a_i^2 labels that pairwise differ by at least p_i to allow the extension to the labeling from $\mathcal{D}$ to $\mathcal{D} \cup \overline{\mathcal{D}}$

Therefore on the set $\mathcal{D}$ the algorithm $\mathcal{B}$ must use label of size at least

$$1 + \max_{i=2,\ldots,k} \left\{ \left(\left\lfloor \frac{(i-1)\sigma + 1}{t\sqrt{2}} + 1 \right\rfloor^2 - 1 \right) p_i \right\},$$

After setting $t = \frac{3}{2\sqrt{2}}$ and suitable estimating, we get that $\mathcal{B}$ needs a label of size at least $1 + \frac{\sigma^2}{9} \max_{i=2,\ldots,k}\{i^2 p_i\}$, although $\chi_{(p_1,\ldots,p_k)}(G_{\mathcal{D}}) = 1$ $\square$

Theorems 4 and 7 imply that:

Corollary 8 *For any $(p_1, \ldots, p_k)$, $k \geq 2$ and any σ, the competitive ratio of algorithm ODL on disk graph (with given representation, diameter ratio at most σ and with at least one edge) is at most $O(\log k)$ times larger than the competitive ratio of* any *on-line $L_{(p_1,\ldots,p_k)}$-labeling algorithm. Therefore, under these restricted conditions and for a fixed number of distance constraints, the algorithm ODL is asymptotically optimal.*

Proof: First, observe that if the set $\mathcal{D}$ from the proof of Theorem 7 is extended by a pair of intersecting disks (but not intersecting the others), then the same argument applied to this new graph $G_{\mathcal{D}}$ gives that the lower bound on the performance ratio of any on-line labeling algorithm for such disk graphs is at least

$$\frac{1 + \frac{\sigma^2}{9} \max_{i=2,\ldots,k}\{i^2 p_i\}}{1 + p_1} \geq c \frac{\sigma^2 \max_{i=2,\ldots,k}\{i^2 p_i\}}{1 + p_1},$$

and the upper bound given by Theorem 4 is

$$\frac{2+12\Big(2p_1-1+\sum_{i=2}^{\lfloor\frac{4k\sigma+4}{3}\rfloor} i\cdot(2p_{\lceil\frac{3i-4}{4\sigma}\rceil}-1)\Big)}{1+p_1} \le c'\frac{\sigma^2\sum_{i=2}^{k} ip_i}{1+p_1}+O(1).$$

for suitable constants c, c' that do not depend on σ nor on $(p_1, \ldots, p_k)$.

(One term σ appears due to the change in number of elements from $\lfloor\frac{4k\sigma+4}{3}\rfloor$ to k, the other follows from the same change and i inside the sum.)

Let j be the index for which the term i^2p_i attains its maximum, i.e. for all $i \geq 2 : p_i \leq \frac{j^2}{i^2}p_j$.

Then

$$\sum_{i=2}^{k} ip_i \leq \sum_{i=2}^{k} \frac{j^2}{i}p_j = j^2p_j\sum_{i=2}^{k}\frac{1}{i} \leq \max_{i=2,\ldots,k}\{i^2p_i\}\cdot O(\log k)$$

and the statement of Corollary 8 is proved. □

As the last remark we propose a slightly different strategy with a smaller channel separation used for disks in the same location, though it does not lead to a better competitive ratio of the final algorithm. The idea is as follows. First scale the cells of $\mathcal{C}$ such that every cell has diameter σ. Then, find a feasible circular l-labeling $\varphi : \bar{\mathcal{C}} \to \{1, \ldots, l\}$. After that, run the algorithm ODL on the set $\mathcal{D}$ of non-unit disks using the scaled cells. As the result, we find a feasible $L_{(p_1,\ldots,p_k)}$-labeling of the disk graph $G_{\mathcal{D}}$, but the disks belonging to the same cell do not induce a complete graph. On the other hand, the area of any cell of diameter σ can be covered by at most $\sigma^2 + t\sigma$ unit cells, where t is a suitable constant that does not depend on σ. Then the number of vertices in any $G_{\mathcal{D}_{i,j}}$ is at most $\sigma^2 + t\sigma$ times more than the maximum clique size $\omega(G_{\mathcal{D}})$. Thus, it follows that

$$\chi_{(p_1,\ldots,p_k)}(G_{\mathcal{D}_{i,j}}) \geq p_1(\omega(G_{\mathcal{D}_{i,j}})-1) \geq p_1\left(\left\lceil\frac{V(G_{\mathcal{D}_{i,j}})}{\sigma^2+t\sigma}\right\rceil-1\right)$$

and we use the same arguments as above.

4 Off-line labeling

In this section we first explore the off-line distance labeling problem on unit disk graphs only for one particular selection of distance constrains

$(p_1, p_2) = (2, 1)$. We start our discussion with the case when the graph and its disk representation are known. Then we present a robust algorithm which do not require the disk representation. At the end we present a similar labeling algorithm for arbitrary many distance constraints and for the class of disk graph with bounded diameter ratio, however we can not turn it into a robust algorithm.

4.1 The approximation algorithm for unit disks

We first discuss the case that the entire graph $G_{\mathcal{D}}$ is given together with the representation $\mathcal{D}$ as the part of the input. The main idea of the off-line approximation algorithm is rather simple: We "cut" the plane into strips of small width, label unit disk graphs induced by strips and combine the labelings of strip graphs.

An unit disk graph $G_{\mathcal{D}}$ is a $\frac{1}{\sqrt{2}}$*-strip graph* if there is a disk representation $\mathcal{D}$ of $G_{\mathcal{D}}$ such that the centers of disks in $\mathcal{D}$ are in a strip of width $\frac{1}{\sqrt{2}}$. In other words, there is a mapping $f\colon V(G_{\mathcal{D}}) \to \mathbb{R} \times [0, \frac{1}{\sqrt{2}}]$ such that $(u, v) \in E(G_{\mathcal{D}})$ if and only if $dist_{\mathcal{E}}(f(u), f(v)) \leq 1$.

Lemma 9 *Let G be a $\frac{1}{\sqrt{2}}$-strip graph and let v be a vertex such that the unit disk corresponding to v (in some representation $\mathcal{D}$) has the smallest x-coordinate. Then the cardinality of the vertex set*

$$N_{G^2}(v) = \{u \in V(G) - \{v\} : \mathrm{dist}_G(u, v) \leq 2\}$$

is at most $3\omega(G) - 1$.

Proof: A rectangle R with sides 2 and $\frac{1}{\sqrt{2}}$ is covered by three squares with sides $\frac{1}{\sqrt{2}}$. Disks with centers in one square form a clique in G; hence the number of disks with centers in R is at most $3\omega(G)$ and the proof follows. $\square$

A vertex ordering $v_1 \prec \cdots \prec v_n$ of a $\frac{1}{\sqrt{2}}$-strip graph G is *increasing* if there is a disk representation $\mathcal{D}$ such that $i < j$ if and only if x-coordinate of a disk in $\mathcal{D}$ corresponding to vertex v_i is at most x-coordinate of a disk corresponding to v_j.

By Lemma 9, the First-fit coloring algorithm that process vertices of the second power G^2 of a $\frac{1}{\sqrt{2}}$-strip graph G in an increasing order, uses at most distinct $3\omega(G)$ colors. This coloring of G^2 is equivalent to an $L_{(1,1)}$-labeling of G. By multiplying all labels of $L_{(1,1)}$-labeling by 2, we obtain an $L_{(2,2)}$-labeling of G. Finally,

$$\chi_{(2,1)}(G) \leq \chi_{(2,2)}(G) \leq 2\chi_{(1,1)}(G) = 2\chi(G^2) \leq 6\omega(G).$$

Using the First-fit approach, the $L_{(2,1)}$-labeling of G can be obtained in $O(nm)$ time ($n = |V(G)|, m = |E(G)|$). Notice, that all labels used for this labeling are *even.*

Now we are ready to describe the approximate labeling algorithm for unit disk graphs and constraints $(2,1)$. Without loss of generality we may assume that G is connected and has at most two vertices (i.e. $\omega(G) \geq 2$).

Algorithm Distance Labeling (DL)
Input: Unit disk representation of a unit disk graph G.
Output: Labeling of c of $V(G)$.

1. Partition a plane into $k = O(n)$ strips $S_1, \ldots, S_k$ of width $\frac{1}{\sqrt{2}}$, numbered from top to bottom, such that S_1 contains a disk with maximal y-coordinate and S_k the one with minimal. This partition induces partition of G into $\frac{1}{\sqrt{2}}$-strip graphs $G_1, \ldots, G_k$. (In a case of disks with centers in two strips ties are broken arbitrarily.)
2. For each $i \in \{1, \ldots, k\}$ find an $L_{(2,1)}$-labeling of G_i using even labels and with the maximum label bounded by $6\omega(G_i) \leq 6\omega(G)$.
3. Change the labels of graph G_i by increasing them by the number $\sharp_{(i \bmod 6)}$, where

$$(\sharp_0, \ldots, \sharp_5) = (0,\ 6\omega(G),\ 12\omega(G),\ -1,\ 6\omega(G)-1,\ 12\omega(G)-1)$$

Theorem 10 *For any unit disk graph G, algorithm DL produces an $L_{(2,1)}$-labeling and the maximal label used by this algorithm is at most $18\omega(G)$.*

Proof: The maximum label used on every G_i is at most $6\omega(G) - 2$. Therefore, the maximal label assigned by the algorithm is at most $6\omega(G) - 2 + 12\omega(G) + 1$.

To verify that DL produces a $L_{(2,1)}$-labeling we observe that for $u, v \in V(G)$, such that u and v belong to different strips and $|c(u) - c(v)| = 1$ then the distance between the centers of the corresponding disks is at least $\frac{2}{\sqrt{2}} > 1$. Similarly, if $c(u) = c(v)$ then the distance between the centers is at least $\frac{5}{\sqrt{2}} > 2$. □

Corollary 11 *The approximation ratio of the algorithm DL is bounded by 12 and tends to 9 as $\omega(G)$ grows to infinity.*

Observe that $\frac{1}{\sqrt{2}}$-strips were used in the description of the algorithm to simplify the explanation. To avoid irrationality, $\frac{1}{\sqrt{2}}$-strips in the algorithm can be replaced by c-strips, where c is any rational number between $\frac{2}{3}$ and $\frac{1}{\sqrt{2}}$. Also the algorithm can be generalized easily to an algorithm producing an $L_{(p,1)}$-labeling ($p \geq 1$) with the maximal label used at most $9p\omega(G)$.

4.2 Robust approximation algorithm

The main purpose of this section is to present the approximation labeling algorithm which doesn't need a geometric representation of a unit disk graph as part of input. (Let us remind that it is *NP*-hard to recognize unit disk graphs.)

In [24] the following notion of *robust algorithms* is discussed: an algorithm which solves an optimization problem on class is called robust if it satisfies the following conditions.

1. Whenever the input is in class C, the algorithm finds the correct solution.
2. If the input is not in the class C, then the algorithm either finds the correct solution, or answers that the input is not in the class.

Based on the ideas of [6], a robust algorithm computing the maximal clique of a unit disk graph is given in [24].

The main idea is that every unit disk graph has an edge ordering $e_1 \prec_e \cdots \prec_e e_m$ such that for every edge e_i the neighbors of its endpoints induce a cobipartite subgraph C_i (i.e. the complement of a bipartite graph) of a graph induced by $\{e_1, \ldots, e_i\}$. Clearly, if there is such an ordering $\prec_e$, then for some edge e_i maximal clique is contained in cobipartite graph C_i. Thus robust algorithm first constructs (if there is any) edge ordering $\prec_e$ (this can be done in $O(m^2 n)$). Then for every graph C_i the algorithm finds the maximal clique in C_i. (This is equivalent to finding the maximum independent set in bipartite graph which can be done in $O(n^{2.5})$.) Therefore, the running time of the algorithm is $O(mn^{2.5})$.

Lemma 12 *Every unit disk graph G has a vertex v such that the set*

$$N_G(v) = \{u \neq v \colon \{u, v\} \in E(G)\}$$

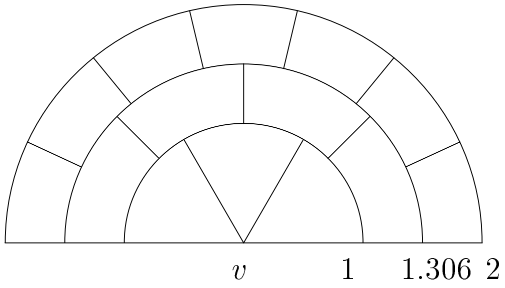


Figure 4: The plane partition around v

has cardinality at most $3\omega(G) - 3$ *and the set*

$$N_{G^2}(v) - N_G(v)$$

contains at most $11\omega(G)$ *vertices.*

Proof: The proof of the lemma is based on the plane partition around vertex v shown in Fig. 4. $\square$

We say that a vertex ordering $v_1 \prec \cdots \prec v_n$ of G is *good* if for every $2 \leq i \leq n$

(i) $|N_G(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq 3\omega(G) - 3$;

(ii) $|(N_{G^2}(v_i) - N_G(v_i)) \cap \{v_1, \ldots, v_{i-1}\}| \leq 11\omega(G)$.

Notice, that by Lemma 12 every unit disk graph has a good vertex ordering. Also for a graph G with n vertices one can in $O(n^3)$ time either find a good vertex ordering, or conclude that there is no good ordering for G.

Now we are ready to describe algorithm RDL which can be regarded as robust distance labeling approximation algorithm. It doesn't use the geometric representation of a graph G and either concludes that G is not unit disk graph, either (we prove it below) it finds an $L_{(2,1)}$-labeling of G with the maximum label at most $\leq 20\omega(G) - 8$.

Algorithm Robust Distance Labeling (RDL)
Input: Graph G in adjacency list.

Output: Either $L_{(2,1)}$-labeling c of $V(G)$, either conclusion that G is not unit disk graph.

```
    1. Run the robust algorithm to compute $\omega(G)$. This algorithm either computes $\omega(G)$, either concludes that $G$ is not unit disk graph.
    2. Find a good vertex ordering $v_1 \prec \cdots \prec v_n$. If there is no such an ordering, then conclude that $G$ is not unit disk graph.
    3. Label vertices sequentially in order $\prec$ as follows:
        3a. Assume that vertices $v_1, \dots, v_{i-1}$ are already labeled.
        3b. Let $\lambda \geq 1$ be the smallest integer which is not used as a label of vertices in $N_{G^2}(v_i) \cap \{v_1, \dots, v_{i-1}\}$ nor is a member of the set $\bigcup_{j \in \{1,\dots,i-1\}: v_j \in N_G(v_i)} \{c(v_j) - 1, c(v_j), c(v_j) + 1\}$.
        3c. Label $v_i$ by $c(v_i) = \lambda$.
```

Theorem 13 *For any graph G, Algorithm RDL either produces an $L_{2,1}$-labeling with maximum label $\leq 20\omega(G) - 8$, or concludes that G is not unit disk graph.*

Proof: Suppose that algorithm output that G is not unit disk graph. If it was done after the first step, then G has no edge ordering $\prec_e$ and therefore is not unit disk graph. If the algorithm halts at the second step, then its conclusion is verified by Lemma 12.

Suppose that RDL outputs a labeling. Let us first show that the maximum label used by the algorithm is not larger than $20\omega(G) - 8$. We proceed by induction. The vertex v_1 is labeled by 1, since both sets declared in 3b are empty. Suppose that we have labeled vertices $v_1, \dots, v_{i-1}$. When we assign a label to v_i, then by (i) it has at most $3\omega(G) - 3$ labeled neighbors and at most $9\omega(G) - 9$ labels are unavailable because of these neighbors. (If one neighbor of v_i has label x then labels $x - 1, x$ and $x + 1$ are forbidden for v_i.) By (ii), at most $11\omega(G)$ labeled vertices are at distance two from v_i. Therefore, the number of unavailable labels is at most $20\omega(G) - 9$. Since we have $20\omega(G) - 8$ labels, $c(v_i) \leq 20\omega(G) - 8$. $\square$

Again, this robust algorithm can be generalized for $L_{(p,1)}$-labeling ($p \geq 1$).

Corollary 14 *The approximation ratio of the algorithm RDL is bounded by $\frac{32}{3} \doteq 10,67$ and tends to 10 as $\omega(G)$ grows to infinity.*

We observe that for the construction of on-line algorithms with constant competitive ratio on unit disk graphs the knowledge of geometrical representation is crucial. For the coloring problem similar robust algorithm on unit disk graphs can be turned into on-line coloring algorithm (with worse competitive ratio). However, this is not the case for the labeling problem. The main reason why RDL cannot be turned into "First-fit" algorithm is that at the moment when we have to select a suitable label for vertex v_i we need information about all vertices from the set $\{v_1, \ldots, v_{i-1}\}$ that are at distance two from v_i in G. Unfortunately this information cannot be fully derived from G restricted onto $\{v_1, \ldots, v_i\}$, as was shown in the beginning of section 3. We show now that if the distance could be derived then such approximation algorithm exists, however it cannot run on-line.

4.3 Off-line labeling with arbitrarily many constraints

In this subsection we briefly discuss an off-line labeling algorithm when the input graph G has a representation $\mathcal{D}$ with diameter ratio bounded by some constant σ. The algorithm doesn't use representation, however it is not robust.

Lemma 15 *Let G be a disk graph and $\sigma(\mathcal{D})$ be the diameter ratio of some of its representation $\mathcal{D}$. Then for each vertex v of G, the subgraph of G induced by vertices at distance at most k from v can be partitioned into $O(k^2\sigma(\mathcal{D})^2)$ disjoint cliques.*

Proof: Without loss of generality assume that the smallest disk in $\mathcal{D}$ has unit diameter. Consider the disk D corresponding to the vertex v in the representation $\mathcal{D}$. If a vertex v' is at distance at most k from v in the graph G, the centers of the corresponding disks D and D' are in the plane at distance $k\sigma(\mathcal{D})$. The disk of diameter $2k\sigma(\mathcal{D})$ could be partitioned into at most $ck^2\sigma(\mathcal{D})^2$ square tiles of diameter one, where c is a positive constant. The disks with centers in the same tile must mutually intersect, hence in the graph G the corresponding vertices induce a clique. Such sets remain cliques, even if we restrict G to the subgraph induced by vertices at distance at most k from v. $\square$

Now we may state the theorem for the off-line labeling problem for the class of disk graphs with diameter ratio bounded by a constant σ.

Theorem 16 *For any k-tuple of distance constraints and for any $\sigma \geq 1$, the First-fit algorithm is $O(k^2\sigma^2)$-approximate $L_{(p_1,\ldots,p_k)}$-labeling algorithm for the class of disk graphs with diameter ratio bounded by σ.*

Proof: Let us think that $\omega(G) \geq 2$. (The case, when G has no edges is trivial.) Observe that in contrast to the case of on-line algorithms we take into account also vertices that are not labeled yet, but are used in computing distance $\mathrm{dist}(v, v')$ for any already labeled vertex v'.

The maximum label used by First-fit is at most

$$1 + (2p_1 - 1)\Big|\{v' \colon \mathrm{dist}_G(v, v') \leq k\}\Big| \leq 1 + (2p_1 - 1)ck^2\sigma^2\omega(G),$$

while $\chi_{(p_1,\ldots,p_k)} \geq p_1(\omega(G) - 1) + 1$. Since $\omega(G) \geq 2$, we have that the approximation ratio is at most

$$\frac{1 + (2p_1 - 1)ck^2\sigma^2\omega(G)}{p_1(\omega(G) - 1) + 1} = O(k^2\sigma^2).$$

□

Notice that the algorithm does not use the disk representation but to guarantee that it reaches the approximation ratio we need to know that the input graph is in DG_σ. This assumption is necessary, since we are not able to perform the membership test for DG_σ. Also it is unclear how to turn the algorithm into a robust one because robust computing or approximating the maximum clique size in a graph from DG_σ seems to be a difficult problem. Thus our robust approach for UD graphs can not be directly transformed for DG_σ.

5 Conclusion

We have shown that there exists on-line labeling algorithms for disk graphs with constant competitive ratio. The value of this ratio depends on the underlying labeling of cells. Our first open problem is to determine the minimum number of distinct labels l such that a circular l-labeling with respect to distance constraints $(p_1, \ldots, p_k)$ and diameter ratio σ exists.

The next open question is whether there exist an off-line approximation algorithm with fixed competitive ratio that do not require the disk representation or that works on disk graphs where diameter ratio is not bounded. If the answer to the last question is negative, we can ask whether the approximation ratio of an labeling algorithm can be bounded by a sublinear function. Negative results in this direction would bring the inapproximability of distance labeling of (unit) disk graphs.

We have already mentioned that the computational complexity of existence of a $L_{(p_1,\ldots,p_k)}$-labeling using bounded number of labels is not fully classified yet. It is known that such problem is solvable in polynomial time for trees and two parameters (p_1, p_2) when $p_2 = 1$ [10]. On the other hand the distance labeling of a tree is still an open problem if p_1 and p_2 have no common divisor and $p_2 > 1$. For the hardness proof of the precoloring version of this problem and related discussion see [11]. Similarly, the computational complexity of the labeling problem remains open for the class of disk graphs and unit disk graphs and at least two distance constraints.

Another direction of further research might consider a different class of graphs that are used in graph theoretic-models of channel assignment, like ball graphs, disk graphs on a sphere, induced subgraph of triangular mesh, etc. It is possible that for some classes of intersection graphs of geometric objects a labeling algorithm derived from the First-fit method behaves as well as for the class of disk graphs.

Finally, the following problem is related to possible construction of robust labeling algorithm for graphs in DG_σ. Is it possible or not to construct an exact or a constant factor approximation algorithm for maximum clique size for graphs in DG_σ?

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