Extremal problems for ordered (hyper)graphs: applications of Davenport–Schinzel sequences

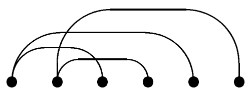
Martin Klazar*

Abstract

We introduce a containment relation of hypergraphs which respects linear orderings of vertices and investigate associated extremal functions. We extend, by means of a more generally applicable theorem, the $n \log n$ upper bound on the ordered graph extremal function of $F = (\{1,3\},\{1,5\},\{2,3\},\{2,4\})$ due to Füredi to the $n(\log n)^2(\log\log n)^3$ upper bound in the hypergraph case. We use Davenport–Schinzel sequences to derive almost linear upper bounds in terms of the inverse Ackermann function $\alpha(n)$. We obtain such upper bounds for the extremal functions of forests consisting of stars whose all centers precede all leaves.

1 Introduction and motivation

In this article we shall investigate extremal problems on graphs and hypergraphs of the following type. Let $G = ([n], E)$ be a simple graph that has the vertex set $[n] = \{1, 2, \ldots, n\}$ and contains no six vertices $1 \le v_1 < v_2 < \cdots < v_6 \le n$ such that $\{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_4\}$, and $\{v_2, v_6\}$ are edges of G, that is, G has no *ordered* subgraph of the form

$G_0 =$  . (1)

*Department of Applied Mathematics (KAM) and Institute for Theoretical Computer Science (ITI), Charles University, Malostranské náměstí 25, 118 00 Praha, Czech Republic. ITI is supported by the project LN00A056 of the Ministery of Education of the Czech Republic. E-mail: `klazar@kam.mff.cuni.cz`

Determine the maximum possible number $g(n) = |E|$ of edges in G.

What makes this task hard is the linear ordering of $V = [n]$ and the fact that G_0 must not appear in G only as an ordered subgraph. If we ignore the ordering for a while, then the problem asks to determine the maximum number of edges in a simple graph G with n vertices and no $2K_{1,2}$ subgraph, and is easily solved. Clearly, if G has two vertices of degrees ≥ 3 and ≥ 5, respectively, or if G has ≥ 6 vertices of degrees 4 each, then a $2K_{1,2}$ subgraph must appear. Thus if G has a vertex of degree ≥ 5 and no $2K_{1,2}$ subgraph, it has at most $(2(n-1)+n-1)/2 = 3n/2 - 1.5$ edges. If all degrees are ≤ 4, the number of edges is at most $(3(n-5)+4\cdot 5)/2 = 3n/2+2.5$. On the other hand, the graph on $[n]$ in which $\deg(n) = n-1$ and $[n-1]$ induces a matching with $\lfloor (n-1)/2 \rfloor$ edges has $n + \lfloor (n-1)/2 \rfloor - 1$ edges and no $2K_{1,2}$ subgraph. We conclude that in the unordered version of the problem the maximum number of edges equals $3n/2 + O(1)$.

The ordered version is considerably more difficult. Later in this section we prove that the maximum number of edges $g(n)$ satisfies

$$n \cdot \alpha(n) \ll g(n) \ll n \cdot 2^{(1+o(1))\alpha(n)^2} \tag{2}$$

where $\alpha(n)$ is the inverse Ackermann function. Recall that $\alpha(n) = \min\{m : A(m) \geq n\}$ where $A(n) = F_n(n)$, the Ackermann function, is defined as follows. We start with $F_1(n) = 2n$ and for $i \geq 1$ we define $F_{i+1}(n) = F_i(F_i(\ldots F_i(1)\ldots))$ with n iterations of F_i. The function $\alpha(n)$ grows to infinity but its growth is extremely slow. We obtain (2) and some generalizations by reductions to *Davenport–Schinzel sequences* (DS sequences). We continue now with a brief review of results on DS sequences that will be needed in the following. We refer the reader for more information and references on DS sequences and their applications in computational and combinatorial geometry to Agarwal and Sharir [1], Klazar [17], Sharir and Agarwal [19], and Valtr [22].

If $u = a_1a_2\ldots a_r$ and $v = b_1b_2\ldots b_s$ are two finite sequences (words) over a fixed infinite alphabet A, where A contains $\mathbf{N} = \{1, 2, \ldots\}$ and also some symbols $a, b, c, d, \ldots$, we say that v *contains* u and write $v \succ u$ if v has a subsequence $b_{i_1}b_{i_2}\ldots b_{i_r}$ such that for every p and q we have $a_p = a_q$ if and only if $b_{i_p} = b_{i_q}$. In other words, v has a subsequence that differs from u only by an injective renaming of the symbols. For example, $v = ccaaccbaa \succ 22244 = u$ because v has the subsequence $cccaa$. On the other hand, $ccaaccbaa \not\succ 12121$. A sequence $u = a_1a_2\ldots a_r$ is called *k-sparse*, where $k \in \mathbf{N}$, if $a_i = a_j, i < j$, always implies $j - i \geq k$; this means that every interval in u of length at most k consists of distinct terms. The length r of u is denoted by $|u|$. For two integers $a \le b$ we write $[a, b]$ for the interval $\{a, a+1, \ldots, b\}$. For two functions $f, g : \mathbf{N} \to \mathbf{R}$ the notation $f \ll g$ is synonymous to the $f = O(g)$ notation; it means that $|f(n)| < c|g(n)|$ for all $n > n_0$ with a constant $c > 0$.

The classical theory of DS sequences investigates, for a fixed $s \in \mathbf{N}$, the function $\lambda_s(n)$ that is defined as the maximum length of a 2-sparse sequence v over n symbols which does not contain the $s+2$-term alternating sequence $ababa\ldots$ ($a \ne b$). The notation $\lambda_s(n)$ and the shift $+2$ are due to historical reasons. The term *DS sequences* refers to the sequences v not containing a fixed alternating sequence. The theory of *generalized DS sequences* investigates, for a fixed sequence u that uses exactly k symbols, the function $\mathrm{ex}(u, n)$ that is defined as the maximum length of a k-sparse sequence v such that v is over n symbols and $v \not\succ u$. Note that $\mathrm{ex}(u, n)$ extends $\lambda_s(n)$ since $\lambda_s(n) = \mathrm{ex}(ababa\ldots, n)$ where $ababa\ldots$ has length $s+2$. In the definition of $\mathrm{ex}(u, n)$ one has to require that v is k-sparse because no condition or even only $k-1$-sparseness would allow an infinite v with $v \not\succ u$; for example, $v = 12121212\ldots \not\succ abca = u$ and v is 2-sparse (but not 3-sparse). An easy pigeon hole argument shows that always $\mathrm{ex}(u, n) < \infty$.

DS sequences were introduced by Davenport and Schinzel [7] and strongest bounds on $\lambda_s(n)$ for general s were obtained by Agarwal, Sharir and Shor [2]. We need their bound

$$\lambda_6(n) \ll n \cdot 2^{(1+o(1))\alpha(n)^2} \tag{3}$$

(recall that $\lambda_6(n) = \mathrm{ex}(abababab, n)$). Hart and Sharir [13] proved that

$$n\alpha(n) \ll \lambda_3(n) \ll n\alpha(n). \tag{4}$$

In Klazar [14] we proved that if u is a sequence using $k \ge 2$ symbols and $|u| = l \ge 5$, then for every $n \in \mathbf{N}$

$$\mathrm{ex}(u, n) \le n \cdot k2^{l-3} \cdot (10k)^{2\alpha(n)^{l-4} + 8\alpha(n)^{l-5}}. \tag{5}$$

It is easy to show that for $k = 1$ or $l \le 4$ we have $\mathrm{ex}(u, n) \ll n$. In particular, for the sequence

$$u(k, l) = 12\ldots k12\ldots k\ldots 12\ldots k \tag{6}$$

with l segments $12\ldots k$ we have, for every fixed $k \ge 2$ and $l \ge 3$,

$$\mathrm{ex}(u(k, l), n) \le n \cdot k2^{kl-3} \cdot (10k)^{2\alpha(n)^{kl-4} + 8\alpha(n)^{kl-5}}. \tag{7}$$

We denote the factor at n in (7) as $\beta(k,l,n)$. Thus

$$\beta(k,l,n) = k2^{kl-3}(10k)^{2\alpha(n)^{kl-4}+8\alpha(n)^{kl-5}}. \tag{8}$$

Let us see now how (3) and the lower bound in (4) imply (2). Let $G = ([n], E)$ be any simple graph not containing G_0 (given in (1)) as an ordered subgraph. Consider the sequence

$$v = I_1 I_2 \dots I_n$$

over $[n]$ where I_i is the decreasing ordering of the list $\{j : \{j,i\} \in E \,\&\, j < i\}$. Note that $I_1 = \emptyset$ and $|v| = |E|$.

Lemma 1.1 *If $v \succ abababab$ then G_0 is an ordered subgraph of G.*

Proof. We assume that v has an 8-term alternating subsequence

$$\dots a_1 \dots b_1 \dots a_2 \dots b_2 \dots a_3 \dots b_3 \dots a_4 \dots b_4 \dots$$

where the appearances of two numbers $a \neq b$ are indexed for further discussion. We distinguish two cases. If $a < b$ then a_2, b_2, a_4, and b_4 lie, respectively, in four distinct intervals I_p, I_q, I_r, and I_s, $p < q < r < s$, (since every I_i is decreasing) and $b < p$ (since b_1 precedes a_2). Hence G_0 is an ordered subgraph of G. If $b < a$ then b_1, a_2, b_3, and a_4 lie, respectively, in four distinct intervals I_p, I_q, I_r, and I_s, $p < q < r < s$, and $a < p$. Again, G_0 is an ordered subgraph of G. □

Thus v has no 8-term alternating subsequence. In v immediate repetitions may appear only on the transitions $I_i I_{i+1}$. Deleting at most $n-1$ (actually $n-2$ because $I_1 = \emptyset$) terms from v we obtain a 2-sparse subsequence w on which we can apply (3). We have

$$|E| = |v| \le |w| + n - 1 \le \lambda_6(n) + n - 1 \ll n \cdot 2^{(1+o(1))\alpha(n)^2}.$$

On the other hand, let $n \in \mathbf{N}$ and v be the longest 2-sparse sequence over $[n]$ such that $v \not\succ ababa$. It uses all n symbols and, by the lower bound in (4), $|v| > cn\alpha(n)$ for an absolute constant $c > 0$. Notice that every $i \in [n]$ appears in v at least twice. We rename the symbols in v so that for every $1 \le i < j \le n$ the first appearance of j in v precedes that of i; this affects neither the property $v \not\succ ababa$ nor the 2-sparseness. By an *extremal term of v* we mean the first or the last appearance of a symbol in v. The sequence v has exactly $2n$ extremal terms. We decompose v uniquely into intervals $v = I_1 I_2 \dots I_{2n}$ so that every I_i ends with an extremal term and contains no other extremal term. Every I_i consists of distinct terms because a repetition $\dots b \dots b \dots$ in I_i would force a 5-term alternating subsequence $\dots a \dots b \dots a \dots b \dots a \dots$ in v. We define a simple (bipartite) graph $G^* = ([3n], E)$ by

$$\{i, j\} \in E \iff i \in [n] \ \&\ j \in [n+1, 3n] \ \&\ i \text{ appears in } I_{j-n}.$$

G^* has $3n$ vertices and $|E| = |v| > cn\alpha(n)$ edges. Suppose that G^* contains the forbidden ordered subgraph G_0 on the vertices $1 \leq a_1 < a_2 < \dots < a_6 \leq 3n$. By the definition of G^*, $z = a_1 a_2 a_1 a_2$ is a subsequence of v and its terms appear in $I_{a_3-n}, \dots, I_{a_6-n}$, respectively. Since $a_2 > a_1$, number a_2 must appear in v before z starts and therefore v contains a 5-term alternating subsequence but this is forbidden. So G^* does not contain G_0 and shows that

$$g(n) \gg n\alpha(n).$$

This concludes the proof of (2).

Problem 1.2 *Narrow the gap $\lambda_3(n) \ll g(n) \ll \lambda_6(n)$ in (2). What is the precise asymptotics of $g(n)$?*

Our illustrative example with $g(n)$ shows that the ordered version of a simple graph extremal problem may differ dramatically from the unordered one. Classical extremal theory of graphs and hypergraphs deals with unordered vertex sets and it produced many results of great variety — see, for example, Bollobás [3, 4], Frankl [9], Füredi [11], and Tuza [20, 21]. However, only little attention has been paid to ordered extremal problems. The only systematic studies devoted to this topic known to us are Füredi and Hajnal [12] (ordered bipartite graphs) and Brass, Károlyi and Valtr [6] (cyclically ordered graphs). We think that for several reasons ordered extremal problems should be studied and investigated more intensively. First, for their intrinsic combinatorial beauty. Second, since they present to us new functions not to be met in the classical theory: nearly linear extremal functions, like $n\alpha(n)$ or $n \log n$, are characteristic for ordered extremal problems and it seems that they cannot appear without ordering of some sort. Third, estimates coming from ordered extremal theory were successfuly applied in combinatorial geometry, where often the right key to a problem turns out to be some linear or partial ordering, and to obtain more such applications we have to understand more thoroughly combinatorial cores of these arguments.

Before summarizing our results, we return to DS sequences and show that the sequential containment $\prec$ can be naturally interpreted in terms of particular hypergraphs, (set) partitions. A sequence $u = a_1a_2\ldots a_r$ over the alphabet A may be viewed as a partition P of $[r]$ where i and j are in the same block of P if and only if $a_i = a_j$. Thus blocks of P correspond to the positions of symbols in u. For example, $u = abaccba$ is the partition $\{\{1,3,7\},\{2,6\},\{4,5\}\}$. If $u = ([r], \sim_u)$ and $v = ([s], \sim_v)$ are two sequences given as partitions by equivalence relations, then $u \prec v$ if and only if there is an *increasing* injection $f : [r] \to [s]$ such that $x \sim_u y \Leftrightarrow f(x) \sim_v f(y)$ holds for every $x, y \in [r]$.

In this article we introduce and investigate a hypergraph containment that generalizes both the ordered subgraph relation and the sequential containment. The containment and its associated extremal functions $\mathrm{ex}_e(F,n)$ and $\mathrm{ex}_i(F,n)$ are given in Definitions 2.1 and 2.2. The function $\mathrm{ex}_e(F,n)$ counts edges in extremal simple hypergraphs H not containing a fixed hypergraph F and the function $\mathrm{ex}_i(F,n)$ counts sums of edge cardinalities. In Theorem 2.3 we show that for many F one has $\mathrm{ex}_i(F,n) \ll \mathrm{ex}_e(F,n)$. Theorem 3.1 shows that if F is a simple graph, then in some cases good bounds on $\mathrm{ex}_e(F,n)$ can be obtained from bounds on the ordered graph extremal function $\mathrm{gex}(F,n)$. We apply Theorem 3.1 to prove in Theorem 3.3 that for $G_1 = (\{1,3\},\{1,5\},\{2,3\},\{2,4\})$ one has $\mathrm{ex}_e(G_1,n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^3$ and the same bound for $\mathrm{ex}_i(G_1,n)$; this generalizes the bound $\mathrm{gex}(G_1,n) \ll n \cdot \log n$ of Füredi. In another application, Theorem 3.5, we prove that the *unordered* hypergraph extremal function $\mathrm{ex}_e^u(F,n)$ of every forest F is $\ll n$. In Theorem 4.1 we generalize the bound (5) to hypergraphs. In Theorem 4.2 we prove that if F is a star forest, then $\mathrm{ex}_e(F,n)$ has an almost linear upper bounds in terms of $\alpha(n)$; this generalizes the upper bound in (2). In the concluding section we introduce the notion of orderly bipartite forests and pose some problems.

This article is a revised version of about one half of the material in the technical report [16]. We present the other half in [18].

2 Definitions and bounding weight by size

A *hypergraph* $H = (E_i : i \in I)$ is a finite list of finite nonempty subsets E_i of $\mathbf{N} = \{1, 2, \ldots\}$, called *edges*. H is *simple* if $E_i \neq E_j$ for every $i, j \in I$, $i \neq j$. H is a *graph* if $|E_i| = 2$ for every $i \in I$. H is a *partition* if $E_i \cap E_j = \emptyset$ for every $i, j \in I$, $i \neq j$. The elements of $\bigcup H = \bigcup_{i \in I} E_i \subset \mathbf{N}$ are called *vertices*. Note that our hypergraphs have no isolated vertices. The *simplification* of H is the simple hypergraph obtained from H by keeping from each family of mutually equal edges just one edge.

Definition 2.1 *Let $H = (E_i : i \in I)$ and $H' = (E'_i : i \in I')$ be two hypergraphs. H* contains *H', in symbols $H \succ H'$, if there exist an* increasing *injection $F : \bigcup H' \to \bigcup H$ and an injection $f : I' \to I$ such that the implication*

$$v \in E'_i \Rightarrow F(v) \in E_{f(i)}$$

*holds for every vertex $v \in \bigcup H'$ and every index $i \in I'$. Else we say that H is H'-*free *and write $H \not\succ H'$.*

The hypergraph containment $\prec$ clearly extends the sequential containment and the ordered subgraph relation. We give an alternative definition of $\prec$. $H = (E_i : i \in I)$ and $H' = (E'_i : i \in I')$ are *isomorphic* if there are an *increasing* bijection $F : \bigcup H' \to \bigcup H$ and a bijection $f : I' \to I$ such that $F(E'_i) = E_{f(i)}$ for every $i \in I'$. H' is a *reduction* of H if $I' \subset I$ and $E'_i \subset E_i$ for every $i \in I'$. Then $H' \prec H$ if and only if H' is isomorphic to a reduction of H. We call that reduction of H an *H'-copy* in H. For example, if $H' = (\{1\}_1, \{1\}_2)$ (H' is a singleton edge repeated twice) then $H \not\succ H'$ iff H is a partition. Another example: If $H' = (\{1, 3\}, \{2, 4\})$ then H is H'-free iff H has no four vertices $a < b < c < d$ such that a and c lie in one edge of H and b and d lie in another edge.

The *order* $v(H)$ of $H = (E_i : i \in I)$ is the number of vertices $v(H) = |\bigcup H|$, the *size* $e(H)$ is the number of edges $e(H) = |H| = |I|$, and the *weight* $i(H)$ is the number of incidences between the vertices and the edges $i(H) = \sum_{i \in I} |E_i|$. Trivially, $v(H) \leq i(H)$ and $e(H) \leq i(H)$ for every H.

Definition 2.2 *Let F be any hypergraph. We associate with F the extremal functions $\mathrm{ex}_e(F), \mathrm{ex}_i(F) : \mathbf{N} \to \mathbf{N}$, defined by*

$$\begin{aligned} \mathrm{ex}_e(F, n) &= \max\{e(H) : H \not\succ F \ \&\ H \text{ is simple} \ \&\ v(H) \leq n\} \\ \mathrm{ex}_i(F, n) &= \max\{i(H) : H \not\succ F \ \&\ H \text{ is simple} \ \&\ v(H) \leq n\}. \end{aligned}$$

We considered $\mathrm{ex}_e(F,n)$ and $\mathrm{ex}_i(F,n)$ implicitly already in Klazar [15]. Except of this article, to our knowledge, this extremal setting is new and was not investigated before. Obviously, for every $n \in \mathbf{N}$ and F, $\mathrm{ex}_e(F,n) \le 2^n - 1$ and $\mathrm{ex}_i(F,n) \le n2^{n-1}$ but much better bounds can be usually given. The *reversal* of a hypergraph $H = (E_i : i \in I)$ with $N = \max(\bigcup H)$ is the hypergraph $\overline{H} = (\overline{E_i} : i \in I)$ where $\overline{E_i} = \{N - x + 1 : x \in E_i\}$. Thus reversals are obtained by reverting the linear ordering of vertices. It is clear that $\mathrm{ex}_e(F,n) = \mathrm{ex}_e(\overline{F},n)$ and $\mathrm{ex}_i(F,n) = \mathrm{ex}_i(\overline{F},n)$ for every F and n.

We give a few comments on Definitions 2.1 and 2.2. Replacing the implication in Definition 2.1 with an equivalence, we obtain an induced hypergraph containment that still extends the sequential containment. Let $\mathrm{iex}_e(F,n)$ be the corresponding extremal function. For $F_k = (\{1\},\{2\},\ldots,\{k\})$ and $k \ge 2$ we have $\mathrm{iex}_e(F_k,n) \ge \binom{n}{k-2}$ because the hypergraph $(E : E \subset [n]\ \&\ |E| = n - k + 2)$ does not contain F_k in the induced sense. But in the hypergraph "DS theory" such uncomplicated hypergraphs like F_k should have linear (or almost linear) extremal functions. Thus the induced containment is not the right generalization, as far as one is interested in "DS theories".

For graphs, if $H_2 \succ H_1$ and H_2 is simple then H_1 is simple as well. But simple hypergraphs may contain hypergraphs that are not simple. In Definition 2.2 H must be simple because allowing all H would produce usually the value $+\infty$ (the simplicity of H may be dropped only for $F = (\{1\}_1,\{1\}_2,\ldots,\{1\}_k)$). On the other hand, we allow for the forbidden F any hypergraph: F need not be simple and may have singleton edges. Such a freedom for F seemed to one referee of [16] "very artificial". This opinion is understandable but the author does not share it. Putting aside psychological inertia, there is no reason why to restrict F from the outset to be simple or in any other way. On the contrary, doing so we might miss some connections and phenomena. Thus we start in the definitions with completely arbitrary F and restrict it later only if circumstances require so.

Another perhaps unusual feature of our extremal theory is that in H and F edges of all cardinalities are allowed; in extremal theories with forbidden substructures it is more common to have edges of just one cardinality. This led naturally to the function $\mathrm{ex}_i(F,n)$ which accounts for edges of all sizes. Trivially, $\mathrm{ex}_i(F,n) \ge \mathrm{ex}_e(F,n)$ for every hypergraph F and $n \in \mathbf{N}$. On the other hand, Theorem 2.3 shows that for many F one has $\mathrm{ex}_i(F,n) \ll \mathrm{ex}_e(F,n)$. In Definition 2.2 we take all H with $v(H) \le n$ that the extremal functions be automatically nondecreasing. Replacing $v(H) \le n$ with $v(H) = n$ would give more information on the functions but also would bring the complication that then extremal functions are not always nondecreasing. It happens for $F = (\{1\}, \{2\}, \ldots, \{k\})$ and we analyze this phenomenon in [16, 18].

Theorem 2.3 *Suppose that the hypergraph F has no two separated edges, which means that $E_1 < E_2$ holds for no two edges of F. Let $p = v(F)$ and $q = e(F) > 1$. Then for every $n \in \mathbf{N}$,*

$$\mathrm{ex}_i(F, n) \leq (2p-1)(q-1) \cdot \mathrm{ex}_e(F, n).$$

Proof. Suppose that H attains the value $\mathrm{ex}_i(F, n)$. We transform H in a new hypergraph H' by keeping all edges with less than p vertices and replacing every edge $E = \{v_1, v_2, \ldots, v_s\}$ of H with $s \geq p$, where $v_1 < v_2 < \cdots < v_s$, by $t = \lfloor |E|/p \rfloor$ new p-element edges $\{v_1, \ldots, v_p\}$, $\{v_{p+1}, \ldots, v_{2p}\}, \ldots, \{v_{(t-1)p+1}, \ldots, v_{tp}\}$. H' may not be simple and we set H'' to be the simplification of H'. Two observations: (i) no edge of H' repeats q or more times and (ii) H'' is F-free. If (i) were false, there would be q distinct edges $E_1, \ldots, E_q$ in H such that $|\bigcap_{i=1}^{q} E_i| \geq p$. But this implies the contradiction $F \prec H$. As for (ii), any F-copy in H'' may use from every $E \in H$ at most one new edge $E'' \subset E$ (the new edges born from E are mutually separated) and so it is an F-copy in H as well. The observations and the definitions of H' and H'' imply

$$\begin{aligned}
\mathrm{ex}_i(F, n) = i(H) &\leq \frac{(2p-1) \cdot i(H')}{p} \leq \frac{(2p-1)(q-1) \cdot i(H'')}{p} \\
&\leq (2p-1)(q-1) \cdot e(H'') \\
&\leq (2p-1)(q-1) \cdot \mathrm{ex}_e(F, n).
\end{aligned}$$

The last innocently looking inequality follows from the fact that $\mathrm{ex}_e(F, n)$ is nondecreasing by definition. $\square$

However, $\mathrm{ex}_i(F, n) \ll \mathrm{ex}_e(F, n)$ does not hold for $F_k = (\{1\}, \{2\}, \ldots, \{k\})$ and $k \geq 2$: $\mathrm{ex}_e(F_k, n) = 2^{k-1} - 1$ for $n \geq k-1$ and $\mathrm{ex}_i(F_k, n) = (k-1)n - (k-2)$ for $n > \max(k, 2^{k-2})$ ([16, 18]). Note that for $F = (\{1\})$ both extremal functions are undefined. F_k is highly symmetric and the ordering of vertices is irrelevant for the containment $H \succ F_k$.

Problem 2.4 *Prove that if F is not isomorphic to $(\{1\}, \{2\}, \ldots, \{k\})$, $k \geq 1$, then $\mathrm{ex}_i(F, n) \ll \mathrm{ex}_e(F, n)$.*

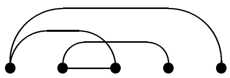
3 Bounding hypergraphs by means of graphs

For a family of simple graphs R and $n \in \mathbf{N}$ we define

$$\begin{aligned}\mathrm{gex}(R,n) \quad = \quad & \max\{e(G):\ G \not\succ G' \text{ for all } G' \in R\ \&\ G \text{ is a simple graph} \\ & \&\ v(G) \le n\}\end{aligned}$$

and for one simple graph G we write $\mathrm{gex}(G,n)$ instead of $\mathrm{gex}(\{G\},n)$. Füredi proved in [10], see also [12], that for

$$G_1 = (\{1,3\},\{1,5\},\{2,3\},\{2,4\}) = \tag{9}$$



one has

$$n\log n \ll \mathrm{gex}(G_1,n) \ll n\log n. \tag{10}$$

(In [10] and [12] the investigated objects are 0-1 matrices, which are ordered bipartite graphs, but in the case of G_1 the bounds are easily extended to all ordered graphs.) Attempts to generalize the upper bound in (10) to hypergraphs motivated the next theorem.

For $k \in \mathbf{N}$ we say that a simple graph G' is a *k-blow-up* of a simple graph G if for every edge coloring $\chi : G' \to \mathbf{N}$ that uses every color at most k times there is a G-copy in G' with totally different colors, that is, χ is injective on the G-copy. For $k \in \mathbf{N}$ and a simple graph G we write $B(k,G)$ to denote the set of all k-blow-ups of G.

Theorem 3.1 *Let F be a simple graph with $p = v(F)$ and $q = e(F) > 1$ and let $B \subset B(\binom{p}{2},F)$. If $f : \mathbf{N} \to \mathbf{N}$ is a nondecreasing function such that*

$$\mathrm{gex}(B,n) < n \cdot f(n)$$

holds for every $n \in \mathbf{N}$, then

$$\mathrm{ex}_e(F,n) < q \cdot \mathrm{gex}(F,n) \cdot \mathrm{ex}_e(F, 2f(n)+1) \tag{11}$$

holds for every $n \in \mathbf{N}$, $n \ge 3$.

Proof. Suppose that the simple hypergraph H attains the value $\mathrm{ex}_e(F,n)$ and $\bigcup H = [m]$, $m \le n$. We put in H' every edge of H with more than 1 and less than p vertices and for every $E \in H$ with $|E| \ge p$ we put in H' an arbitrary subset $E' \subset E$, $|E'| = p$. So $2 \leq |E| \leq p$ for every $E \in H'$ and no edge of H' repeats more then $q-1$ times, for else we would have $H \succ F$. Let H'' be the simplification of H'. We have

$$e(H) \leq n + (q-1)e(H'').$$

Let G be the simple graph consisting of all edges E^* such that $E^* \subset E$ for some $E \in H''$. Observe that if $F' \in B$ and $F' \prec G$, then $F \prec H''$ and thus $F \prec H$. (For the edges $E^* \in G$ forming an F'-copy consider the coloring $\chi(E^*) = E$ where $E \in H''$ is such that $E^* \subset E$. Every color is used at most $\binom{p}{2}$ times and therefore, since F' is a $\binom{p}{2}$-blow-up of F, we have an F-copy in G for which the correspondence $E^* \mapsto E$ is injective.) Hence $F' \prec G$ for no $F' \in B$. Let $v(G) = n'$; $n' \leq n$. We have

$$e(G) \leq \mathrm{gex}(B, n') < n' \cdot f(n').$$

There exists a vertex $v_0 \in \bigcup G$ such that

$$d = \deg_G(v_0) < 2f(n') \leq 2f(n).$$

Fix an arbitrary edge E_0^*, $v_0 \in E_0^* \in G$. Let $X \subset [n]$ be the union of all edges $E \in H''$ satisfying $E_0^* \subset E$ and m be the number of such edges in H''. We have the inequalities

$$m \leq \mathrm{ex}_e(F, |X|) \quad \text{and} \quad |X| \leq d + 1.$$

Thus

$$m \leq \mathrm{ex}_e(F, |X|) \leq \mathrm{ex}_e(F, d+1) \leq \mathrm{ex}_e(F, 2f(n)+1).$$

We see that the two-element set E_0^* is contained in at least 1 but at most $\mathrm{ex}_e(F, 2f(n)+1)$ edges of H''. Deleting those edges we obtain a subhypergraph H_1'' of H'' on which the same argument can be applied. That is, a two-element set E_1^* exists such that $E_1^* \subset E$ for at least 1 but at most $\mathrm{ex}_e(F, 2f(n)+1)$ edges $E \in H_1''$ and clearly $E_1^* \neq E_0^*$. Continuing this way until the whole H'' is exhausted, we define a mapping

$$M : H'' \to \{E^* :\ E^* \subset [n], |E^*| = 2\}$$

such that

$$M(E) \subset E \quad \text{and} \quad |M^{-1}(E^*)| \leq \mathrm{ex}_e(F, 2f(n)+1)$$

holds for every $E \in H''$ and $E^* \subset [n], |E^*| = 2$. Let G' be the simple graph $G' = M(H'')$ and $v(G') = n'$; $n' \leq n$.

The containment $F \prec G'$ implies, by the definition of G', that $F \prec H''$ and hence $F \prec H$, which is not allowed. Thus

$$e(G') \leq \mathrm{gex}(F, n') \leq \mathrm{gex}(F, n).$$

Putting it all together, we obtain (since $\mathrm{gex}(F, n) \geq n - 1$ if $q > 1$)

$$\begin{aligned} \mathrm{ex}_e(F, n) = e(H) &\leq n + (q-1) \cdot e(H'') \\ &\leq n + (q-1) \cdot \mathrm{ex}_e(F, 2f(n)+1) \cdot e(G') \\ &< q \cdot \mathrm{ex}_e(F, 2f(n)+1) \cdot \mathrm{gex}(F, n) \end{aligned}$$

for every $n \geq 3$. $\square$

We give three applications of this theorem. The first one is the promised generalization of the upper bound in (10).

We say that a simple graph G is a *k-multiple* of G_1, where $k \in \mathbf{N}$ and G_1 is defined in (9), if G has this structure: $\bigcup G = A \cup \{v\} \cup B \cup C$ with $A < v < B < C$, $|A| = k$, the vertex v has degree k and is connected to every vertex in A, every vertex in A has degree $2k+1$ and is besides v connected to k vertices in B and to k vertices in C, and G has no other edges. The edges incident with v are called *backward edges* and the edges incident with vertices in $B \cup C$ are called *forward edges*. We denote the set of all k-multiples of G_1 by $M(k)$.

Lemma 3.2 *The sets of graphs $M(k)$, $k \in \mathbf{N}$, have the following properties.*

1. *For every k, $M(3k+1) \subset B(k, G_1)$. In particular, $M(31) \subset B(\binom{5}{2}, G_1)$.*
2. *For every k, $\mathrm{gex}(M(k), n) \ll n \log n$.*

Proof. 1. Let G be a $3k+1$-multiple of G_1 and $\chi : G \to \mathbf{N}$ be an edge coloring using each color at most k times. We select in G two backward edges $E_1 = \{i, v\}$ and $E_2 = \{j, v\}$, $i < j < v$, with different colors. It follows that we can select in G two forward edges $E_3 = \{i, l\}$ and $E_4 = \{j, l'\}$ such that $v < l' < l$ and the colors $\chi(E_1), \ldots, \chi(E_4)$ are distinct. Edges $E_1, \ldots, E_4$ form a G_1-copy on which χ is injective. Thus $G \in B(k, G_1)$.

2. Let $n \geq 2$ and G be any simple graph such that $\bigcup G = [n]$ and $G \not\succ F$ for every $F \in M(k)$. Let $J_i = \{E \in G : \min E = i\}$ for $i \in [n]$. In every J_i

it is possible to mark $\lfloor |J_i|/k \rfloor > |J_i|/k - 1$ edges so that every marked edge is in J_i immediately followed by $k-1$ unmarked ones (we order the edges in J_i by their endpoints). The graph G' formed by the marked edges satisfies

$$e(G') > e(G)/k - n.$$

Also, for every edge $\{i,j\} \in G'$, $i < j$, there are at least $k-1$ edges $\{i,l\} \in G$ with $l > j$, and for every two edges $\{i,j\}, \{i,j'\} \in G'$, $i < j < j'$, there are at least $k-1$ edges $\{i,l\} \in G$ with $j < l < j'$. Now we proceed as in Füredi [10]. We say that $\{i,j\} \in G'$, $i < j$, has *type* (j,m), where $m \ge 0$ is an integer, if there are two edges $\{i,l\}$ and $\{i,l'\}$ in G' such that $j < l < l'$ and $l - j \le 2^m < l' - j$. Consider the partition

$$G' = G^* \cup G^{**}$$

where G^* is formed by edges with at least one type and G^{**} by edges without type. It follows from the definition of type and of G' that if k edges of G^* have the same type, then $F \prec G$ for some $F \in M(k)$ which is forbidden. Thus any type is shared by at most $k-1$ edges. Since the number of types is less than $n(1+\log_2 n)$, we have

$$e(G^*) < (k-1)n + (k-1)n\log_2 n.$$

To bound $e(G^{**})$, we fix a vertex $i \in [n]$ and consider the endpoints $i < j_0 < j_1 < \cdots < j_{t-1} \le n$ of all edges $E \in G'$ which have no type and $\min E = i$. Let $d_r = j_r - j_{r-1}$ for $1 \le r \le t-1$ and $D = d_1 + \cdots + d_{t-1} = j_{t-1} - j_0$. If $d_1 \le D/2$, then $d_1 \le 2^m < D$ for some integer $m \ge 0$ and the edge $\{i,j_0\}$ would have type (j_0,m) because of the edges $\{i,j_1\}$ and $\{i,j_{t-1}\}$. Thus $d_1 > D/2$ and $D - d_1 < D/2$. By the same argument applied to $\{i,j_1\}$, $d_2 > (D-d_1)/2$ and thus $D - d_1 - d_2 < D/4$. In general, $1 \le D - d_1 - \cdots - d_r < D/2^r$ for $1 \le r \le t-2$. Thus $t \le \log_2 D + 2 < 2 + \log_2 n$. Summing these inequalities for all $i \in [n]$, we have

$$e(G^{**}) < 2n + n\log_2 n.$$

Alltogether we have

$$e(G) < kn + k(e(G^*) + e(G^{**})) < (k^2+k)n + k^2 n\log_2 n.$$

We conclude that $\mathrm{gex}(M(k),n) \ll n\log n$ and the constant in $\ll$ depends quadratically on k. $\square$

Theorem 3.3 *Let G_1 be the simple graph given in (9). We have the following bounds.*

1. $n \cdot \log n \ll \mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^3$.
2. $n \cdot \log n \ll \mathrm{ex}_i(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^3$.

Proof. 1. The lower bound follows from the lower bound in (10). To prove the upper bound, we use Theorem 3.1. By 2 of Lemma 3.2, we have $\mathrm{gex}(M(31), n) \ll n \log n$. Also, $\mathrm{gex}(G_1, n) \ll n \log n$ (by the upper bound in (10) or by $\mathrm{gex}(G_1, n) \le \mathrm{gex}(B, n)$). By 1 of Lemma 3.2, we can apply Theorem 3.1 with $B = M(31)$. Starting with the trivial bound $\mathrm{ex}_e(G_1, n) < 2^n$, (11) with $f(n) \ll \log n$ gives

$$\mathrm{ex}_e(G_1, n) \ll n^c$$

where $c > 0$ is a constant. Feeding this bound back to (11), we get

$$\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^{c+1}.$$

Two more iterations of (11) give

$$\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^{c+1}$$

and

$$\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^2 \cdot (\log\log\log n)^{c+1}$$

which is slightly better than the stated bound.

2. The lower bound follows from $\mathrm{ex}_i(G_1, n) \ge \mathrm{ex}_e(G_1, n)$. The upper bound follows from the upper bound in 1 by Theorem 2.3. □

Problem 3.4 *What is the exact asymptotics of* $\mathrm{ex}_e(G_1, n)$*?*

The second application of Theorem 3.1 concerns unordered extremal functions $\mathrm{ex}_e^u(F, n)$ and $\mathrm{gex}^u(G, n)$. They are defined as $\mathrm{ex}_e(F, n)$ and $\mathrm{gex}(G, n)$ except that in the containment the injection need not be increasing. So $\mathrm{gex}^u(G, n)$ is the classical graph extremal function. It is well known, see for example Bollobás [5, Exercise 24 in IV.7], that $\mathrm{gex}^u(F, n) \le (e(F)-1)\cdot n$ for every forest F. We extend the linear bound to unordered hypergraphs. Theorem 3.1 holds also in the unordered case because the proof is independent of ordering. Ordering is crucial only for obtaining linear or almost linear bounds on $\mathrm{gex}(F,n)$ and $\mathrm{gex}(B,n)$ because the inequality (11) is useless if $f(n)$ is not almost constant. The proof of Theorem 3.1 shows also that if F is a forest and all members of B are forests (which is not the case for $B = M(k)$) then $\binom{p}{2}$ can be replaced by $p-1$ (because for $|E| = p$ every p two-element edges $E^* \subset E$ contain a cycle but no $F' \in B$ has a cycle).

Theorem 3.5 *Let F be a forest. Its unordered hypergraph extremal function satisfies*

$$\mathrm{ex}_e^u(F,n) \ll n.$$

Proof. Let $v(F) = p$ and $e(F) = q > 1$ (case $q = 1$ is trivial). It is not hard to find a forest F' with large $e(F') = Q$ — $Q \le (pq^2)^{q+1}$ suffices — that is a $p-1$-blow-up of F. We set $B = \{F'\}$ and use (11) with the bounds $\mathrm{gex}^u(F,n) \le (q-1)n$, $f(n) = Q-1$ (since $\mathrm{gex}^u(B,n) = \mathrm{gex}^u(F',n) \le (Q-1)n$), and $\mathrm{ex}_e^u(F,n) < 2^n$ (trivial):

$$\mathrm{ex}_e^u(F,n) < q \cdot (q-1)n \cdot 2^{2Q-1} = \binom{q}{2} 4^Q \cdot n.$$

$\square$

One can prove the bound $\mathrm{ex}_e^u(F,n) \ll n$ also directly, without Theorem 3.1, by adapting the proof of $\mathrm{gex}^u(F,n) \ll n$ to hypergraphs. The third application of Theorem 3.1 follows in the next section.

4 Partitions and star forests

The bound (5) tells us that if F is any fixed partition with k blocks and H is a k-sparse partition with $H \not\succ F$, then $v(H) = i(H)$ has an almost linear upper bound in terms of $e(H)$. The following theorem bounds $i(H)$ almost linearly in terms of $e(H)$ in the wider class of (not necessarily simple) hypergraphs H. The proof is based on (7).

Theorem 4.1 *Let F be a partition with $p = v(F)$ and $q = e(F) > 1$ and H be a F-free hypergraph, not necessarily simple. Then*

$$i(H) < (q-1) \cdot v(H) + e(H) \cdot \beta(q, 2p, e(H)) \qquad (12)$$

where $\beta(k,l,n)$ is the almost constant function defined in (8).

Proof. Let $\bigcup H = [n]$ and the edges of H be $E_1, E_2, \ldots, E_e$ where $e = e(H)$. We set, for $1 \le i \le n$, $S_i = \{j \in [e] : \ i \in E_j\}$ and consider the sequence

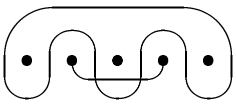
$$v = I_1 I_2 \ldots I_n$$

where I_i is an arbitrary ordering of S_i. Clearly, v is over $[e]$ and $|v| = i(H)$. The sequence v may not be q-sparse, because of the transitions $I_i I_{i+1}$, but it is easy to delete at most $q-1$ terms from the beginning of every I_i, $i > 1$, so that the resulting subsequence w is q-sparse. Thus $|w| \ge |v| - (q-1)(n-1)$. It follows that if w contains $u(q, 2p)$, where $u(k, l)$ is defined in (6), then H contains F but this is forbidden. (Note that the subsequence aab in v forces the first a and the b to appear in two distinct segments I_i and thus it gives incidences of E_a and E_b with two distinct vertices.) Hence $w \not\succ u(q, 2p)$ and we can apply (7):

$$i(H) = |v| < (q-1)n + |w| \le (q-1)n + e \cdot \beta(q, 2p, e).$$

$\square$

We show that for the partition

$$F = H_2 = (\{1, 3, 5\}, \{2, 4\}) =$$ 

the factor at $e(H)$ in (12) must be $\gg \alpha(e(H))$. We proceed as in the proof of $g(n) = \mathrm{gex}(G_0, n) \gg n\alpha(n)$ in (2) and take a 2-sparse sequence v over $[n]$ such that $v \not\succ 12121$, $|v| \gg n\alpha(n)$, and $v = I_1 I_2 \ldots I_{2n}$ where every interval I_i consists of distinct terms. We define the hypergraph

$$H = (E_i : \ i \in [n]) \ \text{ with } \ E_i = \{j \in [2n] : \ i \text{ appears in } \ I_j\}.$$

We have $i(H) = |v| \gg n\alpha(n)$, $\bigcup H = [2n]$, $v(H) = 2n$, and $e(H) = n$. It is clear that $H \not\succ H_2$ because $v \not\succ 12121$.

Taking in Theorem 4.1 H to be simple and with the maximum weight, we obtain as a corollary that if F is a partition, $p = v(F)$, and $q = e(F) > 1$, then

$$\mathrm{ex}_i(F, n) < (q-1)n + \mathrm{ex}_e(F, n) \cdot \beta(q, 2p, \mathrm{ex}_e(F, n)).$$

But Theorem 2.3, when it applies, gives better bounds.

Our last theorem generalizes in two ways the upper bound in (2). We consider a class of forbidden forests that contains G_0 as a member and we extend by means of Theorems 3.1 and 2.3 the almost linear upper bound to hypergraphs. The class consists of *star forests* which are forests F with this structure: $\bigcup F = A \cup B$ for some sets $A < B$ such that every vertex in B has degree 1 and every edge of F connects A and B. Thus F is a star forest iff every component of F is a star and every central vertex of a star is smaller than every leaf.

Theorem 4.2 *Let F be a star forest with $r > 1$ components, p vertices, and $q = p - r$ edges. Let $t = (p-1)(q-1)+1$ and $\beta(k,l,n)$ be the almost constant function defined in (8). We have the following bounds.*

1. $\mathrm{gex}(F, n) < (r-1)n + n \cdot \beta(r, 2q, n)$.
2. $\mathrm{ex}_e(F, n) \ll n \cdot \beta(r, 2tq, n)^3$.
3. $\mathrm{ex}_i(F, n) \ll n \cdot \beta(r, 2tq, n)^3$.

Proof. 1. We mark the centers of the stars in F with $1, 2, \ldots, r$ according to their order and give the leaves of any star the mark of its center. The marks on all leaves then form a sequence u over $[r]$ of length $p - r$. Now let G be any simple graph with $\bigcup G = [n]$ and $G \not\succ F$. We consider the sequence

$$v = I_1 I_2 \ldots I_n$$

where I_j is any ordering of the set $\{i \in [n] : \ \{i, j\} \in G, i < j\}$. As in the previous proof, we select an r-sparse subsequence w of v with length $|w| \geq |v| - (r-1)(n-1)$. Suppose that $w \succ u(r, 2(p-r))$ where $u(k,l)$ is defined in (6). Then w has a (not necessarily consecutive) subsequence z of the form

$$a_1 a_2 \ldots a_r a_1 a_2 \ldots a_r \ldots a_1 a_2 \ldots a_r$$

with $2(p-r)$ segments $a_1 a_2 \ldots a_r$. We have $a_{i_1} < a_{i_2} < \ldots < a_{i_r}$ for a permutation $i_1, i_2, \ldots, i_r$ of $[r]$. We label every term a_{i_j} in z with j. Clearly, if we select one term from the 2-nd, 4-th, ..., $2(p-r)$-th segment of z so that the labels on the selected terms coincide with the sequence u, the selected terms lie in $p - r$ distinct intervals $I_{j_1}, \ldots, I_{j_{p-r}}$, $j_1 < \ldots < j_{p-r}$. Since the selected terms are preceded by one segment $a_1 a_2 \ldots a_r$, we have $a_{i_r} < j_1$. The edges between $a_1, \ldots, a_r$ and $j_1, \ldots, j_{p-r}$ which corespond to the selected terms form an F-copy in G but $G \succ F$ is forbidden. Therefore $w \not\succ u(r, 2(p-r))$ and we can apply (7):

$$e(G) = |v| \le (r-1)n + |w| < (r-1)n + n \cdot \beta(r, 2(p-r), n).$$

2. Suppose that F has the vertex set $[p]$ (so that $[r]$ are the centers of the stars and $[r+1, p]$ are the leaves). For $k \in \mathbf{N}$ we denote $F(k)$ the star forest with the vertex set $[r + (p-r)k]$ in which $[r]$ are again the centers of stars and for $i = 1, 2, \ldots, p-r$ the vertices in $[r + (i-1)k + 1, r + ik]$ are joined to the same vertex in $[r]$ as $r + i$ is joined in F. It is easy to see that $F(t) = F((p-1)(q-1)+1)$ is a $p-1$-blow-up of F. Also, $e(F(k)) = kq$. We set $B = \{F(t)\}$ and use (11) with the bounds $\mathrm{gex}(F, n) \ll n \cdot \beta(r, 2q, n) = n \cdot \beta'$ (bound 1 for F), $f(n) = c\beta(r, 2tq, n) = c\beta$ for a constant $c > 0$ (bound 1 for $F(t)$), and $\mathrm{ex}_e(F, n) < 2^n$ (trivial):

$$\mathrm{ex}_e(F, n) \ll n \cdot \beta' \cdot 2^{2c\beta+1} < n \cdot 2^{2(c+1)\beta}.$$

The second application of (11) gives

$$\mathrm{ex}_e(F, n) \ll n \cdot \beta' \cdot \beta \cdot 2^{2(c+1)\cdot\beta(r, 2tq, 2c\beta+1)} \ll n \cdot \beta^3$$

because $\beta' \le \beta$ and

$$\beta(r, 2tq, x) \ll \log\log x$$

(this is true with any number of logarithms).

3. This follows from 2 by Theorem 2.3. $\square$

The lower bound in (2) shows that in general the factor at n in 1, 2, and 3 of the previous theorem cannot be replaced with a constant and may be as big as $\gg \alpha(n)$. The proved bounds hold also for the reversals of star forests.

5 Concluding remarks

It is reasonable to call a function $f : \mathbf{N} \to \mathbf{R}$ *nearly linear* if $n^{1-\varepsilon} \ll f(n) \ll n^{1+\varepsilon}$ holds for every $\varepsilon > 0$. We identify a candidate for the class of all hypergraphs F with nearly linear $\mathrm{ex}_e(F, n)$. If F is isomorphic to the hypergraph $(\{1\}, \{2\}, \ldots, \{k\})$, then $\mathrm{ex}_e(F, n)$ is eventually constant ([18]) and thus is not nearly linear. For other hypergraphs we have $\mathrm{ex}_e(F, n) \ge n$ because $F \not\prec (\{1\}, \{2\}, \ldots, \{n\})$. An *orderly bipartite forest* is a simple graph F such that F has no cycle and $\min E < \max E'$ holds for every two edges of F. In other words, F is a forest and there is a partition $\bigcup F = A \cup B$ such that $A < B$ and every edge of F connects A and B. We denote the class of all orderly bipartite forests by OBF. We say that F is an *orderly bipartite forest with singletons* if $F = F_1 \cup F_2$ where $F_1 \in \mathrm{OBF}$ and F_2 is a hypergraph consisting of possibly repeating singleton edges. For example, F may be

$$F = (\{8\}, \{6\}_1, \{6\}_2, \{2\}, \{1,6\}, \{3,6\}, \{4,5\}, \{4,7\}).$$

The class OBF subsumes star forests and their reversals. G_1 defined in (9) belongs to OBF but is neither a star forest nor a reversed star forest.

Lemma 5.1 *If the hypergraph F is not an orderly bipartite forest with singletons, then there is a constant $\gamma > 1$ such that*

$$\mathrm{ex}_e(F, n) \gg n^{\gamma}$$

and hence $\mathrm{ex}_e(F, n)$ is not nearly linear.

Proof. If F is not an orderly bipartite forest with singletons, then F has (i) an edge with more than two elements or (ii) two separated two-element edges or (iii) a two-path isomorphic to $(\{1,2\}, \{2,3\})$ or (iv) a repeated two-element edge or (v) an even cycle of two-element edges (odd cycles are subsumed in (iii)). In the cases (i)–(iv) we have $\mathrm{ex}_e(F, n) \gg n^2$ because the complete bipartite graph with parts $[\lfloor n/2 \rfloor]$ and $[\lfloor n/2 \rfloor + 1, n]$ does not contain F. As for the case (v), an application of the probabilistic method (Erdős [8]) provides an unordered graph that has n vertices, $\gg n^{1+1/k}$ edges, and no even cycle of length k. Thus, in the case (v), $\mathrm{ex}_e(F, n) \gg n^{1+1/k}$ for some $k \in \mathbf{N}$. $\square$

We conjecture that $\mathrm{ex}_e(F, n)$ is nearly linear if and only if F is an orderly bipartite forest with singletons that is not isomorphic to $(\{1\}, \{2\}, \ldots, \{k\})$. Since every orderly bipartite forest with singletons is contained in some orderly bipartite forest, it suffices to consider only orderly bipartite forests.

Problem 5.2 *Prove (or disprove) that for every orderly bipartite forest F we have*

$$\mathrm{ex}_e(F, n) \ll n(\log n)^c$$

for some constant $c > 0$.

It is not difficult to find for every $F \in \mathrm{OBF}$ and $k \in \mathbf{N}$ an $F' \in \mathrm{OBF}$ that is a k-blow-up of F. Thus the previous bound would follow by Theorem 3.1 from the graph bound $\mathrm{gex}(F, n) \ll n(\log n)^c$.

It is natural to consider two subclasses $\mathrm{OBF}^l \subset \mathrm{OBF}^\alpha \subset \mathrm{OBF}$ where OBF^l consists of all $F \in \mathrm{OBF}$ with $\mathrm{ex}_e(F, n) \ll n$ and OBF^α consists of all $F \in \mathrm{OBF}$ with $\mathrm{ex}_e(F, n) \ll n \cdot f(\alpha(n))$ for a primitive recursive function $f(n)$. Both inclusions are strict as witnessed by G_0 and G_1 (defined in (1) and (9)). In this article we ignored the class OBF^l completely and showed that OBF^α contains all star forests (and their reversals). It would much interesting to learn more about OBF^l and OBF^α. Does the latter class consist only of star forests and their reversals?

References

[1] P. K. Agarwal and M. Sharir, Davenport–Schinzel sequences and their geometric applications. In: J.-R. Sacks and J. Urrutia (ed.), *Handbook of Computational Geometry,* Elsevier, Amsterdam, 2000; pp. 1–47.

[2] P. K. Agarwal, M. Sharir and P. Shor, Sharp upper and lower bounds on the length of general Davenport–Schinzel sequences, *J. Comb. Theory, Ser. A,* **52** (1989), 228–274.

[3] B. Bollobás, *Extremal Graph Theory,* Academic Press, London, 1978.

[4] B. Bollobás, Extremal graph theory. In: R. L. Graham, M. Grötschel and L. Lovász (ed.), *Handbook of Combinatorics, Volume 2,* Elsevier, Amsterdam, 1995; pp. 1231–1292.

[5] B. Bollobás, *Modern Graph Theory,* Springer, Berlin, 1998.

[6] P. Brass, G. Károlyi and P. Valtr, A Turán-type extremal theory of convex geometric graphs, in preparation.

[7] H. Davenport and A. Schinzel, A combinatorial problem connected with differential equations, *Amer. J. Math.,* **87** (1965), 684–694.

[8] P. Erdős, Graph theory and probability, *Canadian J. Math.,* **11** (1959), 34–38.

[9] P. Frankl, Extremal set systems. In: R. L. Graham, M. Grötschel and L. Lovász (ed.), *Handbook of Combinatorics, Volume 2,* Elsevier, Amsterdam, 1995; pp. 1293–1329.

[10] Z. Füredi, The maximum number of unit distances in a convex n-gon, *J. Comb. Theory, Ser. A,* **55** (1990), 316–320.

[11] Z. Füredi, Turán type problems. In: A. D. Keedwell (ed.), *Surveys in Combinatorics, 1991,* Cambridge University Press, Cambridge, UK, 1991; pp. 253–300.

[12] Z. Füredi and P. Hajnal, Davenport–Schinzel theory of matrices, *Discrete Math.,* **103** (1992), 233–251.

[13] S. Hart and M. Sharir, Nonlinearity of Davenport–Schinzel sequences and of generalized path compression schemes, *Combinatorica,* **6** (1986), 151–177.

[14] M. Klazar, A general upper bound in extremal theory of sequences, *Commentat. Math. Univ. Carol.,* **33** (1992), 737–746.

[15] M. Klazar, Counting pattern-free set partitions II. Noncrossing and other hypergraphs, *Electr. J. Comb.,* **7** (2000), R34, 25 pages.

[16] M. Klazar, Extremal problems (and a bit of enumeration) for hypergraphs with linearly ordered vertex sets, *ITI Series,* technical report **2001-021**, 40 pages.

[17] M. Klazar, Generalized Davenport–Schinzel sequences: results, problems, and applications, *Integers,* **2** (2002), A11, 39 pages.

[18] M. Klazar, Extremal problems for ordered hypergraphs: small configurations and some enumeration, in preparation.

[19] M. Sharir and P. K. Agarwal, *Davenport–Schinzel Sequences and Their Geometric Applications,* Cambridge University Press, Cambridge, UK, 1995.

[20] Zs. Tuza, Applications of the set-pair method in extremal hypergraph theory. In: P. Frankl et al. (ed.), *Extremal Problems for Finite Sets,* János Bolyai Mathematical Society, Budapest, 1994; pp. 479–514.

[21] Zs. Tuza, Applications of the set-pair method in extremal problems, II. In: D. Miklós et al. (ed.), *Combinatorics, Paul Erdős is Eighty, Volume 2,* János Bolyai Mathematical Society, Budapest, 1996; pp. 459–490.

[22] P. Valtr, Generalizations of Davenport–Schinzel sequences. In: R. L. Graham et al. (ed.), *Contemporary Trends in Discrete Mathematics, Štiřín Castle 1997 (Czech Republic),* American Mathematical Society, Providence, RI, 1999; pp. 349–389.