Free Quillen Factorization Systems

Dedicated to Professor Inassaridze at the occasion of his seventieth birthday

Aleš Pultr* and Walter Tholen†

Abstract The notion of Quillen factorization system is obtained by strengthening the left and right lifting properties in a Quillen model category to the unique diagonalization property. An equivalent description of this notion is given in terms of a double factorization system which decomposes each arrow uniquely into three factors. The free category with Quillen factorization system over a given category is described.

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Key words: Quillen factorization system, double factorization system, free Quillen factorization system.

Introduction

In a Quillen model category one has weak factorization systems $(\mathcal{W} \cap \mathcal{D}, \mathcal{N})$, $(\mathcal{D}, \mathcal{W} \cap \mathcal{N})$ such that the class $\mathcal{W}$ (of "weak equivalences") is closed under retracts and has the 2-out-of-3 property (see [AHRT], [H], [Q]). The two players in each of the two systems are linked by the left and right lifting property. Contrary to the situation with factorization systems as studied in Category Theory, the "diagonals" provided by the lifting property generally fail (badly) to be unique. Nevertheless, one may ask: are there examples

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of Quillen model categories with unique "diagonals", the structure of which we call *Quillen factorization system*?

In this paper we describe free such structures over a given category $\mathcal{C}$: they are carried by the category $\mathcal{C}^3$, with $3 = \{\cdot \longrightarrow \cdot \longrightarrow \cdot\}$. The result is a natural extension of the fact that $\mathcal{C}^2$ carries the free (unique-diagonalization) factorization system over $\mathcal{C}$ (see [C], [KT], [RW]). It is proved best by first describing Quillen factorization systems as double factorization systems, which decompose a given arrow into three factors; in Quillen's language: into a trivial cofibration followed by a bifibration followed by a trivial fibration.

Although we reserve the (more tedious) functorial description of the results presented in this paper for a later paper, from the presentation of factorization systems as quotients of free systems (as given in [KT], [RW]) and of weak factorization systems as lax quotients of the same free systems (as given in [RT]) it is clear that double factorization systems emerge as quotients of the system in $\mathcal{C}^3$ described here, and that the pairs of weak factorization systems in a Quillen model category $\mathcal{C}$ are lax quotients of it. Hence, the model structure of $\mathcal{C}^3$ described here is by no means special but indeed universal.

We review some facts and examples of ordinary factorization systems in Section 1, in so far as they are relevant for the double factorizations as introduced in Section 2. Then, in Section 3, we show how they relate to Quillen factorization systems, while in Section 4 we show the freeness over $\mathcal{C}$ of the natural system of $\mathcal{C}^3$.

1. Factorization systems

1.1 An *(orthogonal) factorization system (fs)* in a category $\mathcal{C}$ is usually given by a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms satisfying

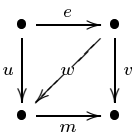
(F1) Iso $\cdot \mathcal{E} \subseteq \mathcal{E}$ and $\mathcal{M} \cdot$ Iso $\subseteq \mathcal{M}$,

(F2) Mor $= \mathcal{M} \cdot \mathcal{E}$,

(F3) $\mathcal{E} \perp \mathcal{M}$.

Here Iso is the class of isomorphisms in $\mathcal{C}$, Mor the class of all morphisms in $\mathcal{C}$, and $\mathcal{M} \cdot \mathcal{E}$ is the class of all composite arrows $m \cdot e$ of $e \in \mathcal{E}$ followed by $m \in \mathcal{M}$. Finally, $\mathcal{E} \perp \mathcal{M}$ stands for $e \perp m$ for all $e \in \mathcal{M}$, $m \in \mathcal{M}$,

which means that for all morphisms u, v with $mu = ve$ there is a unique "diagonal" morphism w with $we = u$ and $mw = v$.



(1)

1.2 Here is a list of well-known properties of the classes comprising a factorization system:

(1) $\mathcal{E} \cap \mathcal{M} =$ Iso.

(2) $(\mathcal{E}, \mathcal{M})$-decompositions of morphisms are unique (up to isomorphism).

(3) $\mathcal{E} = \mathcal{M}^{\perp} := \{e \mid \forall m \in \mathcal{M} : e \perp m\}$, $\mathcal{M} = {}^{\perp}\mathcal{E} := \{m \mid \forall e \in \mathcal{E} : e \perp m\}$.

(4) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

(5) $\mathcal{M}$ is weakly left-cancellable, that is: if $nm \in \mathcal{M}$ and $n \in \mathcal{M}$, then $m \in \mathcal{M}$, and $\mathcal{E}$ is weakly right-cancellable, that is: if $ed \in \mathcal{E}$ and $d \in \mathcal{E}$, then $e \in \mathcal{E}$.

(6) $\mathcal{M}$ is closed under limits, and $\mathcal{M}$ is closed under colimits.

(7) $\mathcal{M}$ is stable under pullback and intersection, and $\mathcal{E}$ is stable under pushout and cointersection.

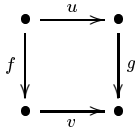
1.3 We also recall some well-known examples.

(1) (Iso, Mor) and (Mor, Iso) are factorization systems of every category $\mathcal{C}$.

(2) Let $P : \mathcal{D} \to \mathcal{C}$ be a fibration, and denoted by Ini the class P-initial (= P-cartesian) morphisms in $\mathcal{D}$. Then, for every fs $(\mathcal{E}, \mathcal{M})$ of $\mathcal{C}$, $(P^{-1}\mathcal{E}, \text{Ini} \cap P^{-1}\mathcal{M})$ is an fs of $\mathcal{D}$. Dually : if P is a cofibration and Fin the class of P-final (=P-cocartesian) morphisms in $\mathcal{D}$, then $(\text{Fin} \cap P^{-1}\mathcal{E},\ P^{-1}\mathcal{M})$ is an fs of $\mathcal{D}$.

(3) An application of (2) to the forgetful functor P: **Top** $\to$ **Set** and the (Epi, Mono) fs for sets yields the (Epi, RegMono) and (RegEpi, Mono)-factorization systems for topological spaces, where of course RegMono is the class of regular monomorphisms, etc.

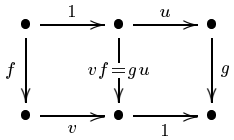
(4) The objects of the functor category $\mathcal{C}^2$ with $2 = \{\cdot \to \cdot\}$ are the morphisms of $\mathcal{C}$, and a morphism $[u, v] : f \to g$ in $\mathcal{C}^2$ is given by the commutative square



(2)

in $\mathcal{C}$; composition is horizontal. The codomain functor cod: $\mathcal{C}^2 \to \mathcal{C}$ is a fibration if and only if $\mathcal{C}$ has pullbacks, with the cod-cartesian morphisms given by the pullback squares in $\mathcal{C}$. An application of (2) and the (Iso, Mor) fs of $\mathcal{C}$ yields the ([Mor, Iso], Pull) fs of $\mathcal{C}^2$ (where, of course, [Mor, Iso] contains all morphisms $[u, v]$ of $\mathcal{C}^2$ with v iso). Dually: the domain functor dom: $\mathcal{C}^2 \to \mathcal{C}$ is a cofibration if and only if $\mathcal{C}$ has pushouts, and with (2) the (Mor, Iso) fs of $\mathcal{C}$ gives the (Push, [Iso, Mor]) fs of $\mathcal{C}^2$.

(5) While [Mor, Iso] is a first factor of a fs of $\mathcal{C}^2$ when $\mathcal{C}$ has pullbacks, it is always a second factor: ([Iso, Mor], [Mor, Iso]) is a fs of $\mathcal{C}^2$, decomposing diagram (2) as



(3)

1.4 Example (5) of 1.3 gives in fact the free category with factorization system over a given category $\mathcal{C}$. Indeed, denoting by $\triangle : \mathcal{C} \to \mathcal{C}^2$ the full embedding $(f : A \to B) \longmapsto ([f, f] : 1_A \longrightarrow 1_B)$, for every category $\mathcal{B}$ with a given fs $(\mathcal{E}, \mathcal{M})$, composition with $\triangle$ is a category equivalence

$$(-)\triangle : (\mathcal{B}^{\mathcal{C}^2})_{\mathcal{E},\mathcal{M}} \longrightarrow \mathcal{B}^{\mathcal{C}}$$

Here the left-hand side denotes the full subcategory of $\mathcal{B}^{\mathcal{C}^2}$ of functors $G : \mathcal{C}^2 \to \mathcal{B}$ with $G([\text{Iso, Mor}]) \subseteq \mathcal{E}$ and $G([\text{Mor, Iso}]) \subseteq \mathcal{M}$.
In fact, any such functor G is, up to isomorphism, determined by $F := G\triangle : \mathcal{C} \to \mathcal{B}$: for every morphism $f : A \longrightarrow \mathcal{B}$ in $\mathcal{C}$, G must map the decomposition

$$1_A \xrightarrow{[1_A, f]} f \xrightarrow{[f, 1_B]} 1_B$$

of $[f, f]$ to the $(\mathcal{E}, \mathcal{M})$-decomposition of Ff in $\mathcal{B}$, which fixes the object Gf in $\mathcal{B}$, up to isomorphism. To see that the value $G[u, v]$ is determined by F, because of the decomposition (3) it suffices to see that this is true for $G[1_A, v]$ and $G[u, 1_D]$; but the decomposition

$$1_A \xrightarrow{[1_A, f]} f \xrightarrow{[1_A, v]} vf \xrightarrow{[vf, 1_D]} 1_D$$

(with $v : B \to D$) shows that $G[1_A, v]$ is determined by the $(\mathcal{E}, \mathcal{M})$-diagonalization property (F3) since $G[1_A, f] \in \mathcal{E}$, $G[vf, 1_D] \in \mathcal{M}$ and the values of

$$G([1_A, v][1_A, f]) = G[1_A, vf]$$

and

$$G([vf, 1_D][1_A, v]) = G[v, v] \cdot G[f, 1_B]$$

have already been fixed. Likewise for $G[u, 1_D]$.

Hence, we have sketched the proof that $(-)\triangle$ is surjective on objects. To see that $(-)\triangle$ is full and faithful, just observe that for any natural transformation $\alpha : G \to G'$ (where G' has the same properties as G), the value of α_f is again determined by the $(\mathcal{E}, \mathcal{M})$-diagonalization property and the values of $\alpha_{1_A}, \alpha_{1_B}$, since the decomposition

$$[f, f] = [f, 1_B][1_A, f]$$

yields the commutative diagram

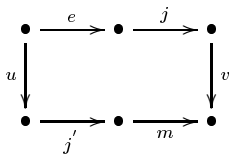
$$\begin{array}{ccccc} G1_A & \xrightarrow{G[1_A, f]} & Gf & \xrightarrow{G[f, 1_B]} & G1_B \\ \downarrow{\scriptstyle \alpha_{1_A}} & & \downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle \alpha_{1_B}} \\ G1_A & \xrightarrow[G'[1_A, f]]{} & G'f & \xrightarrow[G'[f, 1_B]]{} & G'1_B \end{array} \tag{4}$$

2. Double factorization systems

2.1 Definition A *double factorization system (dfs)* in $\mathcal{C}$ is given by a triple $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ of classes of morphisms satisfying

(DF1) Iso $\cdot \mathcal{E} \subseteq \mathcal{E}$, Iso $\cdot \mathcal{J} \cdot$ Iso $\subseteq \mathcal{J}$ and $\mathcal{M} \cdot$ Iso $\subseteq \mathcal{M}$,
(DF2) Mor $= \mathcal{M} \cdot \mathcal{J} \cdot \mathcal{E}$,

(DF3) for any commutative diagram



(5)

in $\mathcal{C}$ with $e \in \mathcal{E}$, $j, j' \in \mathcal{J}$, $m \in \mathcal{M}$ there are uniquely determined "diagonals" s and t with $se = u$, $j's = tj$ and $ut = v$.

¿From (DF3) we obviously obtain that decompositions of morphisms

$$f = mje \ (e \in \mathcal{E},\ j \in \mathcal{J},\ m \in \mathcal{M})$$

as given by (DF2) are unique up to isomorphisms.

2.2 Examples

(1) For every fs $(\mathcal{E}, \mathcal{M})$ of $\mathcal{C}$, $(\mathcal{E}$, Iso, $\mathcal{M})$ is a dfs of $\mathcal{C}$.

(2) For a bifibration $P : \mathcal{D} \to \mathcal{C}$ and any fs $(\mathcal{E}, \mathcal{M})$ of $\mathcal{C}$,

$$(\text{Fin} \cap P^{-1}\mathcal{E},\ P^{-1}\text{Iso},\ \text{Ini} \cap P^{-1}\mathcal{M})$$

is a dfs of $\mathcal{D}$. In the case of the forgetful functor P: **Top** $\to$ **Set** and $(\mathcal{E}, \mathcal{M})$=(Epi, Mono), this yields the dfs

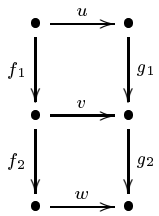
$$(\text{RegEpi},\ \text{Epi} \cap \text{Mono},\ \text{RegMono})$$

of **Top.**

(3) In generalization of the previous example, in any category with (RegEpi, Mono)-and (Epi, RegMono)-factorizations, (RegEpi, Epi $\cap$ Mono, RegMono) is a dfs.

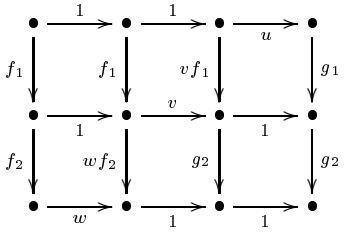
(4) The objects of the functor category $\mathcal{C}^3$ with $3 = \{\cdot \to \cdot \to \cdot\}$ are the composable pairs (f_1, f_2) of morphisms of $\mathcal{C}$, and a morphism $[u, v, w]$:

$(f_1, f_2) \to (g_1, g_2)$ in $\mathcal{C}^3$ is given by a commutative diagram



(6)

in $\mathcal{C}$; composition is horizontal. Diagram (6) may be decomposed as

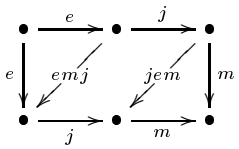


(7)

leading us to the dfs ([Iso, Iso, Mor], [Iso, Mor, Iso], [Mor, Iso, Iso]) of $\mathcal{C}^3$.

2.3 Proposition *For a dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$, Iso $\subseteq \mathcal{E} \cap \mathcal{J} \cap \mathcal{M}$, *and both* $(\mathcal{E}, \mathcal{M} \cdot \mathcal{J})$ *and* $(\mathcal{J} \cdot \mathcal{E}, \mathcal{M})$ *are factorization systems.*

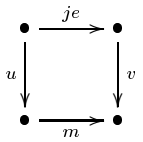
Proof: For the first assertion, by (DF1) it suffices to show that each identity morphism is in $\mathcal{E} \cap \mathcal{J} \cap \mathcal{M}$. But with the decomposition $1 = mje$ we can consider the commutative diagram



(8)

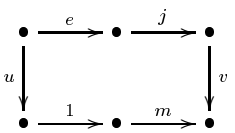
which, with the uniqueness property of (DF3), shows $emj = 1$ and $jem = 1$. Hence, mj, je, m and e are isomorphisms, and by (DF1) $1 = (mj)e \in \mathcal{E}$, $1 = m(je) \in \mathcal{M}$ and $1 = mje \in \mathcal{J}$.
$(\mathcal{J} \cdot \mathcal{E}, \mathcal{M})$ obviously satisfies (F1), (F2). For (F3), consider the commutative

diagram



(9)

and, using Iso $\subseteq \mathcal{J}$, redraw it as



(10)

in order to apply (DF3).

2.4 Corollary *For a dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$, $\mathcal{E} \cap \mathcal{M} \cdot \mathcal{J} = \text{Iso} = \mathcal{J} \cdot \mathcal{E} \cap \mathcal{M}$, *and* $\mathcal{E}$ *and* $\mathcal{M}$ *satisfy properties* (4)-(7) *of Prop.* 1.2.

We also see that any two of the three classes comprising a dfs determine the third:

2.5 Corollary *For a dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$,

$$\mathcal{E} = (\mathcal{M} \cdot \mathcal{J})^{\perp},\ \mathcal{J} = \mathcal{E}_{\perp} \cap \mathcal{M}^{\perp},\ \mathcal{M} = (\mathcal{J} \cdot \mathcal{E})_{\perp}$$

Proof: Only the second equality needs explanation: it follows from

$$\mathcal{J} = \mathcal{J} \cdot \mathcal{E} \cap \mathcal{M} \cdot \mathcal{J},$$

for which "$\subseteq$" is trivial (after Prop. 2.3). For "$\supseteq$", consider $f = je = mj'$ with $e \in \mathcal{E},\ j, j' \in \mathcal{J},\ m \in \mathcal{M}$; then the uniqueness of the double factorization $f = 1je = mj'1$ show that e, m must be isomphisms and $f \in \mathcal{J}$.

The two fs induced by a dfs as in 2.3 are comparable, in the following sense.

2.6 Proportion *The following conditions on two fs* $(\mathcal{E}, \mathcal{N})$ *and* $(\mathcal{D}, \mathcal{M})$ *of* $\mathcal{C}$ *are equivalent; if they hold we call the two fs comparable:*

(i) $\mathcal{E} \subseteq \mathcal{D}$,

(ii) $\mathcal{N} \subseteq \mathcal{M}$,

(iii) $\mathcal{E} \perp \mathcal{M}$,

(iv) $\mathcal{D} = (\mathcal{D} \cap \mathcal{N}) \cdot \mathcal{E}$,

(v) $\mathcal{N} = \mathcal{M} \cdot (\mathcal{D} \cap \mathcal{N})$.

Proof: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is a consequence of the Galois correspondence given by the orthogonality relation and the fact that the classes comprising an fs are closed under this correspondence. Trivially (iv) $\Rightarrow$ (i) since Iso $\subseteq \mathcal{D} \cap \mathcal{N}$ For (i) $\Rightarrow$ (iv), decompose $d \in \mathcal{D}$ as $d = ne$ with $n \in \mathcal{N}, e \in \mathcal{E}$. since $\mathcal{E} \subseteq \mathcal{D}$, with 1.2(5) one concludes $n \in \mathcal{D} \cap \mathcal{N}$. This shows the inclusion "$\subseteq$", with the converse being obvious. (v) $\Leftrightarrow$ (ii) follows dually.

We can now prove a converse proposition to 2.3.

2.7 Theorem

(1) *For every dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ *of* $\mathcal{C}$*, the pairs* $(\mathcal{E}, \mathcal{N} \cdot \mathcal{J})$, $(\mathcal{J} \cdot \mathcal{E}, \mathcal{N})$ *are comparable fs of* $\mathcal{C}$*.*

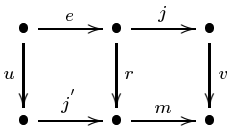
(2) For every comparable pair $(\mathcal{E}, \mathcal{N})$, $(\mathcal{D}, \mathcal{M})$ *of fs of* $\mathcal{C}$*,* $(\mathcal{E}, \mathcal{D} \cap \mathcal{N}, \mathcal{M})$ *is a dfs of* $\mathcal{C}$*.*

(3) The assignments of (1), (2) *constitute a bijective correspondence between all dfs of* $\mathcal{C}$ *and all comparable pairs of fs of* $\mathcal{C}$*.*

Proof:

(1) follows from 2.3.

(2) (DF1) is trivial, and from 2.6 (iv) one has Mor $= \mathcal{M} \cdot \mathcal{D} = \mathcal{M} \cdot (\mathcal{D} \cap \mathcal{N}) \cdot \mathcal{E}$. For (DF3) consider the commutative diagram (5) with $e \in \mathcal{E}$, $j, j' \in \mathcal{D} \cap \mathcal{N}$, $m \in \mathcal{M} \subseteq \mathcal{N}$, and from $\mathcal{E} \perp \mathcal{N}$ obtain a morphism r providing the commutative diagram



(11)

Since $j' \in \mathcal{N}$ and $j \in \mathcal{D}$, both squares have unique "diagonals", as needed.

(3) Since $\mathcal{J} = \mathcal{J} \cdot \mathcal{E} \cap \mathcal{M} \cdot \mathcal{J}$, applying (1) and then (2) to a dfs gives us back the same dfs. Starting with a comparable pair of fs, applying (2) and then (1) leads us to the fs $(\mathcal{E}, \mathcal{M} \cdot (\mathcal{D} \cap \mathcal{N}))$, $((\mathcal{D} \cap \mathcal{N}) \cdot \mathcal{E}, \mathcal{M})$, which are the original systems by 2.6 (iv),(v).

3. The "weak equivalences" of a double factorization system

3.1 Definition For a dfs $(\mathcal{E}, \mathcal{J}, \mathcal{M})$, we also call the morphisms in

- $\mathcal{D} = \mathcal{J} \cdot \mathcal{E}$ the *cofibrations* of the dfs,
- $\mathcal{W} = \mathcal{M} \cdot \mathcal{E}$ the *weak equivalences* of the dfs,
- $\mathcal{N} = \mathcal{M} \cdot \mathcal{J}$ the *fibrations* of the dfs.

Equivalently, in terms of its (essentially unique) $(\mathcal{E}, \mathcal{J}, \mathcal{M})$-decomposition $f = mje$, a morphism f is

- a cofibration iff m is iso,
- a weak equivalence iff j is iso,
- a fibration iff e is iso.

In this terminology, it makes sense to refer to morphisms in $\mathcal{E}$ as to *trivial cofibrations,* and to those in $\mathcal{M}$ as to *trivial fibrations,* as the following lemma shows.

3.2 Lemma *For a dfs as in* 3.1, $\mathcal{E} = \mathcal{W} \cap \mathcal{D}$ and $\mathcal{M} = \mathcal{W} \cap \mathcal{N}$.

Proof: Trivially $\mathcal{E} \subseteq \mathcal{W} \cap \mathcal{D}$. Conversely, for $d = me \in \mathcal{D}$ with $e \in \mathcal{E},\ m \in \mathcal{M}$, since $\mathcal{E} \subseteq \mathcal{D},\ m$ must be iso, hence $d \in \mathcal{E}$.

In order for $(\mathcal{D}, \mathcal{W}, \mathcal{N})$ to constitute a Quillen model structure, $\mathcal{W}$ needs to satisfy two key properties *closure under retracts,* and the *2-out-of-3 property.* The first of these comes for free:

3.3 Proposition *The class of weak equivalences of a dfs is closed under retracts, that is: in any commutative diagram*

$$\begin{array}{ccccc} \bullet & \xrightarrow{p} & \bullet & \xrightarrow{r} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow{q} & \bullet & \xrightarrow{s} & \bullet \end{array} \qquad (12)$$

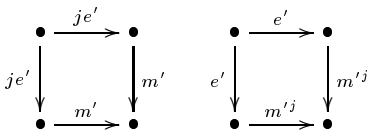
with $rp = 1$ *and* $sq = 1$, $g \in \mathcal{W}$ *implies* $f \in \mathcal{W}$.

Proof: In (12) assume $g = me$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$ and factor f as $f = m'je'$ with $e' \in \mathcal{E}$, $m' \in \mathcal{M}$. Now $je' \perp m$ gives a morphism b with $bje' = ep$, $mb = qm'$, and $e \perp m'j$ gives a morphism $\mathcal{C}$ with $ce = e'r$, $m'jc = gm$, Then we have

$$(jcb)(je') = jcep = je'rp = je',\ m'(jcb) = smb = sqm' = m',$$

$$(cbj)e' = cep = e^r + p = e',\ (m'j)(cbj) = smbj = sqm'j = m'j,$$

so that jcb and cbj serve as "diagonals" in the commutative diagram



(13)

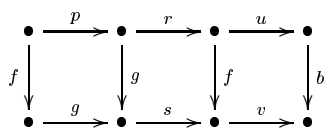
respectively. Consequently, $jcb = 1$ and $cbj = 1$, so that j must be an isomorphism and $f \in \mathcal{W}$.

3.4 Remarks (1) Any class of morphisms closed under retracts satisfies the following two cancellation properties:

(a) if $fr \in \mathcal{B}$ with r split mono, then $f \in \mathcal{B}$;

(b) if $qf \in \mathcal{B}$ with q split epi, then $f \in \mathcal{B}$.

(2) For any class $\mathcal{B}$, the classes $\mathcal{B}^\perp$ and $\mathcal{B}_\perp$ are closed under retracts. Indeed, showing this property for $\mathcal{B}^\perp$, we consider the commutative diagram



(14)

with $b \in \mathcal{B}$ and apply $g \perp b$ to obtain t with $tg = ur,\ bt = vs$. The morphism tq shows $f \perp b$.

3.5 Corollary *For a dfs, each of the classes $\mathcal{E}, \mathcal{J}, \mathcal{M}, \mathcal{D}, \mathcal{W}, \mathcal{N}$ is closed under retracts and satisfies the concellation properties* 3.4(1).

Proof: Apply 2.5, 3.4 and 3.3.

Recall that a class $\mathcal{B}$ has the *2-out-of-3* property if with any two of the three morphism f, g, h with $h = gf$ also the third lives in $\mathcal{B}$; that is: if $\mathcal{B}$ is left and right cancellable and closed under composition. For the latter property, one easily sees:

3.6 Proposition *The class $\mathcal{W}$ of weak equivalences of a dfs $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ is closed under composition if and only if $\mathcal{E} \cdot \mathcal{M} \subseteq \mathcal{M} \cdot \mathcal{E}$.*

Proof: Since $\mathcal{E},\ \mathcal{M} \subseteq \mathcal{W} = \mathcal{M} \cdot \mathcal{E}$, "only if" holds trivially. Conversely, for composable morphisms $w = me, w' = m'e'$, rewrite $e'm$ as $m''e''$ to obtain $w'w = m'm''e''e \in \mathcal{M} \cdot \mathcal{E}$ where $e, e', e'' \in \mathcal{E},\ m, m', m'' \in \mathcal{M}$.

3.7 Proposition *For a dfs as in 3.1, consider the statements*

(i) $\mathcal{W}$ *is weakly left-cancellable;*

(ii) *if* $en \in \mathcal{E}$ *with* $e \in \mathcal{E},\ m \in \mathcal{M}$, *then* $n \in \mathcal{M}$;

(iii) *if* $ej \in \mathcal{E}$ *with* $e \in \mathcal{E},\ j \in \mathcal{J}$, *then* j *is an isomorphism.*

Then (i)⇒ (ii) ⇒ (iii), *while all are equivalent when* $\mathcal{W}$ *is closed under composition.*

Proof: (i)⇒ (ii) If $en \in \mathcal{E}$ with $e \in \mathcal{E},\ n \in \mathcal{N}$, since $\mathcal{E} \subseteq \mathcal{W}$ one obtains $n \in \mathcal{W}$ with (i), hence $n = me^{'}$ with $e^{'} \in \mathcal{W},\ m \in \mathcal{M} \subseteq \mathcal{N}$. But then e' must be iso, and $n \in \mathcal{M}$ follows.
(ii)⇒ (iii) If $ej \in \mathcal{E}$ with $e \in \mathcal{E},\ j \in \mathcal{J} \subseteq \mathcal{N}$, then $j \in \mathcal{M}$ by (ii). Since $\mathcal{J} \subseteq \mathcal{D},\ j \in \mathcal{D} \cap \mathcal{M} =$ Iso.
(iii)⇒ (i) Assume $wv = m_o e_o \in \mathcal{W}$ with $w = me \in \mathcal{W}$ and decompose v as $v = m'je'$, with $e_o, e, e' \in \mathcal{E},\ j \in \mathcal{J},\ m_o, m, m' \in \mathcal{M}$. When $\mathcal{W}$ is closed under composition, rewrite em' as $em' = m^{''}e^{''}$ with $e^{''} \in \mathcal{E},\ m^{''} \in \mathcal{M}$. We then have $m_o e_o = mv = (mm^{''})(e^{''}je^{'})$ with $e_o, e^{''}je^{'} \in \mathcal{D}$ and $m_o,\ mm^{''} \in \mathcal{M}$. Since the two $(\mathcal{D}, \mathcal{M})$-decomposition are isomorphic, $e^{''}je^{'} \in \mathcal{E}$ follows. Weak right cancellation of $\mathcal{E}$ then gives $e^{''}j \in \mathcal{E}$ which, by (iii), implies j iso. This shows $v \in \mathcal{W}$.
Dualization of 3.7 gives:
3.8 Corollary *For a dfs as in* 3.1 consider the statements:

(i) $\mathcal{W}$ *is weakly right-cancellable;*

(ii) *if* $dm \in \mathcal{M}$ *with* $d \in \mathcal{D},\ m \in \mathcal{M}$, *then* $d \in \mathcal{E}$;

(iii) *if* $jm \in \mathcal{M}$ *with* $j \in \mathcal{J},\ m \in \mathcal{M}$, *then* j *is an isomorphism.*

Then (i)⇒(ii)⇒(iii), *while all are equivalent when* $\mathcal{W}$ *is closed under composition.*
3.9 Definition *An (orthogonal) Quillen factorization system* of a category $\mathcal{C}$ is given by morphism classes $\mathcal{D}, \mathcal{W}, \mathcal{N}$ such that

(DF1) $(\mathcal{W} \cap \mathcal{D},\ \mathcal{N}), (\mathcal{D}, \mathcal{W} \cap \mathcal{N})$ are factorization systems,

(DF2) $\mathcal{W}$ has the 2-out-of-3 property.

3.10 Theorem (1) *For every dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ satisfying the conditions

(a) $\mathcal{E} \cdot \mathcal{M} \subseteq \mathcal{M} \cdot \mathcal{E}$,

(b) $j \in \mathcal{J}$ *is an isomorphism whenever there is* $e \in \mathcal{E}$ *with* $ej \in \mathcal{E}$ *or* $m \in \mathcal{M}$

with $jm \in \mathcal{M}$,

the triple $(\mathcal{J}\cdot\mathcal{E},\ \mathcal{M}\cdot\mathcal{E},\ \mathcal{M}\cdot\mathcal{J})$ *is a Quillen factorization system.*

(2) *For every Quillen factorization system* $(\mathcal{D},\mathcal{W},\mathcal{N})$, *the triple* $(\mathcal{W}\cap\mathcal{D},\ \mathcal{D}\cap\mathcal{N},\ \mathcal{W}\cap\mathcal{N})$ *forms a dfs* $(\mathcal{E},\mathcal{J},\mathcal{M})$ *satisfying properties* (a),(b).

(3) *The assignments of* (1),(2) *constitute a bijective correspondence between all Quillen factorization systems of* $\mathcal{C}$ *and the dfs of* $\mathcal{C}$ *satisfying* (a), (b).

Proof: (1) By 2.3 the dfs $(\mathcal{E},\mathcal{J},\mathcal{M})$ gives the fs $(\mathcal{E},\mathcal{N}),(\mathcal{D},\ \mathcal{M})$ which by 3.2, satisfy $\mathcal{E}=\mathcal{W}\cap\mathcal{D}$, $\mathcal{M}=\mathcal{W}\cap\mathcal{N}$, with $\mathcal{D},\mathcal{W},\mathcal{N}$ as in 3.1. Hence, (QF1) holds, and (QF2) follows from 3.6, 3.7, 3.8.

(2) With the given Quillen factorization system, one has the comparable fs as in (QF1) and therefore the dfs $(\mathcal{E},\mathcal{J},\mathcal{M})=(\mathcal{W}\cap\mathcal{D},\mathcal{D}\cap\mathcal{N},\mathcal{W}\cap\mathcal{N})$ by 2.7(2). To be able to derive properties (a),(b) from 3.6, 3.7, 3.8 we should know that, when forming $\mathcal{E}\cdot\mathcal{M}$ we get back the original class $\mathcal{W}$, i.e. $\mathcal{W}=(\mathcal{W}\cap\mathcal{N})\cdot(\mathcal{W}\cap\mathcal{D})$. Indeed, "$\supseteq$" is trivial, and for $w\in\mathcal{W}$ decomposed as $w=nd$ with $n\in\mathcal{M}$ and $d\in\mathcal{D}$ one has in fact $n\in\mathcal{W}$ iff $d\in\mathcal{W}$.

(3) This is just a restriction of the bijective correspondence described in 2.7.

3.11 Remark Just like for a dfs (see 2.5), also for a Quillen factorization system $(\mathcal{D},\mathcal{W},\mathcal{N})$, any two of the three morphism classes determine the third:

$$\mathcal{D}=(\mathcal{W}\cap\mathcal{N})^{\perp},\ \mathcal{W}=\mathcal{D}_{\perp}\cdot\mathcal{N}^{\perp},\ \mathcal{N}=(\mathcal{W}\cap\mathcal{D})_{\perp}.$$

3.12 Examples
(1) For any fs $(\mathcal{E},\mathcal{M})$ of $\mathcal{C}$, 2.2(1) and 3.10(1) give the trivial Quillen factorization system ($\mathcal{E}$, Mor, $\mathcal{M}$) of $\mathcal{C}$.

(2) The dfs (RegEpi, Epi $\cap$ Mono, RegMono) of 3.10(3) has as its weak equivalences the class $\mathcal{W}=\text{RegMono}\cdot\text{RegEpi}$, which rarely has the 2-out-of-3 property, unless Epi $\cap$ Mono = Iso. More precisely:
1. If $\mathcal{C}$ has finite coproducts with the coproduct injections being regular monomorphisms, then $\mathcal{W}$ is closed under composition if and only if Epi $\cap$ Mono = Iso; equivalently, if $\mathcal{W}$ = Mor;
2. If $\mathcal{C}$ has a terminal object 1, with all morphisms with codomain 1 being regular epimorphisms, then $\mathcal{W}$ is weakly left-cancellable if and only if Epi $\cap$ Mono =Iso, equivalently, if $\mathcal{W}$ = Mor;

3. If $\mathcal{C}$ has an initial object 0, with all morphisms with domain 0 being regular monomorphisms, then $\mathcal{W}$ is weakly right-cancellable if and only if Epi $\cap$ Mono = Iso; equivalently, if $\mathcal{W}$ = Mor.
Statements 2 and 3 follow immediately from 3.7 and 3.8. For statement 1, one considers the "cograph factorization" of a morphism $j : A \to B$ through $A + B$ to see that, under closure of $\mathcal{W}$ under composition, by 3.6 there is a factorization $j = m \cdot e$ with e regularly epic and m regularly monic, which renders all three morphisms being isomorphisms. Of course, the last argument in fact proves a general fact, as follows: *for a dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ *in a category* $\mathcal{C}$ *in which every arrow admits a decomposition into an* $\mathcal{M}$*-morphism followed by an* $\mathcal{E}$*-morphism, the class* $\mathcal{W} = \mathcal{M} \cdot \mathcal{E}$ *is closed under composition if and only if* $\mathcal{J}$ = Iso *if and only if* $\mathcal{W}$ = Mor.

(3) An application of 3.10 (2) to the dfs 2.2(4) gives that ([Iso,Mor,Mor], [Mor,Iso,Mor], [Mor,Mor,Iso]) is a Quillen factorization system of $\mathcal{C}^3$. Its universal role is described in the next section.

4. Free Systems

4.1 For a category $\mathcal{B}$ with dfs $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ and any category $\mathcal{C}$, we denote by $(\mathcal{B}^{\mathcal{C}^3})_{\mathcal{E},\mathcal{J},\mathcal{M}}$ the full subcategory of $\mathcal{B}^{\mathcal{C}^3}$ of functors $G : \mathcal{C}^3 \to \mathcal{B}$ with (*)

$$G([\text{Iso,Iso,Mor}]) \subseteq \mathcal{E},\ \ G([\text{Iso,Mor,Iso}]) \subseteq \mathcal{J},\ \ G([\text{Mor,Iso,Iso}]) \subseteq \mathcal{M}.$$

If $(\mathcal{D}, \mathcal{W}, \mathcal{M})$ is the Quillen factorization system associated with $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ via 3.10, then these conditions translate equivalently into (**)

$$G([\text{Iso,Mor,Mor}]) \subseteq \mathcal{D},\ \ G([\text{Mor, Iso, Mor}]) \subseteq \mathcal{W},\ \ G([\text{Mor,Mor,Iso}]) \subseteq \mathcal{W}.$$

4.2 Theorem *For a category* $\mathcal{B}$ *with dfs* $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ *and any category* $\mathcal{C}$*, pre-composition with the diagonal embedding* $\Delta : \mathcal{C} \to \mathcal{C}^3$ *yields an equivalence of categories*

$$(-)\Delta : (\mathcal{B}^{\mathcal{C}^3})_{\mathcal{E},\mathcal{J},\mathcal{M}} \longrightarrow \mathcal{B}^{\mathcal{C}}.$$

Proof: Let us first show that $(-)\Delta$ is surjective on objects, by constructing from $F : \mathcal{C} \to \mathcal{B}$ a functor $G : \mathcal{C}^3 \to \mathcal{B}$ with $G\Delta = F$ and (*). For a morphism $f : A \to B$ in $\mathcal{C}$, such a functor G must map the decomposition

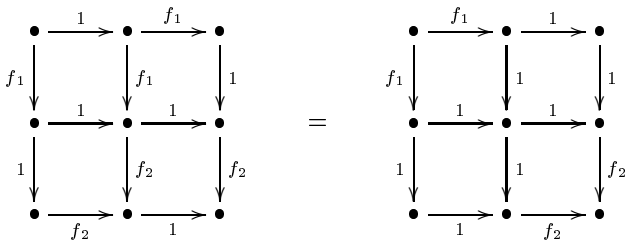
$$(1_A, 1_A) \xrightarrow{[1_A, 1_A, f]} (1_A, f) \xrightarrow{[1_A, f, 1_B]} (f, 1_B) \xrightarrow{[f, 1_B, 1_B]} (1_B, 1_B)$$

of Δf to the $(\mathcal{E}, \mathcal{J}, \mathcal{M})$-decomposition of Ff in $\mathcal{B}$, which (up to isomorphism) fixes the objects $G(1_A, f)$, $G(f, 1_B)$ and the morphisms $G[1_A, 1_A, f]$,

$G[1_A, f, 1_B]$, $G[f, 1_B, 1_B]$. In order to define G on arbitrary objects (f_1, f_2) in $\mathcal{C}^3$, consider the decomposition

$$[f_1, 1, f_2] = [f_1, 1, 1][1, 1, f_2] = [f_1, 1, 1][1, 1, f_2],$$

depicted by

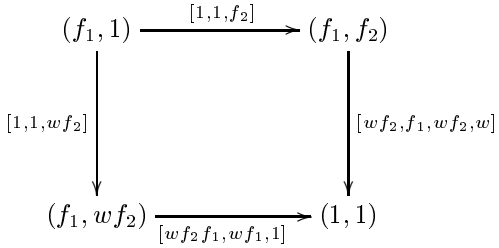


(15)

Since the values of the RHS under G have already been fixed, and since the LHS represents the double factorization of the morphism $[f_1, 1, f_2]$ (with trivial middle factor), $G(f_1, f_2)$ must be the middle object occurring in the $(\mathcal{E}, \mathcal{M})$-factorization of the morphism $G(1, 1, f_2)G(f_1, 1, 1)$ in $\mathcal{B}$. Next we must define G on morphisms, and it suffices to do so for each of the three players in the decomposition (7) of $[u, v, w] : (f_1, f_2) \to (g_1, g_2)$. We outline the procedure only for the first of the three, i.e. for the morphism

$$[1, 1, w] : (f_1, f_2) \longrightarrow (f_1, wf_2).$$

This morphism is the unique diagonal in the square



(16)

whose top arrow must be mapped by G into $\mathcal{E}$ and whose bottom arrow must be mapped into $\mathcal{N} = \mathcal{M} \cdot \mathcal{J}$. Hence, the $(\mathcal{E}, \mathcal{N})$-diagonalization propery fixes the value of $G[1, 1, w]$, provided that we make sure that the G-images of the

four sides of the square (16) are already fixed. This is certainly true for the top and the left arrow, via (15). The bottom arrow is decomposed as

$$(f_1, wf_2) \xrightarrow{[f_1,1,1]} (1, wf_2) \xrightarrow{[1,wf_2,1]} (wf_2, 1) \xrightarrow{[wf_2,1,1]} (1,1) \ ,$$

and the right arrow as

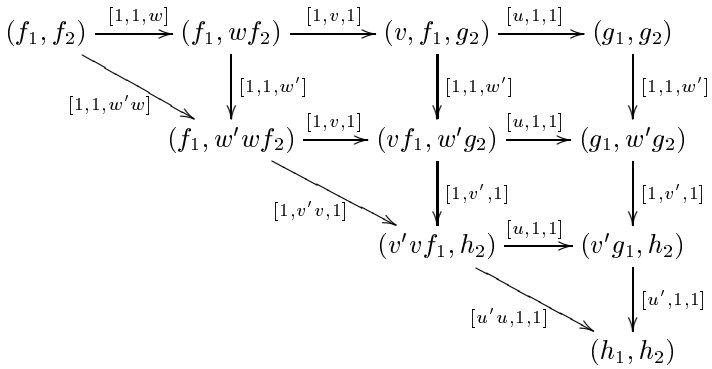
$$(f_1, f_2) \xrightarrow{[f_1,1,1]} (1, f_2) \xrightarrow{[1,f_2,1]} (f_2, 1) \xrightarrow{[f_2,1,1]} (1,1) \xrightarrow{[w,w,w]} (1,1)$$

where the G-values of each of the occuring factors are indeed already determined.

It is clear that G thus defined preserves identity morphisms. Showing that it preserves composition of arrows is a lot more tedious. The following diagram shows how the decomposition (7) of a composite arrow

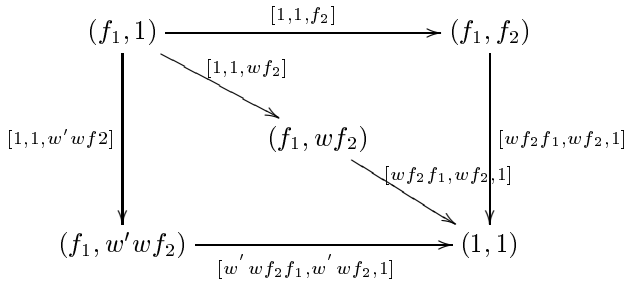
$$(f_1, f_2) \xrightarrow{[u,v,w]} (g_1, g_2) \xrightarrow{[u',v',w']} (h_1, h_2)$$

relates to the decompositions of its two factors:



(17)

One must now show that the G-image of this diagram commutes in $\mathcal{B}$. We show this for the upper left-hand triangle of (17) and consider the commutative diagram (18), the outer square of which determines the G-value of $[1, 1, w'w]$



(18)

Diagram (16) is inscribed in diagram (18) as the upper triangle and determines the value of $G[1,1,w]$ by the diagronalization property, while the lower triangle determines the value of $G[1,1,w']$, giving two inscribed diagonal arrows whose composite must be the top-right-to-bottom-left diagonal arrow of the G-image of the outer square, i.e. $G[1,1,w',w]$.

Let us now turn to the proof that $(-)\triangle$ is full and faithful. For that if suffices to show that a natural transformation $\alpha : G \to G'$ (with G' satisfying the same conditions (*) as G) is completely determined by $\alpha\triangle$, i.e. by its values $\alpha_{(1,1)}$. But the commutative diagram

$$\begin{array}{ccccccc} G(1_A,1_A) & \xrightarrow{G[1,1,f]} & G(1_A,f) & \xrightarrow{G[1,f,1]} & G(f,1_B) & \xrightarrow{G[f,1,1]} & G(1_B,1_B) \\ \downarrow{\scriptstyle \alpha_{(1_A,1_A)}} & & \downarrow{\scriptstyle \alpha_{(1_A,f)}} & & \downarrow{\scriptstyle \alpha_{(f,1_B)}} & & \downarrow{\scriptstyle \alpha_{(1_B,1_B)}} \\ G'(1_A,1_A) & \xrightarrow{G'[1,1,f]} & G'(1_A,f) & \xrightarrow{G'[1,f,1]} & G'(f,1_B) & \xrightarrow{[G'[f,1,1]} & G'(1_B,1_B) \end{array} \tag{19}$$

shows that the values of $\alpha_{(1_A,f)}, \alpha_{(f,1_B)}$ are determined by the double diagonalization property, and the left part of (15) shows that these determine the value of $\alpha_{(f_1,f_2)}$, via the $(\mathcal{E},\mathcal{M})$-diagonalization property:

$$\begin{array}{ccccc} G(f,1) & \xrightarrow{G[1,1,f_2]} & G(f_1,f_2) & \xrightarrow{G[f_1,1,1]} & G(1,f_2) \\ \downarrow{\scriptstyle \alpha_{(f_1,1)}} & & \downarrow{\scriptstyle \alpha_{(f_1,f_2)}} & & \downarrow{\scriptstyle \alpha_{(1,f_2)}} \\ G'(f_1,1) & \xrightarrow[G'[f,1,1]]{} & G'(f_1,f_2) & \xrightarrow[G'[f_1,1,1]]{} & G'(f_1,f_2) \end{array} \tag{20}$$

We can leave off the routine check that $\alpha_{(f_1,f_2)}$ thus defined is really a natural transformation.

Corollary *For a category $\mathcal{C}$ and a category $\mathcal{B}$ with Quillen factorization system, any functor $F : \mathcal{C} \to \mathcal{B}$ admits an extension $G : \mathcal{C}^3 \to \mathcal{B}$ along the embedding $\triangle : \mathcal{C} \to \mathcal{C}^3$ which maps fibrations to fibrations, cofibrations to cofibrations and weak equivalences to weak equivalences. The extension is uniquely determined, up to isomorphism.*

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