Cyclic, Diagonal and Facial Colorings

Daniel Král'[1,*] Tomáš Madaras[2,†] Riste Škrekovski[1,3]

[1]Institute for Theoretical Computer Science, Charles University,
Malostranské náměstí 25, 118 00 Prague, Czech Republic.
E-mail: {kral,skreko}@kam.mff.cuni.cz

[2]Institute of Mathematics, P. J. Šafárik University,
Jesenná 5, 04154 Košice, Slovakia.
E-mail: madaras@science.upjs.sk

[3]Department of Mathematics, University of Ljubljana
Jadranska 19, 1111 Ljubljana, Slovenia.
E-mail: skreko@fmf.uni-lj.si

Abstract

In an l-facial coloring, any two different vertices joined by a facial walk of length at most l receive distinct colors. The concept of facial coloring extends the well-known concept of cyclic coloring. We prove that $\left\lfloor \frac{18l}{5} \right\rfloor + 2$ colors suffice for an l-facial coloring of a plane graph. For $l = 2$, 3 and 4, the upper bounds of 8, 12 and 15 colors are shown. We use our results on facial coloring to decrease to 16 the upper bound on the number of colors needed for 1-diagonal coloring of plane quadrangulations.

1 Introduction

In this paper, we introduce an extension of cyclic coloring of plane graphs which we call facial coloring. A cyclic coloring of a plane graph is a coloring

*Institute for Theoretical Computer Science is supported as project LN00A056 by Ministry of Education of the Czech Republic.

†Supported by Slovak VEGA grant 1/7467/20 and Institute for Theoretical Computer Science of Charles University.

of its vertices such that any two vertices incident with the same face receive distinct colors. The number of used colors have clearly to be at least the size $\Delta^\star$ of the largest face (measured as the number of its vertices) of a plane graph. Hence any upper bounds have always to be restricted to plane graphs with bounded maximum face sizes (we survey known results later).

Two vertices u and v are l-facially adjacent if there exists a facial walk of length (measured as its number of edges) at most l between them. In an l-facial coloring, we demand that any two different l-facially adjacent vertices receive distinct colors. If $\Delta^\star \leq 2l+1$, any cyclic coloring is an l-facial coloring and moreover if a plane graph is 2-connected, any l-facial coloring is a cyclic coloring (note that each plane graph can be changed to a 2-connected graph only by adding some edges without increasing its cyclic chromatic number). The bounds for the l-facial coloring do not restrict to plane graphs with bounded maximum face sizes but they hold for all plane graphs regardless their maximum face sizes. Hence the concept of facial coloring may be viewed as an extension of the concept of cyclic coloring (which restricts to plane graph with bounded face sizes) to all plane graphs. In addition, our results on the 2-facial coloring are used to prove a better upper bound for the number of colors needed in 1-diagonal coloring of plane quadrangulations.

We survey results on cyclic coloring in this paragraph: The best known lower bound $\lfloor \frac{3}{2}\Delta^\star \rfloor$ for cyclic coloring is also conjectured to be the best possible; see a well-known monograph [12] (p. 37) on graph coloring problems by Jensen and Toft. Ore and Plummer proved the first upper bound of $2\Delta^\star$ for the problem in [13]. Borodin slightly improved the bound to $2\Delta^\star - 3$ for $\Delta^\star \geq 8$ in [3]. Significant progress has been made recently: Borodin, Sanders and Zhao managed to prove the bound of $\lfloor \frac{9}{5}\Delta^\star \rfloor$ was in [5] and the currently best known general bound $\lceil \frac{5}{3}\Delta^\star \rceil$ is due to Sanders and Zhao [16]. Better results are known for graphs with small maximum face sizes, i.e., for small values of $\Delta^\star$. Let $f_c(\Delta^\star)$ be the number of colors needed for a cyclic coloring of a plane graph with maximum face size $\Delta^\star$. The case of cyclic coloring of plane triangulations, i.e., $\Delta^\star = 3$, is equivalent to the famous Four Color Theorem which was proved in [1] (see also [15] for a recent refinement of its proof). Hence $f_c(3) = 4$. The case of $\Delta^\star = 4$ is Ringel's problem: The problem was solved and it was shown that $f_c(4) = 6$ by Borodin in [4, 2]. The values of f_c for $\Delta^\star = 3$ and $\Delta^\star = 4$ are the only ones which are currently known exactly. The upper bounds $f_c(5) \leq 8$ and $f_c(6) \leq 10$ were proved in [5] and $f_c(7) \leq 12$ in [3]. The lower bound $7 \leq f_c(5)$ is known. There is a related conjecture by Plummer and Toft

[14] on cyclic coloring of 3-connected plane graphs which have been recently proved for graphs with large maximum face sizes [7, 8, 11] by Enomoto, Horňák and Jendrol'.

We define an l-facial coloring in a formal way and state its basic properties in Section 2. Notation related to plane graphs used in this paper is also introduced in Section 2. Again, let $f_f(l)$ be the number of colors needed to l-facial coloring of a plane graph. It clearly holds that $f_c(2l+1) \leq f_f(l)$. We do not know a single value of l for which the inequality can be proved to be strict. We extend all the results for cyclic coloring except for those of [16] to facial colorings: A structural result of [5] is used to get an upper bound $f_f(l) \leq \lfloor \frac{18}{5} l \rfloor + 2$ in Section 3 (Theorem 3, see Corollary 2 for $l \leq 4$). This bound corresponds to the bound on f_c from [5]. Better results can be obtained for small values of l. Since the 1-facial coloring is the usual colorings of plane graphs, the case $l = 1$ is also equivalent to the Four Color Theorem. Hence $f_f(1) = 4$. The bound $f_f(2) \leq 8$ is proved in Section 4; this matches the corresponding best known bound $f_c(5) \leq 8$ for the cyclic coloring. The bounds $f_f(3) \leq 12$ and $f_f(4) \leq 15$ are proved in Section 5. Both these bounds match the corresponding best known bounds $f_c(7) \leq 12$ and $f_c(9) \leq 15$ for the cyclic coloring.

The obtained results are applied to the 1-diagonal coloring of plane quadrangulations. A 1-diagonal coloring of plane triangulations was studied (in a dual form) in [6]. The concept was generalized to a d-diagonal coloring of plane graphs in [10]. Two vertices u and v are d-diagonally adjacent if there exists a set S of at most d edges such that the removal of the edges of S makes u and v to be incident with the same face. In the d-diagonal coloring, any two vertices which are d-diagonally adjacent have to receive distinct colors. A 0-diagonal coloring is clearly a cyclic coloring. A d-diagonal coloring of plane quadrangulations was first studied in [9]; Horňák and Jendrol' proved in [9] that 21 colors are always sufficient to 1-diagonal color a plane quadrangulation. Sanders and Zhao improved this upper bound to 19 in [17]. Related results on diagonal coloring of plane graphs with small maximum face sizes can be found in [18]. We decrease the upper bound on the number of colors needed for a 1-diagonal coloring of plane quadrangulations to 16. The lower bound of 11 for this type of coloring can be found in [17].

2 Basic Properties

Throughout the paper, we write $V(G)$, $E(G)$ and $F(G)$ for the vertex set, the edge set and the face set of a plane graph G. We call a vertex v a k-vertex if its degree is k; similarly, a k-face is a face of size k. We write $\leq k$ ($\geq k$) for integers smaller or equal (greater or equal) to k; e.g., a ≤ 4-vertex is a vertex of degree at most 4. A vertex v is a $(f_1, \ldots, f_d)$-vertex if it is a d-vertex and the faces incident with v have sizes (in either a clockwise or anti-clockwise order around v) $f_1, \ldots, f_d$. In addition, $*$ denotes all integers in this notation, e.g., a $(4, \geq 5, *)$-vertex is a vertex of degree 3 incident with a face of size 4 and a face of size at least 5.

A *facial walk* of a face f is a sequence of vertices of it in the order obtained when traversing a part of the boundary of the face. If a plane graph is not 2-connected, then some of its facial walks contain some vertices several times. A facial walk not containing any vertex twice is a *facial path.* The *length* of a facial walk (path) is the number of its edges. In particular, two vertices u and v are *l-facially adjacent* if they are connected by a facial walk of length of at most l. Thus, two vertices are 1-facially adjacent iff they are adjacent. We call a coloring of vertices of a plane graph an *l-facial coloring* if any two l-facially adjacent vertices receive distinct colors. The *l-facial degree* of a vertex v is the number $\deg_l(v)$ of vertices which are l-facially adjacent to it. An *(l, k)-minimal graph* is a plane graph with the smallest number of vertices (and among all such graphs with the smallest number of edges) which has no l-facial coloring using at most k colors. An l-facial coloring using k colors is called an *(l, k)-coloring* for brevity.

Several basic lemmas on l-facial coloring are stated in what follows. Each of Lemmas 2–6 has its counterpart for the case of the cyclic coloring [3]. Their proofs are straightforward extensions of the proofs of the corresponding lemmas from [3] and hence they are omitted. The only exception is Lemma 2 which turned out to be surprisingly much more difficult to prove than its counterpart and hence we decided to include its proof.

Lemma 1 *Let v be a $(f_1, \ldots, f_d)$-vertex and $l \geq 1$ an integer. The l-facial degree of v is at most $\left(\sum_{i=1}^{d} \min\{f_i, 2l+1\}\right) - 2d$.*

Proof: The number of the l-facial neighbors of v among vertices incident with an f_i-face is $\min\{f_i - 1, 2l\}$. Then, the bound of the lemma follows by summing up these expressions and decreasing the sum by 1 because each neighbor of v is counted twice in it. (Note that even this sum is only an upper bound on the l-facial degree because some l-facial neighbors of v may be counted in it several times.)

■

Lemma 2 *Each (l,k)-minimal graph (for $k \geq 3l+1$) is 2-connected.*

Proof: Let G be an (l,k)-minimal graph. G is clearly connected. Assume that G is not 2-connected. Let G_1 be one of end-blocks of G, v the corresponding separating vertex and G_2 the rest of G (including v). Let $v_{-l},\ldots,v_{-1},\ v,\ v_1,\ldots,v_l$ be a facial walk of G_1 and $w_{-l},\ldots,w_{-1},\ v,\ w_1,\ldots,w_l$ a facial walk of G_2 such that $v_{-l},\ldots,v_{-1},\ v,\ w_{-1},\ldots,w_{-l}$ and $v_l,\ldots,v_1,\ v,\ w_1,\ldots,w_l$ are facial walks of G. It might happen that some vertices of $v_{-l},\ldots,v_{-1},\ v,\ v_1,\ldots,v_l$ are actually the same (if the length of the border of the outer face is smaller than $2l+1$).

Both G_1 and G_2 have an (l,k)-coloring due to the minimality of G; let c_1 be an (l,k)-coloring of G_1 and c_2 an (l,k)-coloring of G_2. Assume that c_1 and c_2 use completely different colors. Let $V^- = \{v_{-1},\ldots,v_{-l}\}$, $V^+ = \{v_1,\ldots,v_l\}$, $W^- = \{w_{-1},\ldots,w_{-l}\}$ and $W^+ = \{w_1,\ldots,w_l\}$. The sets V^- and V^+ need not to be disjoint; similarly, the sets W^- and W^+. We consider all the four sets to be multisets, i.e. if a single vertex appears twice among $v_{-l},\ldots,v_{-1},v_1,\ldots,v_l$, then we pretend that there are two vertices colored with its color in $V^- \cup V^+$ even if it is the only vertex of $V^- \cup V^+$ with the particular color.

Some colors used by c_1 and c_2 will be identified. First, we identify the colors $c_1(v)$ and $c_2(v)$. For each $1 \leq i \leq l$, if the vertex v_{-i} is the only vertex among the vertices $V^- \cup V^+$ colored with the color $c_1(v_{-i})$ and w_{i-l-1} is the only vertex among the vertices $W^- \cup W^+$ colored with the color $c_2(w_{i-l-1})$, then we identify the colors $c_1(v_{-i})$ and $c_2(w_{i-l-1})$. We proceed in the same manner, for the pair of vertices v_i and w_{l-i+1}.

Next, we proceed as follows: If there is a color γ such that it is used by c_1 to color some vertices of G_1 and no vertex of $\{v\} \cup V^- \cup V^+$ is colored with the color γ, we identify the color γ with any color used by c_2 and not by c_1 (if there is any). Similarly, we identify colors used by c_2 avoiding the vertices $\{v\} \cup W^- \cup W^+$ with colors used by c_1 and not by c_2. These rules are applied till any such colors exist. The obtained coloring is clearly an l-coloring. It uses at most k colors unless more than k colors are used on the vertices $U := \{v\} \cup V^- \cup V^+ \cup W^- \cup W^+$.

We prove that at most k colors are used by the final coloring to color the vertices of U. This will assure that at most k colors are used by the final coloring at all. Let $k(x)$ for $x \in U$ be the number of vertices of U having the same color as x. The number of colors used by the coloring on the vertices of U is equal to $\sum_{x\in U}\frac{1}{k(x)}$. The following inequality will be proven:

$$\sum_{x\in U}\frac{1}{k(x)} \leq 3l+1 \leq k.$$

If the colors of v_{-i} and w_{i-l-1} were not identified, then either $k(v_{-i}) \geq 2$ or $k(w_{i-l-1}) \geq 2$ and hence:

$$\frac{1}{k(v_{-i})} + \frac{1}{k(w_{i-l-1})} \leq \frac{3}{2}.$$

If the colors were identified, then $k(v_{-i}) = k(w_{i-l-1}) \geq 2$ and this gives:

$$\frac{1}{k(v_{-i})} + \frac{1}{k(w_{i-l-1})} \leq 1.$$

The same inequalities hold for the pair of vertices v_i and w_{l-i+1}. For each $i \in \{-l,\ldots,-1,1,\ldots,l\}$, the sum of the inverse values of the two corresponding vertices is at most $\frac{3}{2}$. Hence the whole sum is bounded as follows:

$$\sum_{x\in U\setminus\{v\}}\frac{1}{k(x)} \leq \frac{3}{2}\cdot 2l \leq 3l$$

which gives the desired inequality and consequently finishes the whole proof.

■

Lemma 3 *No (l,k)-minimal graph (for $k \geq 2l+1$) contains a separating cycle of length at most $2l+1$.*

Lemma 4 *No (l,k)-minimal graph contains an edge separating an f_1-face and an f_2-face with $f_1+f_2 \leq 2l+3$.*

Lemma 5 *No (l,k)-minimal graph contains a vertex with the l-facial degree at most $k-1$.*

Lemma 6 *No (l,k)-minimal graph (for $l \geq 2$ and $3l+1 \leq k$) contains an edge uv separating two ≥ 4-faces such that u and v are ≥ 3-vertices, the l-facial degree of u is at most k and the l-facial degree of v is at most $k+1$.*

3 The Bound for the General Case

We first state a structural theorem whose proof can be found in [5] (Theorem 3.1 of [5]). Theorem 3.1 in [5] states that each plane graph with maximum face size $\Delta^\star \geq 8$ contains a vertex which is cyclically adjacent to at most $\left\lfloor \frac{9}{5}\Delta^\star \right\rfloor - 1$ vertices (two vertices are cyclically adjacent if they are incident with the same face). The proof of Theorem 3.1 in [5] is actually a proof of the following structural result on plane graphs:

Theorem 1 *Each connected plane graph contains at least one of the following configurations:*

1. *a* ≤ 2*-face.*
2. *a* 1*-vertex.*
3. *a* $(\leq 15, *)$*-vertex.*
4. *a* 3*-vertex which is a* $(3, \leq 10, *)$*-vertex, a* $(4, \leq 7, *)$*-vertex or a* $(5, \leq 6, *)$*-vertex.*
5. *a* 4*-vertex which is of one of the following types: a* $(3, \leq 4, *, \leq 4)$*-vertex, a* $(3, \leq 4, \leq 4, *)$*-vertex, a* $(3, 3, 5, *)$*-vertex or a* $(3, 5, 3, *)$*-vertex.*
6. *a* $(3, 3, 3, 3, *)$*-vertex.*
7. *a* k*-face (*$k \geq 16$*) with a facial path of length at least* $\left\lceil \frac{k-11}{5} \right\rceil$ *formed by* 2*-vertices.*

Theorem 1 can be used to bound the minimum l-facial degree of a plane graph:

Corollary 1 *Let* $l \geq 5$ *be a fixed integer. Each plane graph without loops and parallel edges contains a vertex of the* l*-facial degree at most* $\left\lfloor \frac{18l}{5} \right\rfloor + 1$ *colors.*

Proof: Let G be a plane graph and $l \geq 5$ a fixed integer throughout the proof. We may assume that G is connected. The proofs consists of a careful examination of the configurations of Theorem 1:

1. This is forbidden by the assertion of the lemma.

2. If G contains a 1-vertex v, then the l-facial degree of v is at most $2l$.

3. If G contains a $(\leq 15, *)$-vertex v, then the l-facial degree of v is at most $15 + 2l - 3 = 2l + 12 \leq \lfloor \frac{18l}{5} \rfloor + 1$ (if $l \geq 7$). If $l = 5$ or $l = 6$, respectively, the l-facial degree of v is at most $4l - 2$, i.e. 18 or 22, respectively, which is at most $\lfloor \frac{18l}{5} \rfloor + 1$.

4. If v is a $(3, \leq 10, *)$-vertex, a $(4, \leq 7, *)$-vertex or a $(5, \leq 6, *)$-vertex, then the l-facial degree of v is at most $13+2l+1-6 = 2l+8 \leq \lfloor \frac{18l}{5} \rfloor + 1$.

5. If v is a $(3, \leq 4, *, \leq 4)$-vertex or a $(3, \leq 4, \leq 4, *)$-vertex, then v has the l-facial degree at most $3+4+4+2l+1-8 = 2l+4 \leq \lfloor \frac{18l}{5} \rfloor + 1$. If v is a $(3, 5, 3, *)$-vertex or a $(3, 3, 5, *)$-vertex, then the l-facial degree of v is at most $3 + 3 + 5 + 2l + 1 - 8 = 2l + 4 \leq \lfloor \frac{18l}{5} \rfloor + 1$.

6. If v is a $(3, 3, 3, 3, *)$-vertex, then it has the l-facial degree at most $3 + 3 + 3 + 3 + 2l + 1 - 10 = 2l + 3 \leq \lfloor \frac{18l}{5} \rfloor + 1$.

7. If G contains a k-face ($k \geq 16$) with a facial path of length at least $\lceil \frac{k-11}{5} \rceil$ formed by 2-vertices, then we proceed as follows: Let v be any of 2-vertices of the facial path. If $k \leq 2l$, then the l-facial degree of v is at most:

$$k + (2l + 1) - 3 - \left\lceil \frac{k - 11}{5} \right\rceil = 2l - 2 + \left\lfloor \frac{4k + 11}{5} \right\rfloor \leq$$

$$2l - 2 + \left\lfloor \frac{8l + 11}{5} \right\rfloor \leq \left\lfloor \frac{18l}{5} \right\rfloor + 1.$$

   If $k \geq 2l + 1$, then the facial path has length at least $\lceil \frac{2l}{5} \rceil - 2$. Hence the l-facial degree of v is at most $2(2l+1) - 3 - \left(\lceil \frac{2l}{5} \rceil - 2\right) = \lfloor \frac{18l}{5} \rfloor + 1$.

■

Theorem 1 provides also upper bounds of 5, 9, 12 and 15, on minimum 1-facial, 2-facial, 3-facial and 4-facial degree, respectively, of plane graphs. We did not state these bounds explicitly because we do not need them. The proofs of these bounds are essentially the same as the proof of Corollary 1.

We use the obtained upper bound on the minimum l-facial degree to get a bound on the number of colors needed for an l-facial coloring of a plane graph:

Theorem 2 *Let $l \geq 5$ be a fixed integer. Each plane graph has an l-facial coloring using at most $\lfloor \frac{18l}{5} \rfloor + 2$ colors.*

Proof: Let G be a $(l, \lfloor \frac{18l}{5} \rfloor + 2)$-minimal graph. Since G contains neither a loop nor a pair of parallel edges, G contains a vertex of the l-facial degree at most $\lfloor \frac{18l}{5} \rfloor + 1$ (Corollary 1) which is impossible due to Lemma 5.

■

4 The Bound for 2-Facial Coloring

We first prove two lemmas which forbid the existence of certain edges and paths in $(2,8)$-minimal graphs:

Lemma 7 *Let G be a $(2,8)$-minimal graph and let x and y be two adjacent vertices of G. Then the following holds:*

a) $\deg_2(x) + \deg_2(y) \geq 18$*;*

b) If $\deg_2(x) \leq 10$ *and* $\deg_2(y) \leq 10$*, then one of the faces incident with the edge xy is a ≤ 5-face.*

Proof: By Lemma 1 and Lemma 2, the graph G is 2-connected and with minimum degree at least 3 (the 2-facial degree of each vertex is at least 8). Let f_1 and f_2 be the two faces incident with xy (note that $f_1 \neq f_2$). For $i = 1, 2$, let $P_i = x_i x y y_i$ be the facial walk of f_i.

In the proof of each of the cases (a) and (b), we construct from the graph G a smaller graph G'. By the minimality of G, G' has a $(2,8)$-facial coloring. Afterwards, we modify this coloring to a $(2,8)$-coloring of G which yields a contradiction.

a) Suppose that the claim is false. Since each vertex has its 2-facial degree at least 8, we may assume that $d_2(x) = 8$ and $d_2(y)$ is equal either to 8 or to 9.

   Suppose first that both f_1 and f_2 are ≥ 4-faces. By Lemma 3, the vertices x, y, x_1, x_2, y_1 and y_2 are pairwise distinct (note that if there is a triangle $x_1 x x_2$ or $y_1 y y_2$ and this triangle is not separating, then the 2-facial degree of x or y, respectively, is less than 8). Moreover, the vertices x_1 and y_2 are at distance 3. Otherwise, we obtain either a separating cycle of length ≤ 5 or one of the vertices x and y is a 3-vertex incident with a 3-face and thus its 2-facial degree is less than 8.

Let G' be the graph obtained from G by contraction of the path x_1xyy_2 to a single vertex z and c a $(2,8)$-coloring of G'. We extend c to a $(2,8)$-coloring of G: The vertices of $V(G)\setminus\{x,y,x_1,y_2\}$ preserve their colors, the vertices x_1 and y_2 are colored with the color of the vertex z (this is possible because these two vertices are not 2-facially adjacent). By the assumption, there is at least one available color for y ($\deg_2(y) \leq 9$) and at least two available colors for x ($\deg_2(x) \leq 8$). So, we can easily obtain a $(2,8)$-coloring of G.

Suppose now that f_1 or f_2 is a triangle, say f_1. Then, by Lemma 4, f_2 is a ≥ 5-face. Since both the vertices x and y have 2-facial degrees at least 8, they are ≥ 4-vertices. If x_1 and x_2 are 2-facially adjacent, then either they are joined by an edge or they have another common neighbor different from x. Since x has degree at least 4 (its 2-facial degree is at least 8), in both the cases, there is a separating cycle of length at most 4 which is impossible due to Lemma 3. So, we may assume that x_1 and x_2 are not 2-facially adjacent. The contraction of the path x_1xx_2 to a single vertex z yields the graph G' which has a $(2,8)$-coloring c due to minimality of G. Again we use this coloring to obtain a $(2,8)$-coloring of G: The vertices of $V(G) \setminus \{x,x_1,x_2\}$ preserve their colors, the vertices x_1 and x_2 are colored with the color of the vertex z in G'. Finally, we color x (recall $\deg_2(x) \leq 8$).

b) Suppose the claim is false. Then, the sizes of both the faces f_1 and f_2 are at least six. Suppose first that $\deg_2(x) \leq 9$. The vertices x_1 and y_1 are not 2-facially adjacent, otherwise there is a separating ≤ 5-cycle which is impossible by Lemma 3. Similarly, x_2 and y_2 are not 2-facially adjacent. Now, we create a graph G': Add to G the edges x_1y_1 and x_2y_2 and contract them to the vertices z_1 and z_2, respectively. The $(2,8)$-coloring c of G', which exists due to the minimality of G, is used to get a $(2,8)$-coloring of G. Let the colors of all the vertices except for x,y,x_1,y_1,x_2,y_2 be the same, we color x_1,y_1 with the color of z_1 and x_2,y_2 with the color of z_2. Since $\deg_2(y) \leq 10$ and $\deg_2(x) \leq 9$, one available color for y and two available colors for x are left. Now, one can easily extend c to y and x.

Suppose now that $\deg_2(x) = 10$. By the symmetry, it is possible to assume that also $\deg_2(y) = 10$. Note that the degrees of both x and y have to be at least 4 in G. Hence, one can easily show that no two vertices from x_1, x_2, y_1, y_2 are 2-facially adjacent (unless G contains a separating ≤ 5-cycle which is impossible due to Lemma 3). The graph G' is obtained by contracting all the edges of the paths P_1 and P_2 into a single vertex z. We extend the $(2,8)$-coloring c of G', which exists due to the minimality of G, to a $(2,8)$-coloring of G. All the non-contracted vertices preserve their colors and the vertices x_1, x_2, y_1 and y_2 are colored with the color $c(z)$. There are two available colors for x and y and so a $(2,8)$-coloring of G can be easily obtained.

■

Lemma 8 *Let G be a $(2,8)$-minimal graph and abc a path in G comprised of three 3-vertices. Then, b is a $(5,5,5)$-vertex.*

Proof: Suppose that the lemma is false. We may assume that G is 2-connected. Let f be the face which is incident with edges ab and bc. By Lemma 1 and Lemma 7(a), f is a ≥ 5-face.

Suppose first that f is a 5-face. Let $f = abcde$ and let f' be the other face incident with the edge ab. Assume that f' is a ≥ 6-face. By Lemma 7(a), the vertices a, b, c are $(5, \geq 5, \geq 5)$-vertices and the (usual) degrees of the vertices d and e are at least three. Let a', b', c', respectively, be the neighbors of a, b, c, respectively, which are not incident with the face f. Note that no two vertices from a', b', d are 2-facially adjacent due to Lemma 3.

We obtain the graph G' from G by contracting the connected subgraph of G induced by the vertex set $\{a, b, c, d, a', b'\}$ to a single vertex x. By the minimality of G, G' has a $(2,8)$-coloring. We obtain a $(2,8)$-coloring of G from it: All the non-contracted vertices preserve their colors. The vertices a', b' and d are colored with the color of the vertex x. Then, we color the vertices c, a, b one by one (in this order) with available colors (each time there is at least one available color). Due to symmetry, we may conclude that if f is a 5-face, then b is a $(5,5,5)$-vertex.

Suppose now that f is a ≥ 6-face. The vertex b is a $(5, 5, \geq 6)$-vertex due to Lemma 7(b). Hence, a and c are $(5, \geq 5, \geq 6)$-vertices due to Lemma 6. Let d be the other neighbor of c incident with f and $bcehg$ the other face (i.e., not f) incident with the edge bc. By Lemma 3, all the vertices a, b, c, d, e, h and g are mutually distinct. Moreover, no two vertices from a, d, h

are 2-facially adjacent. Contract the connected subgraph of $G + bh$ induced by the vertex set $\{a, b, c, d, h\}$ to a single vertex x; let G' be the obtained graph. By the minimality of G, G' has a $(2, 8)$-coloring. Now, we extend this coloring to G. All the non-contracted vertices preserve their colors. The vertices a, d and h are colored with the color of x and then the vertices b and c are colored with available colors.

■

The proofs of the following two lemmas are the same as their counterparts for Lemma 5.5 and Lemma 5.6 in [5].

Lemma 9 *No* $(2, 8)$*-minimal graph contains a* $(3, \geq 5, 3, \geq 5)$*-vertex.*

Lemma 10 *No* $(2, 8)$*-minimal graph contains any of the following configurations (consult Figure 1):*

1. 5*-faces abwyv, bcxzw and abcut such that the vertices a and c are 3-vertices and the following holds:*

$$\deg_2(w) \leq 11 \vee (\deg_2(w) \leq 12 \wedge (\deg_2(y) \leq 9 \vee \deg_2(x) \leq 10)),$$

2. *a* 5*-face xybua and a* 3*-face xys such that x is a 4-vertex, a is a 3-vertex, b is a* $(4, 5, \geq 5)$*-vertex, and either y is a 4-vertex or a is a* $(4, 5, \geq 5)$*-vertex,*

3. *a* 5*-face vabxc such that the vertices a and b are 3-vertices, x is a 4-vertex, c is a* $(4, 5, \geq 5)$*-vertex and ybx is a subwalk of a* ≥ 5*-face.*

We alter the proof of Theorem 5.1 of [5] to the case of facial colorings:

Theorem 3 *Each plane graph has a* 2*-facial coloring using at most* 8 *colors.*

Proof: We use the discharging method to prove the theorem. The initial charge c_0 is given by the following assignment:

- The charge of a k-vertex v is $c_0(v) := 6 - k$.
- The charge of a k-face f is $c_0(f) := 6 - 2k$.

It is easy to verify (using Euler's formula) that the sum of the initial charges assigned to the vertices and the faces of a graph is 12.

The charge is then distributed using the following set of rules (in Rules R2-R10, we assume that $x_1x_2x_3x_4x_5$ is a facial 5-path of a face f; in particular f is a ≥ 5-face):

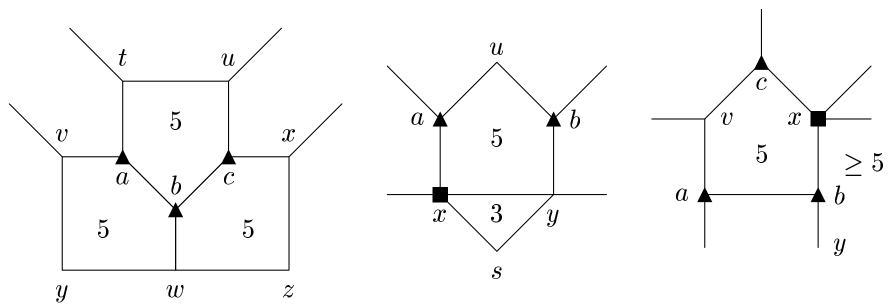

Figure 1: Three basic configurations of Lemma 10. The 3-vertices are depicted as triangles and the 4-vertices as squares. The numbers placed in the faces represent their sizes.

Rule R1: Each ≤ 5-vertex sends $\frac{1}{2}$ to each incident 4-face.

Rule R2: If x_2 is a $(4, \geq 5, \geq 5)$-vertex, then it sends $\frac{5}{4}$ to f.

Rule R3: If x_1 or x_3 is of degree ≥ 4 and x_2 is a $(5,5,5)$-vertex adjacent to two $(5,5,\geq 5)$-vertices, then x_2 sends $\frac{5}{4}$ to f.

Rule R4: If both the vertices x_1 and x_3 are $(5,5,\geq 5)$-vertices and the vertex x_2 is a $(5,5,5)$-vertex, then x_2 sends $\frac{1}{2}$ to f.

Rule R5: If x_2 is a 3-vertex and f receives no charge from x_2 by Rules R2, R3 or R4, then x_2 sends 1 to f.

Rule R6: If x_2 is a 4-vertex and x_1x_2 or x_2x_3 is incident with a 3-face, then x_2 sends $\frac{3}{4}$ to f.

Rule R7: If x_2 is a 4-vertex and f receives no charge from x_2 by Rule R6, then x_2 sends $\frac{1}{2}$ to f.

Rule R8: If x_2 is a 5-vertex and each of x_1x_2, x_2x_3 is incident with a 3-face, then x_2 sends $\frac{1}{2}$ to f.

Rule R9: If x_2 is a 5-vertex and f receives no charge from x_2 by Rule R8, then x_2 sends $\frac{1}{4}$ to f.

Rule R10: If x_1 and x_4 are $(5, 5, \geq 5)$-vertices and x_2 and x_3 are $(5, 5, 5)$-vertices, then f receives $\frac{1}{4}$ from the 5-face adjacent to f which is incident with $x_2 x_3$.

Now, we prove that for every $x \in V(G) \cup F(G)$, the final charge $c^*(x)$ is non-positive. We consider several cases to do so:

Case that x is a vertex of G: If x is a 3-vertex, it is not incident with a 3-face due to Lemma 1 and Lemma 5. If x is incident with a 4-face, then other two faces incident with it are $\geq$ 5-faces due to Lemma 4. Hence by Rule R2 x sends $\frac{5}{4}$ to each of the $\geq$ 5-faces and by Rule R1 it sends $\frac{1}{2}$ to the 4-face. We infer $c^*(x) = 0$.

Suppose now that x is a $(\geq 5, \geq 5, \geq 5)$-vertex. If x is adjacent to three 3-vertices, then x is a $(5, 5, 5)$-vertex due to Lemma 8. But then Configuration 1 of Lemma 10 is present in the graph (x being the vertex b of the configuration) which is impossible. If x is adjacent to exactly two 3-vertices, then x is a $(5, 5, 5)$-vertex due to Lemma 8; it sends out $\frac{5}{4}$ by Rule R3 to each of the two faces incident with its neighbor which is $\geq$ 4-vertex and it sends out $\frac{1}{2}$ to the remaining 5-face. Hence $c^*(x) = 0$. If x is adjacent to at most one 3-vertex, then it sends out 1 to each of the incident faces by Rule R5. Hence $c^*(x) = 0$ as well.

If x is a 4-vertex, then x is incident with at most one 3-face by Lemma 4, Lemma 7 and Lemma 9. If x is incident with a 3-face, then it sends out twice $\frac{3}{4}$ by Rule R6 and once $\frac{1}{2}$ either by Rule R1 or by Rule R7. In both the cases, $c^*(x) = 0$ as desired.

If x is a 5-vertex, then it cannot be incident to more than two 3-faces by Lemma 4. If it is incident to two 3-faces, then it is a $(\geq 5, 3, \geq 5, 3, \geq 5)$-vertex and it sends out once $\frac{1}{2}$ by Rule R8 and twice $\frac{1}{4}$ by Rule R9. Otherwise, x is incident with at least four $\geq$ 4-faces, and it sends out altogether 1 to all them by Rule R1 and Rule R9. In all the cases $c^*(x) = 0$.

If x is a $\geq$ 6-vertex, then it neither sends nor receives any charge, and hence $c^*(x) \leq 0$.

Case that x is a face of G: If x is a 3-face or a 4-face, it is easy to see that $c^*(x) \leq 0$ (no rule applies to a 3-face and the only rule which applies to a 4-face is Rule R1).

We first deal with x being a ≥ 6-face. Vertices of any facial path uvw of x may send at most 3 (in total) to x: By Lemma 8, at least one of them is a ≥ 4-vertex and so it sends at most $\frac{3}{4}$ to x. If the total contribution is bigger than 3, then other two vertices are 3-vertices each sending $\frac{5}{4}$ to x. But in such case v sends at most $\frac{1}{2}$ to x (Rule R6 cannot apply) because of Lemma 1 and Lemma 5 used to u and w. Hence the total contribution is at most 3 also in this case. From this, it immediately follows that $c^*(x) \leq 2 - 2r + 3\lceil \frac{r}{3} \rceil \leq 0$ where r is the size of x.

The remaining case is that the face x is a 5-face $x_1x_2x_3x_4x_5$. Let $S \subseteq \{x_1, \ldots, x_5\}$ consist of those vertices which are $(5, \geq 5, \geq 5)$-vertices. If a vertex x_i is a 3-vertex, then its unique neighbor not incident with x is denoted by x_i'. Regarding the possible content of S, the following cases are enough to be considered:

Subcase $S = \{x_1, x_2, x_3, x_4, x_5\}$**:** By Lemma 8, all the vertices x_i have to be $(5, 5, 5)$-vertices. Hence the charge is sent to x only by Rules R4 and R10 and so $c^*(x) \leq -4 + \frac{5}{2} + \frac{5}{4} \leq 0$.

Subcase $S = \{x_1, x_2, x_3, x_4\}$**:** Both the vertices x_2 and x_3 by Lemma 8 are $(5, 5, 5)$-vertices and each of them sends $\frac{1}{2}$ to x by Rule R4. By Rule R10, x receives $\frac{1}{4}$. If $c^*(x) > 0$, then each of x_1 and x_4 have to send $\frac{5}{4}$ to x by Rule R3 and x_5 $\frac{1}{2}$ by Rule R7. This implies that x_1 and x_4 are $(5, 5, 5)$-vertices and x_5 is a 4-vertex. But this is Configuration 1 of Lemma 10 with $w = x_5$ and $y = x_1$ ($\deg_2(x_5) \leq 12$ and $\deg_2(x_1) \leq 9$); note that x_4' is a 3-vertex, otherwise Rule R3 does not apply to x_4.

Subcase $S = \{x_1, x_2, x_3\}$**:** By Lemma 8, x_2 is a $(5, 5, 5)$-vertex (thus Rule R4 applies to it). Each of x_1 and x_3 sends at most $\frac{5}{4}$ to x. Neither x_4 nor x_5 is a 3-vertex incident with a ≤ 4-face due to Lemma 7. Hence both of them are ≥ 4-vertices and they send at most $\frac{1}{2}$ to x each. Thus $c^*(x) \leq -4 + \frac{10}{4} + \frac{3}{2}$.

Subcase $S = \{x_2, x_4, x_5\}$**:** As in the previous case, it is possible to conclude that both x_1 and x_3 are ≥ 4-vertices and all the faces incident with x are ≥ 5-faces. The vertex x_2 sends 1 to x by Rule R5.

Assume that the vertices x_1 and x_5 send together to x more than $\frac{3}{2}$. Then, x_1 is a 4-vertex sending $\frac{1}{2}$ to x by Rule R7 and x_5 is 3-vertex sending $\frac{5}{4}$ to x by Rule R3. The latter implies that x_5' is a 3-vertex. Configuration 1 of Lemma 10 may be seen in the graph with $b = x_5$, $w = x_1$, $y = x_2$ and $c = x'5$ (note that $\deg_2(x_1) \leq 12$ and $\deg_2(x_2) \leq 9$) which is impossible.

The symmetric argument yields that the vertices x_3 and x_4 send together at most $\frac{3}{2}$ to x. Hence, $c^*(x) \leq -4 + 1 + 2 \cdot \frac{3}{2} = 0$.

Subcase $S = \{x_4, x_5\}$**:** By Lemma 7, both the vertices x_1 and x_3 are ≥ 4-vertices. If the vertices x_4 and x_5 together send more than $\frac{9}{4}$ to x, then Rule R3 applies to both of them and they send the charge of $\frac{10}{4}$ to x. Hence x'_4 and x'_5 are 3-vertices and x_4 and x_5 are $(5,5,5)$-vertices by Lemma 8. Then x sends to the other face incident with the edge x_4x_5 the charge of $\frac{1}{4}$ by Rule R10. It is possible to conclude that the charge sent to x by the vertices x_4 and x_5 minus the possible charge sent out by the face x itself is at most $\frac{9}{4}$.

We first deal with the case that x_2 is a 3-vertex. In such case, x_2 is a $(4, 5, \geq 5)$-vertex because $x_2 \notin S$ and it sends $\frac{5}{4}$ to x by Rule R2. If x_1 is a 4-vertex, then Configuration 3 of Lemma 10 with $x = x_1$, $a = x_4$, $b = x_5$ and $c = x_2$ is present in the graph which is impossible. Similarly, x_3 is not a 4-vertex. Since both x_1 and x_3 are ≥ 5-vertices, each of them sends to x at most $\frac{1}{2}$. Hence $c^*(x) \leq -4 + \frac{9}{4} + \frac{5}{4} + 2 \cdot \frac{1}{2} = 0$.

The remaining case is x_2 being a ≥ 4-vertex. Each of the vertices x_1, x_2 and x_3 sends to x at most $\frac{3}{4}$. If all of them send $\frac{3}{4}$, then Rule R6 applies to all them and x_2 is a $(3, 5, 3, \geq 5)$-vertex which is impossible by Lemma 9. Hence these three vertices send x in total at most 2. If neither x_4 nor x_5 sends $\frac{5}{4}$ to x, then the total charge received by x is at most 4. If $c^*(x) > 0$, then it is possible to assume that x_4 sends $\frac{5}{4}$ to x (using Rule R3), two vertices of x_1, x_2 and x_3 send $\frac{3}{4}$ to x (using Rule R6) and the remaining one sends $\frac{1}{2}$ to x (using Rule R7 or Rule R8). Hence, x'_4 is a 3-vertex and x_4 is a $(5, 5, 5)$-vertex by Lemma 8.

If x_3 sends $\frac{3}{4}$, then it has to be by Rule R6. This implies that x_3 is a 4-vertex, the edge x_2x_3 is incident with 3-face and $\deg_2(x_3) \leq 10$. On the other hand, if x_3 contributes $\frac{1}{2}$, then it is sent by Rule R7. This gives that x_3 is a 4-vertex and $\deg_2(x_3) \leq 12$. Moreover, in such case Rule R6 has to apply to both x_1 and x_2 (and hence they both are 4-vertices). This gives that the edge x_1x_2 is incident with a 3-face and $\deg_2(x_1) \leq 10$ and $\deg_2(x_2) \leq 10$. In either of the cases, Configuration 1 of Lemma 10 with $x = x_1$, $w = x_3$ and $b = x_4$ is present in the graph which is impossible.

Subcase $S = \{x_1, x_3\}$: The vertex x_2 is a $\geq$ 4-vertex and so it sends at most $\frac{1}{2}$ to x. Each of x_1 and x_3 sends 1 to x by Rule R5. The vertices x_4 and x_5 are also $\geq$ 4-vertices by Lemma 7 and each of them sends at most $\frac{3}{4}$ to x. Hence $c^*(x) \leq -4 + \frac{1}{2} + 2 \cdot 1 + 2 \cdot \frac{3}{4} = 0$.

Subcase $S = \{x_2\}$: The vertex x_2 sends 1 to x by Rule R5. The vertices x_1 and x_3 are $\geq$ 4-vertices by Lemma 7. If none of the remaining vertices is a $(4, 5, \geq 5)$-vertex, then $c^*(x) \leq -4 + 1 + 4 \cdot \frac{3}{4} = 0$. Otherwise, either x_4 or x_5 is a $(4, 5, \geq 5)$-vertex (they cannot be both again due to Lemma 7). Assume x_5 is the $(4, 5, \geq 5)$-vertex. Then x_5 sends $\frac{5}{4}$ to x by Rule R2 and x_1 sends at most $\frac{1}{2}$.

If $c^*(x) > 0$, then the vertices x_3 and x_4 sends together $\frac{5}{4}$ to x. This implies that x_3 and x_4 are 4-vertices and the edge x_3x_4 is incident with a 3-face. Configuration 2 with $a = x_2$, $b = x_5$, $x = x_3$ and $y = x_4$ of Lemma 10 can be seen present in the graph which is impossible.

Subcase $S = \emptyset$: If any of the vertices incident with x sends to it more than $\frac{3}{4}$, then that vertex must be a $(4, 5, \geq 5)$-vertex which sends $\frac{5}{4}$ to x by Rule R2. There are no two such adjacent vertices by Lemma 7 and hence x is incident with at most two such vertices. If x is incident with no such vertex, then $c^*(x) \leq -4 + 5 \cdot \frac{3}{4} < 0$. Assume that x is incident with exactly one $(4, 5, \geq 5)$-vertex sending $\frac{5}{4}$ to x, x_2 is this vertex and x_2x_3 is incident with a 4-face. The vertex x_3 sends at most $\frac{1}{2}$ to x (otherwise, x_3 is a $(3, 5, 4, *)$-vertex which is impossible by Lemma 7 used to the edge x_2x_3). Hence $c^*(x) \leq -4 + 3 \cdot \frac{3}{4} + \frac{5}{4} + \frac{1}{2} = 0$.

The case which remains is that the face x is incident with two $(4, 5, \geq 5)$-vertices. Suppose that x_1 and x_3 are these two vertices. Then x_2 may send at most $\frac{1}{2}$ to x. If x_4 or x_5 sends $\frac{3}{4}$ to x (and hence it is a 4-vertex), then we obtain a copy of Configuration 2 with $\{a, b\} = \{x_1, x_3\}$ in the graph. Thus $c^*(x) \leq -4 + 2 \cdot \frac{5}{4} + 3 \cdot \frac{1}{2} = 0$.

■

5 The Bounds for 3-Facial and 4-Facial Colorings

We first restate Lemma 5 for the case of 3-vertices and Lemma 6 for the case of edges whose both end vertices are 3-vertices:

Lemma 11 *Let v be an (f_1, f_2, f_3)-vertex of an (l, k)-minimal graph G. Let $s_i = \min\{2l+1, f_i\}$ for $i = 1, 2, 3$. Then, $s_1 + s_2 + s_3 \geq k + 6$.*

Lemma 12 *Let v_1 and v_2 be two adjacent 3-vertices of an (l, k)-minimal graph G; let v_1 be an (f_1, f_2, f_3)-vertex and v_2 an (f_1, f_2, f_4)-vertex. Let $s_i = \min\{2l+1, f_i\}$ for $i = 1, 2, 3, 4$. Suppose that $f_1 \geq 4$ and $f_2 \geq 4$. If $s_1 + s_2 + s_3 \leq k + 6$, then $s_2 + s_3 + s_4 \geq k + 8$.*

We can now state and prove the main theorem of this section:

Theorem 4 *Each plane graph has a 3-facial coloring using at most 12 colors and a 4-facial coloring using 15 colors.*

Proof: Let $l \in \{3, 4\}$ be a fixed integer and G be an $(l, 3l+3)$-minimal graph. The minimum degree of G is at least three by Lemma 5 and each edge has face-weight at least $2l+4$ by Lemma 4. The proof is an application of a discharging method. The initial charge c_0 is the following:

- The charge of a k-vertex v is $c_0(v) := 4 - k$.
- The charge of a k-face f is $c_0(f) := 4 - k$.

It is easy to verify (using Euler's formula) that that the total charge distributed to the vertices and faces is equal to 8.

We apply the following two rules:

Rule R1: A 3-face sends $\frac{1}{3}$ to each incident face.

Rule R2: A 3-vertex v sends 0, 0, $\frac{1}{5}$ and $\frac{1}{3}$, respectively, to each incident face of size 3, 4, 5 and 6, respectively. The remaining positive charge of v is equally distributed between other incident faces.

Let c^* be the final charge. We prove for the sake of contradiction that $c^*(x) \leq 0$ for all $x \in V(G) \cup F(G)$. Several cases are distinguished:

Case A: Let $v \in V(G)$. If v is a ≥ 4-vertex, it neither receives nor sends out any charge and hence $c^*(v) \leq 0$. If v is a 3-vertex, then it is either incident with three 6-faces or a $\geq$ 7-face (Lemma 11) and hence it sends out all its charge by Rule R2.

Case B: Let $f \in F(G)$. Let r be the size of f.

Case B.1: $r \leq 2l$. The face f is not incident with a 3-face due to Lemma 4 and hence it receives no charge by Rule R1. If $r = 3$, the face f sends all its charge out by Rule R1 and it receives no charge by Rule R2; hence $c^*(f) = 0$. If $r = 4$, f neither sends nor receives any charge; so $c^*(f) = 0$. If $r = 5$ or $r = 6$, then each incident vertex sends to f at most $\frac{1}{5}$ or $\frac{1}{3}$ by Rule R2, respectively. Thus $c^*(f) \leq -1 + 5 \cdot \frac{1}{5} = 0$ if $r = 5$ and $c^*(f) \leq -2 + 6 \cdot \frac{1}{3} = 0$ if $r = 6$.

Suppose now that $6 < r \leq 2l$. In this case, $l = 4$. If $r = 7$, then f could receive from an incident vertex v more than $\frac{1}{3}$ only if v is incident with a ≤ 5-face. Due to Lemma 11, f is incident with neither a $(\leq 5, \leq 6, r)$-vertex nor a $(\leq 4, r, *)$-vertex (recall $r = 7$). Hence no vertex v sends more than $\frac{2}{5}$ to f and if v sends $\frac{2}{5}$ to f, then v is a $(5, \geq 7, r)$-vertex. By parity argument, f can receive $\frac{2}{5}$ from at most six vertices. Thus, $c^*(f) \leq -3 + 6 \cdot \frac{2}{5} + \frac{1}{3} < 0$. If $r \geq 8$, then each vertex sends to f at most $\frac{1}{2}$, and so $c^*(f) \leq 4 - r + r \cdot \frac{1}{2} \leq 0$.

Case B.2: $r \geq 2l + 1$. If $\frac{1}{3}$ is sent to f through an edge xy by Rule R1, we pretend that this charge was sent to f through the vertices x and y (through each of them the charge of $\frac{1}{6}$). If r is an odd number, then there is a ≥ 4-vertex incident with the face f or a 3-vertex incident with f which is a $(\geq 6, \geq 6, r)$-vertex. Assume the opposite in order to see this, i.e., that each vertex incident with f is a $(\leq 5, *, r)$-vertex. Then there is a $(\leq 5, \leq 5, r)$-vertex v incident with f (recall that r is odd). The l-facial degree of v is at most $2l + 11 - 6 = 2l + 5$ contradicting minimality of G by Lemma 5. Let x_0 be such a vertex, i.e. a 4-vertex or a $(\geq 6, \geq 6, r)$-vertex. Note that x_0 must exist only if r is odd. If x_0 is a 4-vertex, then it sends no charge to f by Rule R2. Hence the face f receive through x_0 the charge of at most $\frac{1}{3}$ by Rule R1. In the case that x_0 is a $(\geq 6, \geq 6, r)$-vertex, then no charge is sent from x_0 to f by Rule R1 and the charge sent by Rule R2 is also at most $\frac{1}{3}$.

Case B.2.1: $l = 3$. By Lemma 11, no 3-vertex is incident with a 3-face. Thus each vertex sends to f charge by at most one of the rules R1 and R2. In particular, any vertex sends at most $\frac{1}{2}$ to f (it sends $\frac{1}{2}$ iff it is a $(4, r, *)$-vertex). Let u, v and w be any three consecutive vertices on the boundary of f. We prove that the charge sent to f together by these three vertices is at most $\frac{4}{3}$. If one of them is not a 3-vertex, then the total contribution is at most $\frac{4}{3}$. Assume next that all u, v and w are 3-vertices. By Lemma 12, there are no two consecutive $(4, r, *)$-vertices on the boundary of f. So, if two vertices of u, v and w send to f the charge of $\frac{1}{2}$, then these two vertices are u and w. In that case, v is either a 4-vertex or a $(\geq 6, \geq 6, r)$-vertex (again using Lemma 12). In either of the cases, v sends at most $\frac{1}{3}$ to f and the three vertices altogether send to f at most $\frac{4}{3}$. If only one vertex out of u, v and w sends $\frac{1}{2}$ to f, then the others may send each at most $\frac{2}{5}$ to f. So, f receives at most $\frac{13}{10} < \frac{4}{3}$ in total.

If $r \geq 8$, then, by the above claim, $c^*(f) \leq 4 - r + \frac{4}{3} \cdot \frac{r}{3} \leq 0$. If $r = 7$, split vertices of f different from x_0 to two parts formed by three consecutive vertices. Each of these parts sends to f at most $\frac{4}{3}$. We infer that $c^*(v) \leq -3 + \frac{4}{3} \cdot 2 + \frac{1}{3} = 0$.

Case B.2.2: $l = 4$. Any ≥ 4-vertex can send to f the charge of at most $\frac{1}{3}$ (only Rule R1 may apply). Any 3-vertex can send to f the charge of at most $\frac{1}{2} + \frac{1}{6}$ by the rules R1 and R2 (no 3-vertex is a $(3, 3, r)$-vertex). Let u, v, w and z be any four consecutive vertices on the boundary of f. We claim that these four vertices send to f altogether the charge of at most $\frac{7}{3}$. If at least one of u, v, w and z is a ≥ 4-vertex, then f receives from them the charge at most $3 \cdot \left(\frac{1}{2} + \frac{1}{6}\right) + \frac{1}{3} \leq \frac{7}{3}$. So, assume that all of them are 3-vertices.

In order to obtain a contribution bigger than $\frac{7}{3}$ at least three of these four vertices contribute $\frac{1}{6}$ to f by R1. By Lemma 12, there are no three consecutive $(3, r, *)$-vertices. So, we may assume that u, w and z contribute $\frac{1}{2} + \frac{1}{6}$. In that case, v is a $(\geq 6, \geq 6, \geq 6)$-vertex (Lemma 12 applied to u and w), so v sends $\frac{1}{3}$ to f. The charge at most $\frac{7}{3}$ is sent to f in total. Thus, the claim is established.

By the above claim, if $r \geq 10$, then $c^*(f) \leq 4 - r + \frac{7}{3} \cdot \frac{r}{4} \leq 0$. If $r = 9$, then the existence of the vertex x_0 implies (similarly to the case B.2.1) that $c^*(f) \leq -5 + \frac{7}{3} \cdot 2 + \frac{1}{3} = 0$.

■

Theorem 3 and Theorem 4 extend Theorem 2 to l-facial colorings for small values of l:

Corollary 2 *Let $l \geq 1$ be a fixed integer. Each plane graph has an l-facial coloring using at most $\lfloor \frac{18l}{5} \rfloor + 2$ colors.*

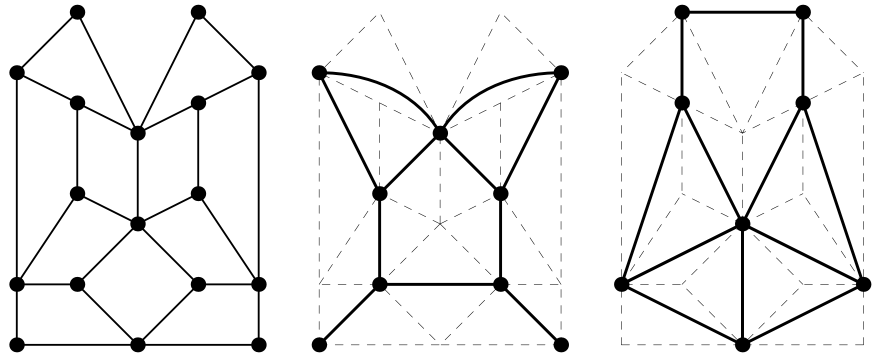

Figure 2: Construction of graphs G_1 and G_2 from a quadrangulation Q described in the proof of Theorem 5.

6 1-Diagonal Coloring of Quadrangulations

Recall that we demand in a 1-diagonal coloring that any two vertices incident with the same face or with the pair of neighboring faces (i.e., faces sharing an edge) have distinct colors. We use the bound from Theorem 3 to improve the upper bound on the number of colors needed for 1-diagonal coloring of plane quadrangulations:

Theorem 5 *Every plane quadrangulation has a* 1*-diagonal coloring using at most* 16 *colors.*

Proof: Let Q be a plane quadrangulation. Q is a bipartite graph and let V_1 and V_2 be its two parts of vertices. We form plane graphs G_1 and G_2 as follows: The vertex set of G_i is V_i, $i \in \{1, 2\}$, and two vertices are joined by an edge in G_i iff they are incident with the same face of Q (see Figure 2). Which vertices of V_1 (or V_2) have to get distinct colors in a 1-diagonal coloring of Q? The vertices which are incident with the same face (i.e., adjacent in G_1 or G_2) and the vertices whose faces are neighboring (i.e., those connected by a facial walk of length 2 in G_1 or in G_2). We may conclude that two vertices in V_1 (V_2, resp.) have to be colored in Q by different colors iff they are 2-facially adjacent in G_1 (G_2, resp.). Both G_1 and G_2 have 2-facial colorings using at most 8 colors due to Theorem 3. Let c_1 be such a coloring of G_1 and c_2 of G_2; assume that c_1 and c_2 use

l	1	2	3	4	5	6	7	8	9	10
$\Delta^\star = 2l+1$	3	5	7	9	11	13	15	17	19	21
Upper bound on $f_f(l)$	4	8	12	15	20	23	27	30	34	38
Upper bound on $f_c(\Delta^\star)$	4	8	12	15	19	22	25	29	32	35
Lower bound	4	7	10	13	16	19	22	25	28	31

Table 1: Summary of known results for cyclic and facial coloring.

completely different colors. Let c be a coloring of Q such that $c = c_i(v)$ for i such that $v \in V_i$. The coloring c (using at most 16 colors) is clearly a 1-diagonal coloring of Q.

■

7 Open Problems

We discuss relation between the concept of cyclic coloring and its extension facial coloring and pose several open problems in this section. Recall that $f_f(l)$ is the maximum number of colors needed for an l-facial coloring of a plane graph and $f_c(\Delta^\star)$ is the maximum number of colors needed for a cyclic coloring of a plane graph with maximum face size at most $\Delta^\star$. Our upper bounds on $f_f(l)$, the best known upper bounds on $f_c(2l+1)$ and the best lower bounds for $1 \le l \le 10$ can be found in Table 1 (remember that $f_c(2l+1) \le f_f(l)$). We match the bound $f_c(\Delta^\star) \le \left\lfloor \frac{9\Delta^\star}{5} \right\rfloor$ by proving $f_f(l) \le \left\lfloor \frac{18l}{5} \right\rfloor + 2$. The bound $f_c(\Delta^\star) \le \left\lceil \frac{5\Delta^\star}{3} \right\rceil$ from [16] bids the following problem: Does the inequality $f_f(l) \le \left\lceil \frac{10l+5}{3} \right\rceil$ hold for all $l \ge 1$? However, it would be more interesting to bring some light into the relation between the two studied concepts of coloring (cyclic and facial colorings). We do not know a single example of a graph for which the concept of facial coloring would be more general (in sense it requires more colors) than the concept of cyclic coloring:

Problem 1 *Is it true that $f_f(l) = f_c(2l+1)$ for all $l \ge 1$?*

Ore and Plummer [13, 12] conjectured that $f_c(\Delta^\star) = \left\lfloor \frac{3}{2}\Delta^\star \right\rfloor$; we state as a problem the counterpart of this famous conjecture for facial coloring:

Problem 2 *Does the equality $f_f(l) = 3l + 1$ hold for all $l \geq 1$?*

The equality in the last problem is known only when $l = 1$ for which it is equivalent to the Four Color Theorem. The equality in the case of $l = 2$ ($f_f(2) = 7$) would have several interesting corollaries: Besides improving the best upper bound for the cyclic coloring of plane graphs with maximum face size at most 5 (and hence finding the value of $f_c(5)$) and decreasing the upper bound on the number of colors needed for 1-diagonal coloring of plane quadrangulations to 14 (from 16 which we proved in Theorem 5), it would also prove Wegner's conjecture on 2-distance coloring (i.e., coloring of squares of graphs) restricted to plane cubic graphs because 2-distance colorings (colorings of the square) of a plane cubic graph are precisely their 2-facial colorings. Due to its interest, we state this as a separate problem:

Problem 3 *Does each plane graph have a 2-facial coloring using at most 7 colors?*

References

[1] K. Appel, W. Haken: *Every planar map is four colorable*, Contemp. Math. 98 (1989).

[2] O. V. Borodin: *A new proof of 6 color theorem*, J. Graph Theory **19** (1995), 507–521.

[3] O. V. Borodin: *Cyclic coloring of plane graphs*, Discrete Math. **100** (1992), 281–289.

[4] O. V. Borodin: *Solution of Ringel's problems on vertex-face coloring of plane graphs and coloring of 1-planar graphs (in Russian)*, Met. Diskret. Anal. **41** (1984), 12–26.

[5] O. V. Borodin, D. P. Sanders, Y. Zhao: *On cyclic colorings and their generalizations*, Discrete Math. **203** (1999), 23–40.

[6] A. Bouchet, J. L. Fouquet, J. L. Jolivet, M. Riviere: *On a special face colouring of cubic graphs*, Ars Combin. **24** (1987), 67–76.

[7] H. Enomoto, M. Horňák, S. Jendrol': *Cyclic chromatic number of 3-connected plane graphs*, SIAM J. Discrete Math. **14(1)** (2001), 121–137.

[8] M. Horňák, S. Jendrol': *On a conjecture by Plummer and Toft*, J. Graph Theory **30(3)** (1999), 177–189.

[9] M. Horňák, S. Jendrol': *On some properties of 4-regular plane graphs*, J. Graph Theory **20** (1995), 163–175.

[10] M. Horňák, S. Jendrol': *On the d-distance face chromatic number of plane graphs*, Discrete Math. **164** (1997), 171–174.

[11] M. Horňák, S. Jendrol': *On vertex types and cyclic colourings of 3-connected plane graphs*, Discrete Math. **212** (2000), 101–109.

[12] T. R. Jensen, B. Toft: *Graph Coloring Problems*, Wiley (1995), New York.

[13] O. Ore, M. D. Plummer: *Cyclic coloration of plane graphs*, in: W. T. Tutte: Recent progress in combinatorics (Proceedings of the Third Waterloo Conference on Combinatorics, May 1968), Academic Press 1969, 287–293.

[14] M. D. Plummer, B. Toft: *Cyclic coloration of 3-polytopes*, J. Graph Theory **11(4)** (1987), 507–515.

[15] N. Robertson, D. P. Sanders, P. D. Seymour, R. Thomas: *The Four-Colour Theorem*, J. Comb. Theory Ser. B **70** (1999), 2–44.

[16] D. P. Sanders, Y. Zhao: *A new bound on the cyclic chromatic number*, J. Comb. Theory Ser. B **83** (2001), 102–111.

[17] D. P. Sanders, Y. Zhao: *On d-diagonal colorings*, J. Graph Theory **22** (1996), 155–166.

[18] D. P. Sanders, Y. Zhao: *On d-diagonal colorings of embedded graphs of low maximum face size*, Graphs and Combin. **14** (1998), 81–94.