A CONSTRUCTIVE VIEW OF COMPLETE REGULARITY

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Abstract. This note introduces the largest interpolative relation contained in the rather below(= well inside) relation in a frame as a constructive version of the familiar completely below (= really inside) relation.It establishes several constructively valid results for this whose analogues for the latter are only proved by means of the Axiom of Countable Dependent Choice.

The familiar proof that any compact regular frame is completely regular depends on the Axiom of Countable Dependent Choice (CDC), and while it is not clear whether, in Zermelo–Fraenkel Set Theory treated in classical logic, a weaker (or no) choice principle would be sufficient for this result, one does know that it is not constructively valid in the sense of Topos Theory. This raises the question whether there is an alternative to complete regularity, presumably somewhat weaker, for which it can be proved constructively that it retains the familiar connection of the latter, given CDC, with uniformities, compactifications, and normality, as well as its coreflectiveness. It is the aim of this note to show that such an alternative does indeed exist; in fact it is the analogue of complete regularity based on just the largest interpolative relation contained in the familiar rather below relation.

In the following, a binary relation S on a set will be called *interpolative* if $S \subseteq S \circ S$, that is, whenever xSy there exist z such that $xSzSy$. Now, the union of any set of interpolative relations on some set E is clearly interpolative, and hence any binary relation R on E contains a largest interpolative relation, to be called its *interpolative part* $R_\circ$.

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Concerning binary relations R on some pseudocomplemented bounded lattice A, with zero 0, unit e, and pseudocomplementation $a \mapsto a^*$, we call R an *entailment* whenever

(1) aRb implies $a \leq b$.
(2) If $a' \leq a$, aRb, and $b \leq b'$ then $a'Rb'$.
(3) R is a bounded sublattice of $A \times A$.
(4) aRb implies b^*Ra^*.

Now we have

Lemma. *The interpolative part of any entailment is an entailment.*

Proof. We begin with some more general remarks concerning an entailment R and any interpolative $S \subseteq R$.

(i) The relation

$$T = \{(a,b) \mid (a \leq a' \text{ and } b' \leq b \text{ for some } a'Sb'\}$$

is interpolative: if aTb so that $a \leq a'$, $a'Sb'$, $b' \leq b$ for some a' and b' then $a'ScSb'$ for some c and hence $aTcTb$.

(ii) Similarly

$$U = \{(a,b) \mid a \leq a_1 \wedge a_2,\ a_iSb_i,\ b_1 \wedge b_2 \leq b \text{ for some } a_1, a_2, b_1, b_2\}$$

is interpolative: given aUb as in the above definition, there exist c_1 and c_2 such that $a_iSc_iSb_i$ and hence $aU(c_1 \wedge c_2)Ub$.

(iii) The case for $\vee$ is formally the same and hence omitted. Further, the relation $\{(0,0),(e,e)\}$ is trivially interpolative.

(iv) Finally,

$$V = \{(b^*, a^*) \mid aSb\}$$

is interpolative: that aSb implies $aScSb$ for some c says that $b^*Sc^*Sa^*$ whenever b^*Va^*.

Now, all the above new relations are contained in R and for $S = R_\circ$ it then follows that (2)–(4) hold for $R_\circ$, and (1) for $R_\circ$ trivially follows from (1) for R. $\square$

Remark 1. Note that (1) is never used in the above proof except for the final step, and hence the same lemma already holds for the relations on A which only satisfy (2)–(4). On the other hand, various other properties of R carry over to $R_\circ$; thus for instance transitivity, as follows by the same kind of argument as used in the above proof.

Remark 2. For any relation on a set E, let R_* be the relation such that

$$\begin{aligned} xR_*y \text{ iff there exists a sequence } &(z_{nk})_{n=0,1,\ldots;\ k=0,1,\ldots,2^n} \\ &\text{such that } z_{00}=x, z_{01}=y, z_{nk}=z_{n+1,2k}, z_{nk}Rz_{n,k+1} \\ &\text{for all } n=0,1,\ldots \text{ and } k=0,1,\ldots,2^n. \end{aligned}$$

Then xR_*y trivially implies xRy (the case $n=0$) and similarly $xR_*z_{11}R_*y$, that is, R_* is interpolative; as a result, $R_* \subseteq R_\circ$. On the other hand, if CDC is assumed then succsessive choices of interpolating elements show that

(∗) *$S_* = S$ for any interpolative relation S*

and consequently, for any relation R,

$$R_\circ = (R_\circ)_* \subseteq R_*$$

so that $R_\circ = R_*$. Thus, the relation $R_\circ$ may be viewed as the choice-free (and, in fact, constructive) version of R_*. We note in passing that the question whether the above condition (∗) implies CDC seems to be open.

Regarding R_*, it should be added that this already occurs in Banaschewski [1] where it is described as the largest interpolative relation contained in R, for arbitrary R, with CDC assumed tacitly.

As indicated earlier, our ultimate concern are *frames*. For general background, see Johnstone [7] and Vickers [9]; regarding the specific facts relevant here we recall the following for any frame L.

(i) $a \prec b$ (a is *rather below*, or: *well inside* b) means that $a \wedge c = 0$ and $b \vee c = e$ for some c or, equivalently, $a^* \vee b = e$ for the pseudocomplement a^*, and L is called *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for all $a \in L$.

(ii) $a \prec\!\!\prec b$ (a is *completely below*, or: *really inside* b) means, in the present notation, that $a(\prec)_*b$, and L is called *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for all $a \in L$.

(iii) A frame L is called normal if, for any $a \vee b = e$, there are u, v such that $a \vee v = e = u \vee b$ and $u \wedge v = 0$. In particular, in a normal frame $\prec$ interpolates (if $a^* \vee b = e$ we have a v such that $a^* \vee v = e$ and $v^* \vee b = e$). Any compact regular frame is normal; with CDC this makes it completely regular.

(iv) Both, the regular and the completely regular, compact frames are coreflective in the category **Frm** of all frames where the coreflections are given, for any frame L, by suitable subframes of the frame $\mathfrak{J}L$

of ideals of L, with the coreflections given by taking joins in L (Banaschewski - Mulvey [4]).

Note that, in the present terminology, the relation $\prec$ on any frame is an entailment so that, by the Lemma, $(\prec)_\circ$ is also an entailment; given Remark 2, we consider this as the constructive form of $\prec\!\!\prec$. Further, a frame L will be called *strongly regular* whenever

$$\text{(A)} \qquad a = \bigvee\{x \in L \mid x(\prec)_\circ a\} \text{ for all } a \in L.$$

We call (A) the *admissibility condition* and use the corresponding terminology for any relation on a frame L.

Clearly, any regular frame L with interpolating $\prec$ is strongly regular. We recall this is the case for compact, or more generally normal, regular L; in particular this then holds for normal subfit L because these are regular (Banaschewski [3]); we include a short proof for completeness sake. For this recall that L is subfit if

$$\text{(SF)} \qquad a \leq b \text{ whenever } a \vee c = e \text{ implies } b \vee c = e, \text{ for all } c \in L.$$

Now, if L is also normal then, for any $a \in L$, $a \vee c = e$ implies $b \vee c = e$ for some $b \prec a$ and hence $\bigvee\{x \in L \mid x \prec a\} \vee c = e$, proving regularity by (SF).

On the other hand, constructively, strong regularity is strictly weaker than complete regularity, by the fact mentioned at the beginning that a compact regular frame need not be completely regular in that setting.

For the following, recall that a *compactification* of a frame L is a dense onto homomorphism $M \to L$ with compact regular M. Also, see Isbell [6] for the notion of uniformity.

Proposition. *The following are equivalent for any frame L:*

(1) *L is strongly regular.*
(2) *L has a compactification.*
(3) *L has a uniformity.*

Proof. (1)⇒(2). Recalling the notion of *strong inclusion* $\lhd$ on a frame L (Banaschewski [2]), one observes that it is exactly that of an admissible interpolative entailment contained in $\prec$. Hence strong regularity means $(\prec)_\circ$ is a strong inclusion, and by Proposition 1 of [2] a frame has a strong inclusion iff it has a compactification.

(2)⇒(3). It is a familiar fact that, for any compact regular frame M, the covers of M form a uniformity of M (and indeed its only one) – obviously generated by the finite covers, and hence any compactification $M \to L$ of a frame L provides L with a uniformity, consisting of the images of the covers of M

(3)$\Rightarrow$(1). For any uniformity $\mathfrak{U}$ on a frame L we have the admissible relation $\lhd_{\mathfrak{U}}$ such that $a \lhd_{\mathfrak{U}} b$ whenever there exist a cover $C \in \mathfrak{U}$ for which $s \wedge a \neq 0$ implies $s \leq b$ for all $s \in C$, and it is known from Banaschewski - Pultr [5] that $\lhd_{\mathfrak{U}}$ is contained in $(\prec)_\circ$ which makes L strongly regular. $\square$

Remark 3. As a direct proof of (1)$\Rightarrow$(3), we note that any admissible interpolative relation $\sqsubset$ on a frame L which is contained in $\prec$ determines a uniformity on L as follows (we refer to Pultr [8] for some of the technical details.).

For any sequence $a_0 = 0 \sqsubset a_1 \sqsubset \cdots \sqsubset a_n \sqsubset a_{n+1} = e$

$$A(a_1, \ldots, a_n) = \{a_i \wedge a_{i-2}^* \mid i = 2, \ldots, n+1\}$$

is a cover: one shows inductively that $\bigvee\{a_i \wedge a_{i-2}^* \mid i = 2, \ldots, , k\} = a_k$ using that $a_i \sqsubset a_{i+1}$ implies $a_i \prec a_{i+1}$. In the same way,

$$(a_i \wedge a_{i-2}^*) \wedge (a_k \wedge a_{k-2}^*) = 0$$

whenever $i \leq k-2$ or $i \geq k+2$ and consequently

$$\begin{aligned} &A(a_1, \ldots, a_n)(a_k \wedge a_{k-2}^*) = \\ &\quad = (a_{k-1} \wedge a_{k-3}^*) \vee (a_k \wedge a_{k-2}^*) \vee (a_{k+1} \wedge a_{k-1}^*) = a_{k+1} \wedge a_{k-3}^* \end{aligned}$$

for all $k \geq 3$ while $A(a_1, \ldots, a_n)(a_2 \wedge a_0^*) = a_3$. Now consider any sequence $a_0 = 0 \sqsubset a_1 \sqsubset \cdots \sqsubset a_{3n} \sqsubset a_{3n+1} = e$. Then $a_3 \leq a_6 \vee a_0^*$ trivially, and for $k \geq 3$

$$a_{k+1} \wedge a_{k-3}^* \leq a_{3(i+1)} \wedge a_{3(i-1)}^*$$

whenever $k = 3i, 3i+1, 3i+2$; hence

$$A(a_1, \ldots, a_{2n})A(a_1, \ldots, a_{2n}) \leq A(a_3, a_6 \ldots, a_{3n-3}, a_{3n})$$

and since $\sqsubset$ is interpolative this shows that any of the covers considered here is star refined by a cover of the same kind. Further, the set of these covers is admissible because $x \sqsubset a$ implies $A(x,a)x = \{a, x^*\}x \leq a$ and $\sqsubset$ is an admissible relation. Hence, since in general $A_iA_i \leq B_i$ for $i = 1, \ldots, k$ implies $(A_1 \wedge \cdots \wedge A_k)(A_1 \wedge \cdots \wedge A_k) \leq B_1 \wedge \cdots \wedge B_k$, the covers $A(a_1, \ldots, a_n)$ generate a uniformity on L.

Regarding compactification, let L be a strongly regular frame and call an ideal $J \subseteq L$ *strongly regular* if, for each $a \in J$, there exist $b \in J$ such that $a(\prec)_\circ b$. Since $(\prec)_\circ$ is a strong inclusion it follows from Banaschewski [2] that these ideals form a regular subframe $\mathfrak{S}L$ of $\mathfrak{J}L$ and the map $\bigvee : \mathfrak{S}L \to L$

taking joins in L is a compactification. Moreover, since $(\prec)_\circ$ is clearly the largest strong inclusion of L we have

Corollary 1. *$\mathfrak{S}L$ is the compact regular coreflection of L, with coreflection map $\bigvee : \mathfrak{S}L \to L$.*

Remark 4. At the time Banaschewski–Mulvey [4] was written, the authors felt quite keenly that it would be desirable to characterize the existence of compactifications of a frame L, and especially its universal compactification (= its compact regular coreflection), by some simple, suggestive admissible relation, making this case entirely parallel to that of completely regular compactifications, but despite a good deal of effort they were unable to do this. Thus, the present proposition and the above corollary finally fill a long perceived gap.

Given that the existence of uniformities on frames is preserved by coproducts and homomorphic images, the proposition further implies the following

Corollary 2. *The strongly regular frames form a coreflective subcategory of* **Frm**.

Explicitly, one sees that any frame has a largest subframe which has a uniformity, and this provides the coreflection. On the other hand, the coreflectiveness of the strongly regular frames can also be obtained directly, without using the equivalence of (1) and (3) of the proposition: for this it is sufficient to show that any homomorphic image of a strongly regular frame is strongly regular and that any frame generated by strongly regular subframes is itself strongly regular. Here, the first point is quite obvious since any homomorphism preserves $\prec$ and takes any admissible interpolating relation to such relation whenever it is onto. The second part is obtained by a refinement of the proof of the corresponding result for regularity; we omit the details.

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