# More about subcolorings *

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**Abstract**

A subcoloring is a vertex coloring of a graph in which every color class induces a disjoint union of cliques. We derive a number of results on the combinatorics, the algorithmics, and the complexity of subcolorings.

On the negative side, we prove that 2-subcoloring is NP-hard for comparability graphs, and that 3-subcoloring is NP-hard for AT-free graphs and for complements of planar graphs. On the positive side, we derive polynomial time algorithms for 2-subcoloring of complements of planar graphs, and for $r$-subcoloring of interval and of permutation graphs. Moreover, we prove asymptotically best possible upper bounds on the subchromatic number of interval graphs, chordal graphs, and permutation graphs in terms of the number of vertices.



**Keywords:** graph coloring; subcoloring; special graph classes; polynomial time algorithm; computational complexity.



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# 1 Introduction

We denote by $G = (V, E)$ a finite undirected and simple graph. The *complement* $\overline{G}$ of $G = (V, E)$ is the graph on $V$ with edge set $\overline{E}$ such that $\{u, v\} \in \overline{E}$ if and only if $\{u, v\} \notin E$. For a set of graphs $\mathcal{G}$, we denote by $\overline{\mathcal{G}}$ the set of complements of graphs from $\mathcal{G}$; hence $G \in \mathcal{G}$ if and only if $\overline{G} \in \overline{\mathcal{G}}$. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs with $V_G \cap V_H = \emptyset$. The *disjoint union* $G \dot{\cup} H$ is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. The *join* $G \vee H$ of $G$ and $H$ is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H \cup \{\{u, v\} : u \in V_G, v \in V_H\}$. Finally, $G \bowtie H$ denotes the graph that results by adding a new vertex $v$ to the disjoint union of $G$ and $H$, and by joining $v$ to all the vertices in $G$ and $H$.

For every non-empty $W \subseteq V$, the subgraph of $G = (V, E)$ induced by $W$ is denoted by $G[W]$. A *clique* $C$ of a graph $G = (V, E)$ is a non-empty subset of $V$ such that all the vertices of $C$ are pairwise adjacent, i.e., $G[C]$ is a complete graph. The maximum size of a clique in $G$ is denoted by $\omega(G)$. A subset of vertices $I \subseteq V$ is *independent* if no two of its elements are adjacent. An *$r$-coloring* of the vertices of a graph $G = (V, E)$ is a partition $V_1, V_2, \ldots, V_r$ of $V$; the $r$ sets $V_j$ are called the *color classes* of the $r$-coloring. An $r$-coloring is *proper* if every color class is an independent set. The *chromatic number* $\chi(G)$ is the minimum value $r$ for which a proper $r$-coloring exists.

Evidently, an $r$-coloring is proper if and only if for every color class $V_j$ the induced subgraph $G[V_j]$ is the union of complete graphs of cardinality one. This awkward reformulation leads to several interesting generalizations of the classical chromatic number.

- An $r$-coloring $V_1, V_2, \ldots, V_r$ is an *$r$-subcoloring*, if for every color class the induced subgraph $G[V_i]$ is the disjoint union of complete graphs (there is no restriction on the sizes of these complete graphs).
- An $r$-coloring is a *cocoloring*, if for every color class the induced subgraph $G[V_i]$ either is a clique or an independent set.
- Let $F$ be some fixed graph. An $r$-coloring is an *$F$-free coloring*, if for every color class the induced subgraph $G[V_i]$ does not contain $F$ as an induced subgraph.

The *subchromatic number* $\chi_{sub}(G)$, the *cochromatic number* $\chi_{co}(G)$, and the *$F$-free chromatic number* $\chi(F, G)$ of a graph $G$, is the smallest number $r$ for which $G$ has an $r$-subcoloring, an $r$-cocoloring, and an $F$-free $r$-coloring, respectively. Note that a coloring is a subcoloring if and only if it is a $P_3$-free coloring (where $P_k$ denotes the path on $k$ vertices).

In this paper, we study the algorithmic and combinatorial behavior of the subchromatic number on various classes of specially structured graphs. See Appendix A for definitions of these graph classes.

### 1.1 Known results

Finding proper colorings for various classes of perfect graphs is a long studied and well understood problem. We refer to the book [11] of Golumbic for a classical source on algorithmic aspects of perfect graphs. By a celebrated result of Grötschel, Lovász & Schrijver [12], the chromatic number of a perfect graph can be computed in polynomial time. Simple and fast algorithms are known for different subclasses of perfect graphs like chordal graphs, comparability graphs, permutation graphs etc. However, even small steps away from proper coloring towards more general concepts like subcoloring and cocoloring increase the computational complexity of coloring enormously.

For instance, the cochromatic number is NP-hard to compute even for permutation graphs (Wagner [17]). Gimbel, Kratsch & Stewart [9] proved the NP-hardness of computing the cochromatic number for circle graphs and line graphs of comparability graphs, and they derived a polynomial time algorithm for chordal graphs. Achlioptas [1] proved that for any graph $F$ with at least three vertices and for any fixed integer $r \geq 2$, the problem of deciding whether a given input graph has an $F$-free $r$-coloring is NP-hard. By putting $F = P_3$, we get that $r$-subcoloring is NP-hard for any fixed integer $r \geq 2$. Fiala, Jansen, Le & Seidel [8] strengthened this hardness result to input graphs that are triangle-free, planar, and have maximum vertex degree four. On the positive side, [8] gave polynomial time algorithms for the subcoloring problem on cographs and on graphs of bounded treewidth.

The literature also contains a number of results on $P_4$-free colorings: Gimbel & Nešetřil [10] showed that $P_4$-free $r$-coloring in $r = 2$ or $r = 3$ colors is NP-hard even for planar input graphs. Since $P_4$ is isomorphic to its complement $\overline{P}_4$, we conclude that $P_4$-free coloring in 2 or 3 colors is NP-hard for complements of planar graphs. Hoàng & Le [13] proved that $P_4$-free 2-coloring is NP-hard for comparability and cocomparability graphs.

Now let us list a number of useful combinatorial results from the literature on subcolorings and cocolorings.

**Proposition 1.1.** *For any graph $G$, $\chi_{sub}(G) \leq \chi_{co}(G) \leq \min\{\chi(G), \chi(\overline{G})\}$.*

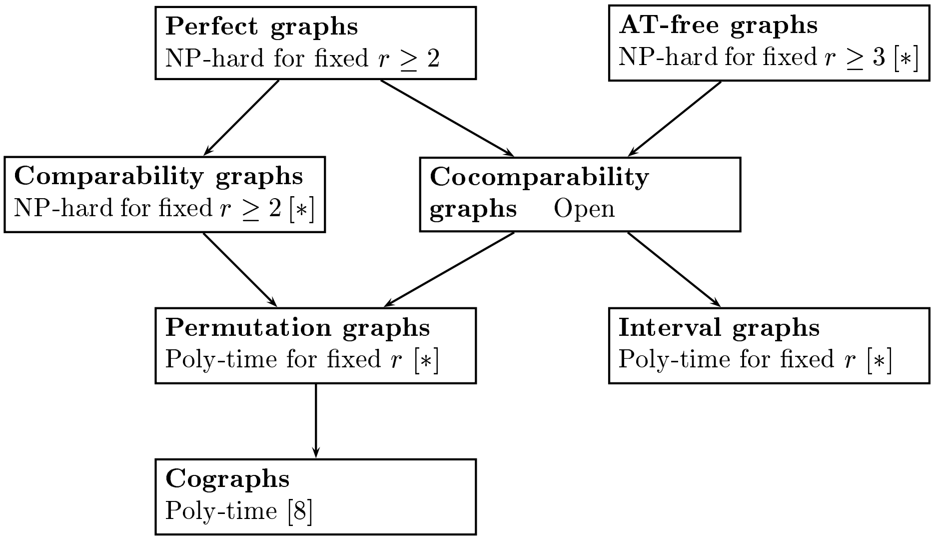


Figure 1: Summary of some of our results on $r$-subcoloring for special graph classes, and the containment relations between these classes. [*] denotes a contribution of this paper.

**Proposition 1.2.** *(Mynhardt & Broere [16])*
*Let $K_{m,m,\ldots,m}$ be the complete $m$-partite graph containing $m$ classes of $m$ vertices. Then $\chi_{sub}(K_{m,m,\ldots,m}) = \chi_{co}(K_{m,m,\ldots,m}) = \chi(K_{m,m,\ldots,m}) = m$.*

**Proposition 1.3.** *(Albertson, Jamison, Hedetniemi & Locke [2])*
*Let $G$ and $H$ be graphs with $\chi_{sub}(G) \geq k$ and $\chi_{sub}(H) \geq k$. Then $\chi_{sub}(G \bowtie H) \geq k+1$.*

## 1.2 Our results

We study combinatorial, algorithmic and complexity aspects of the subcoloring problems. In particular, we derive the following results.

- For general $n$-vertex graphs the subchromatic number may be $\Theta(n/\log n)$; for perfect graphs, permutation graphs, and cographs, it may be $\Theta(\sqrt{n})$; for chordal graphs and interval graphs, it may be

$\Theta(\log n)$. All these bounds are best possible up to constant factors. These results are proved in Section 2.

- For complements of planar graphs 2-subcoloring is polynomially solvable (Section 4) whereas 3-subcoloring is NP-hard (Section 3).
- For AT-free graphs, $r$-subcoloring is NP-hard for any fixed $r \geq 3$ (Section 3). For comparability graphs, $r$-subcoloring is NP-hard for any fixed $r \geq 2$ (Section 5.1).
- For interval graphs (Section 5.2) and for permutation graphs (Section 5.3) $r$-subcoloring is polynomially solvable for any fixed $r \geq 2$.

Figure 1 summarizes some of our results and illustrates the relations between some of the graph classes studied in this paper. Appendix A contains the definitions of these graph classes.

# 2 Upper and lower bound results

In this section we derive several bounds on the subchromatic number of graphs in terms of their number of vertices. We first state two useful results from the literature.

**Proposition 2.1.** *(Albertson, Jamison, Hedetniemi & Locke [2])*
*For any graph $G$ on $n$ vertices, $\chi_{sub}(G) \leq 2n/(\log_2 n - 2) + O(n/(\log_2 n)^2)$.*

**Proposition 2.2.** *(Erdős, Gimbel & Kratsch [6])*
*Every* perfect *graph $G$ on $\binom{k+2}{2} - 1$ vertices has cochromatic number at most $k$. Therefore, $\chi_{co}(G) \leq \lfloor \sqrt{2n + 1/4} - 1/2 \rfloor$.*

As our first result, we observe that up to constant factors the upper bound stated in Proposition 2.1 is best possible.

**Lemma 2.3.** *For every $n$, there exists a graph $G$ on $n$ vertices with $\chi_{sub}(G) \geq n/(2\log_2 n + 1)$.*

*Proof.* We slightly modify the famous argument of Erdős [5]. Consider the random graph on $n$ vertices that contains every edge independently with probability $1/2$.

A subset $X$ of $k = 2\log_2 n + 1$ vertices is called *good*, if it induces a disjoint union of cliques, and thus constitutes a feasible color class for a subcoloring. Let us estimate the probability that some fixed set $X$ is good. Altogether, there are $2^{\binom{k}{2}}$ possibilities for the edges in $X$. Out of these exactly $B_k$ are good, where $B_k$ denotes the $k$th Bell number that is the number of ways a set of $n$ elements can be partitioned into nonempty subsets. The crude upper bound $B_k \leq k!$ yields that the probability that $X$ is good is at most $k!/2^{\binom{k}{2}}$.

Therefore, the expected total number of good subsets of cardinality $k$ is at most $\binom{n}{k} \cdot k!/2^{\binom{k}{2}}$. Since $\binom{n}{k} \leq n^k/k!$ and since $k = 2\log_2 n + 1$, a straightforward calculation reveals that this expected number is strictly less than 1. Hence, there exists a graph $G$ in the probability space that does not contain any good subset. In any subcoloring of $G$ all color classes contain fewer than $k$ vertices, and thus $\chi_{sub}(G) > n/k$. □

For perfect graphs, the subchromatic number is much smaller than $n/\log n$: Propositions 2.2 and 1.1 yield that for every perfect graph $G$ on $n$ vertices, $\chi_{sub}(G) \leq \chi_{co}(G) \leq \lfloor\sqrt{2n+1/4}-1/2\rfloor$. Erdős, Gimbel & Kratsch [6] observed that the disjoint union of cliques $H = K_1 \dot{\cup} K_2 \dot{\cup} \cdots \dot{\cup} K_k$ (this is a graph on $\binom{k+1}{2}$ vertices) has cochromatic number $k$. In every subcoloring of $\overline{H}$ every color class is either a clique, or an independent set. Thus $\chi_{sub}(\overline{H}) = \chi_{co}(\overline{H}) = \chi_{co}(H) = k$. Since $\overline{H}$ is a perfect graph (in fact, it is even a cograph and a permutation graph), we get that up to additive constant the bound $\sqrt{2n}$ is the best possible upper bound for the subchromatic number of perfect graphs.

In the rest of this section, we will discuss interval graphs. We will show that for interval graphs the subchromatic number is bounded by $O(\log n)$.

**Lemma 2.4.** *For every interval graph $G$ on $n$ vertices, $\chi_{sub}(G) \leq \lfloor\log_2(n+1)\rfloor$. This bound is best possible.*

*Proof.* For the upper bound we use induction on $n$. The statement is clearly true for $n = 1$. Consider an interval representation of an interval graph $G$ on $n$ vertices; without loss of generality we assume that the left endpoints of the intervals are the integers $1, 2, \ldots, n$. If $n$ is odd, we take an arbitrary maximal clique $C$ that contains the interval with left endpoint $(n+1)/2$. Then every component of $G - C$ contains at most $(n-1)/2$ vertices. We color $C$ by one color, and we use $\lfloor\log_2((n+1)/2)\rfloor$ additional colors to color all these components inductively. If $n$ is even, a similar analysis goes through.

For showing that the bound $\lfloor\log_2(n+1)\rfloor$ is best possible, we consider the following graphs $G_i$: For $k = 1$, the graph $G_1$ consists of one vertex.

For $k > 1$, we set $G_k = G_{k-1} \bowtie G_{k-1}$ where the new vertex is called $v$. Note that $G_k$ has $2^k - 1$ vertices, and that by Proposition 1.3 $\chi_{sub}(G_k) = k$. Moreover, $G_k$ is an interval graph; its interval representation can be obtained by putting two disjoint interval representations of $G_{k-1}$ next to each other, and by adding one long interval that corresponds to the vertex $v$. $\square$

We mention without proof that a similar inductive argument yields $\chi_{sub}(G) = O(\log n)$ for any *chordal* graph $G$. Albertson, Jamison, Hedetniemi & Locke [2] observed that the interval graphs $G_k$ in the proof of Lemma 2.4 form a class of interval graphs with unbounded subchromatic number. We now present a stronger result on the coloring of interval graphs with forbidden subgraphs.

**Lemma 2.5.** *For any $m$ and $r$, there exists an interval graph $\mathcal{I}(m, r)$ that does not have a $P_m$-free $r$-coloring.*

*Proof.* Let $N = R(K_{m+1}; r)$ be the Ramsey number that specifies the smallest number of vertices in a complete graph such that every $r$-coloring of the edges of this graph induces a monochromatic clique $K_{m+1}$ (that is, a clique in which all edges have the same color). Let $K_N$ be the complete graph on vertex set $\{1, 2, \ldots, N\}$. Let $\mathcal{I}_N$ be the intersection graph of all closed intervals with integer endpoints from $\{1, 2, \ldots, N\}$. Every interval $[a, b]$ in $\mathcal{I}_N$ naturally corresponds to the edge $\{a, b\}$ in $K_N$.

Now consider an arbitrary $r$-coloring of the intervals in $\mathcal{I}_N$. This induces a corresponding $r$-coloring of the edges in $K_N$, and hence there exists a monochromatic clique $K_{m+1}$ with vertex set $X$ with $|X| = m + 1$. In $\mathcal{I}_N$, the intervals with both endpoints in $X$ also form a monochromatic set $\mathcal{I}_X$. Since $\mathcal{I}_X$ contains an induced path $P_m$, every $r$-coloring of $\mathcal{I}_N$ contains an induced monochromatic path $P_m$. $\square$

# 3 Negative results: AT-free graphs

In this section we derive a generic NP-hardness result. As corollaries to this result, we derive the NP-hardness of 3-subcoloring for graphs with independence number two (and hence for AT-free graphs).

For an integer $p \geq 1$ and a graph $G$, we denote by $pG$ the disjoint union of $p$ copies of $G$. We start with an auxiliary lemma.

**Lemma 3.1.** *For any graph $G$ and for any integer $p \geq \chi(G)$, the chromatic number $\chi(G)$ of $G$ coincides with the subchromatic number $\chi_{sub}(\overline{pG})$ of the graph $\overline{pG}$.*

*Proof.* It is trivial to see $\chi_{sub}(\overline{pG}) \leq \chi(G)$: Any independent set $I$ in the graph $G$ translates into a clique with $p|I|$ vertices in the graph $\overline{pG}$. Hence, any color class in a proper coloring of $G$ corresponds to a feasible color class in a subcoloring of $\overline{pG}$.

On the other hand, we claim $\chi(G) \leq \chi_{sub}(\overline{pG})$. This statement is trivial if $\chi(G) = 1$, and from now on we assume that $G$ contains at least one edge. Let $G_1, G_2, \ldots, G_p$ denote the $p$ disjoint copies of $G$ in the graph $pG$. Consider any coloring of $pG$ in $k < \chi(G)$ colors. Then every subgraph $G_i$ contains a monochromatic edge. Since $k < p$, there are two subgraphs $G_i$ and $G_j$ with $i \neq j$ that both contain a monochromatic edge of the same color. In $\overline{pG}$, the endpoints of these two monochromatic edges induce a monochromatic $C_4$, and thus $\overline{pG}$ contains a monochromatic $P_3$. $\square$

The following theorem is the main result of this section. It is an immediate consequence of Lemma 3.1.

**Theorem 3.2.** *Let $\mathcal{G}$ be a graph class that is closed under taking disjoint unions (that is, $G, H \in \mathcal{G}$ implies $G \dot{\cup} H \in \mathcal{G}$). Let $r$ be an integer.*

*If the proper $r$-coloring problem is NP-hard for graphs from $\mathcal{G}$, then the $r$-subcoloring problem is NP-hard for graphs from $\overline{\mathcal{G}}$.*

**Corollary 3.3.** *The 3-subcoloring problem is NP-hard even when restricted to*

*(a) graphs with independence number at most two,*

*(b) AT-free graphs,*

*(c) complements of planar graphs.*

*Proof.* Maffray & Preissmann [15] proved that proper 3-coloring is NP-hard even for triangle-free graphs. The class of triangle-free graphs is closed under taking disjoint unions, and a graph is triangle-free if and only if its complement has independence number at most two. With this, (a) follows from Theorem 3.2.

Since the graphs with independence number at most two form a subclass of the AT-free graphs, (b) is a consequence of (a). Finally, the class of planar graphs is closed under taking disjoint unions, and it is well-known that proper 3-coloring of planar graphs is an NP-hard problem. Thus, Theorem 3.2 implies (c). □

We conclude this section with some consequences of Lemma 3.1 on the hardness of approximation of the subcoloring problem for general graphs. We rely on the results of Feige & Kilian [7] on the hardness of approximating the chromatic number of a graph: For any $\varepsilon > 0$, the chromatic number of $n$-vertex graphs cannot be approximated within a factor of $n^{1-\varepsilon}$, unless NP $\subseteq$ ZPP.

**Corollary 3.4.** *For any $\varepsilon > 0$, the subchromatic number of $n$-vertex graphs cannot be approximated within a factor of $n^{1/2-\varepsilon}$, unless* NP $\subseteq$ ZPP.

*Proof.* Let $G$ be an arbitrary graph on $n$ vertices. Then the graph $\overline{nG}$ has $n^2$ vertices, and by Lemma 3.1 we have $\chi(G) = \chi_{sub}(\overline{nG})$. Now the result of Feige & Kilian [7] completes the argument. □

# 4 Positive results: Complements of planar graphs

Gimbel & Nešetřil [10] showed that deciding $P_4$-free 2-colorability of a planar graph is NP-hard. Since the complement of $P_4$ is again $P_4$, this implies that $P_4$-free 2-subcolorability is NP-hard for complements of planar graphs, too. Fiala, Jansen, Le & Seidel [8] showed that deciding 2-subcolorability of a planar graph is NP-hard. Surprisingly, we will show in this section that 2-subcolorability is polynomially solvable for complements of planar graphs. This will follow as a corollary from the main theorem of this section.

**Theorem 4.1.** *Let $\ell \geq 2$, and let $\mathcal{G}$ be a class of graphs that do not contain $K_{\ell,\ell}$ as an (induced or non-induced) subgraph. Then 2-subcolorability of a graph $G = (V, E)$ in $\overline{\mathcal{G}}$ can be decided in polynomial time $O(|V|^{3\ell})$.*

Before giving the proof of Theorem 4.1, we state two auxiliary lemmas.

**Lemma 4.2.** *Let $H_i$ $(1 \leq i \leq m)$ be graphs with $|H_i| \leq \ell - 1$ such that the join $G = H_1 \vee H_2 \vee \cdots \vee H_m$ does not contain $K_{\ell,\ell}$ as an (induced or non-induced) subgraph. Then $|H_1| + |H_2| + \cdots + |H_m| \leq 3\ell - 3$.*

*Proof.* Suppose otherwise, and consider a counterexample with $|H_1|+|H_2|+\cdots+|H_m| \geq 3\ell-2$. Let $k$ be the smallest index for which $q := |H_1|+|H_2|+$

$\cdots + |H_k| \geq \ell$. Since $|H_k| \leq \ell - 1$ and since $|H_1| + |H_2| + \cdots + |H_{k-1}| \leq \ell - 1$ by the definition of $k$, we conclude that $q \leq 2\ell - 2$. Therefore $p := |H_{k+1}| + \cdots + |H_m| \geq \ell$ holds. But then the $q \geq \ell$ vertices in $H_1, \ldots, H_k$ on one side and the $p \geq \ell$ vertices in $H_{k+1}, \ldots, H_m$ on the other side span a $K_{\ell,\ell}$, the desired contradiction. □

**Lemma 4.3.** *Let $G$ be a graph that does not contain $K_{\ell,\ell}$ as an (induced or non-induced) subgraph. Then for every $\overline{P_3}$-free coloring of $G$, each color class $C$ is of one of the following two types:*

*(A) the join $Q \vee I$ of a graph $Q$ with at most $\ell - 1$ vertices and an independent set $I$;*

*(B) a graph $R$ with at most $3\ell - 3$ vertices.*

*Proof.* We use the fact that every $\overline{P_3}$-free graph is also $P_4$-free, and hence a cograph. We consider the cograph $G[C]$ induced by the color class $C$.

If $G[C]$ is not connected, then every connected component of this $\overline{P_3}$-free graph must consist of a single vertex. Hence, we are in case (A) with $Q = \emptyset$. In the remaining cases, $G[C]$ is a connected cograph and hence can be presented as the join $H_1 \vee H_2$ of two cographs $H_1$ and $H_2$ with $|H_1| \leq |H_2|$. Clearly, $|H_1| \leq \ell - 1$ since $G[C]$ does not contain $K_{\ell,\ell}$. If $H_2$ is not connected, then its vertex set induces an independent set. If $H_2$ is connected, then it again can be written as the join of two cographs. By repeating this procedure over and over again, we eventually find that $G[C]$ can be written as the join $H_1 \vee H_2 \vee \cdots \vee H_m$, where the graphs $H_1, \ldots, H_{m-1}$ all have at most $\ell - 1$ vertices, and where $H_m$ is either a graph with at most $\ell - 1$ vertices or an independent set.

If $|H_m| \leq \ell - 1$, then we get from Lemma 4.2 that $|C| \leq 3\ell - 3$, and we are in case (B). If $|H_m| \geq \ell$, then $H_m$ is an independent set. Moreover, $|H_1| + |H_2| + \cdots + |H_{m-1}| \leq \ell - 1$, since $G[C]$ does not contain $K_{\ell,\ell}$ as a subgraph. Hence, we are in case (A) with $I = H_m$ and $Q = H_1 \vee H_2 \vee \cdots \vee H_{m-1}$. □

*Proof of Theorem 4.1.* For the ease of exposition, we will deal with $\overline{P_3}$-free colorings of graphs from the class $\mathcal{G}$, instead of $P_3$-free colorings of graphs from the class $\overline{\mathcal{G}}$. Clearly, these two problems are equivalent. Now let $G = (V, E)$ be a graph from $\mathcal{G}$.

According to Lemma 4.3, in any $\overline{P_3}$-free 2-coloring of $G$ both color classes must be of type (A) or (B). This yields the three possible cases (AA), (AB), and (BB). The two cases (AB) and (BB) involve a color class of type (B). They can be solved simultaneously by enumerating all $O(|V|^{3\ell-3})$ candidate sets for $R$. In any feasible solution, the graphs $G[R]$ and $G[V-R]$ both are $\overline{P_3}$-free, and this can be checked in $O(|V|^3)$ time.

We are left with the case (AA) where both color classes are of type (A). In this case we are looking for a partition of the vertex set $V$ into four sets $Q_1$, $Q_2$, $I_1$, $I_2$ such that for $i = 1, 2$

- $0 \leq |Q_i| \leq \ell - 1$ holds,
- $G[Q_i]$ is a $\overline{P_3}$-free graph,
- $G[I_i]$ is an independent set,
- $G[Q_i] \vee G[I_i]$ is a subgraph of $G$.

Note that if $G[Q_i]$ is a $\overline{P_3}$-free graph and if $G[I_i]$ is an independent set, then also $G[Q_i] \vee G[I_i]$ is a $\overline{P_3}$-free graph.

We enumerate all $O(|V|^{2\ell-2})$ possible pairs of candidate sets for $Q_1$ and $Q_2$. For each such pair, we check whether $G[Q_1]$ and $G[Q_2]$ are $\overline{P_3}$-free. If they are not, then we move on to the next pair. If they are, then we try to split the remaining vertices in $Z := V - Q_1 - Q_2$ into the independent sets $I_1$ and $I_2$. A vertex $z \in Z$ is called *good for* $I_i$ if it is adjacent to all vertices in $Q_i$. In any feasible solution, $I_i$ only contains vertices that are good for it.

If there is a vertex $z \in Z$ that is neither good for $I_1$ nor for $I_2$, then we stop and move on to the next pair. Whenever a vertex $z \in Z$ is good for $I_i$ and not good for the other independent set $I_{1-i}$, then we precolor $z$ with color $i$. Then we check whether this precoloring can be extended to a proper 2-coloring of $G[Z]$; this can be done by standard methods in $O(|V|+|E|)$ time. If the precoloring can not be extended, then we move on to the next pair. If it can be extended, then the two proper color classes $I_1$ and $I_2$ together with $Q_1$ and $Q_2$ constitute a solution to the $\overline{P_3}$-free coloring problem on $G$. □

Since planar graphs do not contain $K_{3,3}$ as a subgraph, we have the following corollary to Theorem 4.1. Corollaries 3.3.(c) and 4.4 together provide a complete classification of the complexity of subcolorings of complements of planar graphs.

**Corollary 4.4.** *The 2-subcoloring problem on complements of planar graphs is polynomially solvable.*

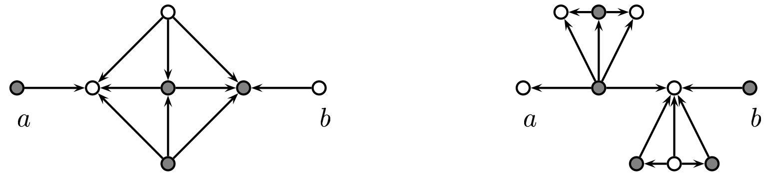


Figure 2: The gadget graphs *Source-Source* and *Source-Sink*.

# 5 Perfect graphs

In this section, we discuss three classes of perfect graphs. In Section 5.1 we prove NP-hardness of $r$-subcolorability for every fixed $r \geq 2$ on comparability graphs. In Sections 5.2 and 5.3, we give polynomial time dynamic programming algorithms for $r$-subcolorability for every fixed $r \geq 2$, on interval graphs and on permutation graphs, respectively.

## 5.1 Comparability graphs

In this section we prove the NP-hardness of $r$-subcolorings on comparability graphs for every fixed $r \geq 2$. Our NP-hardness reduction is based on the two graphs *Source-Source* (depicted to the left) and *Source-Sink* (depicted to the right) in Figure 2. Both graphs have two contact vertices $a$ and $b$.

**Lemma 5.1.** *(a) The graphs* Source-Source *and* Source-Sink *are comparability graphs.*

*(b) The graphs* Source-Source *and* Source-Sink *possess a* 2*-subcoloring, in which no contact point receives the same color as its neighbor.*

*(c) In every* 2*-subcoloring of* Source-Source *and* Source-Sink*, the contact vertices $a$ and $b$ must receive different colors.*

*Proof.* Proof of (a). The orientations depicted in Figure 2 are transitive. Proof of (b). The 2-colorings depicted in Figure 2 are subcolorings. Proof of (c). By checking all possible cases. □

Statements (b) and (c) in Lemma 5.1 are extremely useful for our NP-hardness proofs: Consider a graph $G$, and let $x, y$ be a pair of vertices in $G$. Let the graph $G^+$ result from $G$ by adding an independent copy of a *Source-Source* or a *Source-Sink* gadget to $G$, and by identifying vertex $x$

with contact point $a$, and vertex $y$ with contact point $b$. Then the graph $G^+$ has a 2-subcoloring, if and only if $G$ has a 2-subcoloring in which $x$ and $y$ receive different colors.

**Theorem 5.2.** *The* 2*-subcoloring problem is NP-hard for comparability graphs.*

*Proof.* The proof is by a reduction from the NP-complete SET SPLITTING problem [14]: Given a finite set $S$ and a collection $C$ of triples over $S$, decide whether there is a partition of $S$ into two subsets $S_1$ and $S_2$ such that every triple in $C$ has a non-empty intersection with $S_1$ and with $S_2$.

Now let $C = \{c_1, c_2, \ldots, c_m\}$ be a collection of triples over a finite set $S = \{s_1, s_2, \ldots, s_n\}$. We construct the following graph $G_C$ from $C$: For every $s_j \in S$, there is a corresponding vertex $x_j \in X$. For every triple $c_i = (s_i^1, s_i^2, s_i^3) \in C$, there are three corresponding vertices $y_i^1, y_i^2, y_i^3$ that form a $P_3$; there is an edge between $y_i^1$ and $y_i^2$, and there is an edge between $y_i^2$ and $y_i^3$. Vertex $y_i^2$ is called the *middle vertex* of this path, and vertices $y_i^1$ and $y_i^3$ are called the *end vertices*. Moreover, we introduce the following copies of the *Source-Source* and *Source-Sink* gadgets:

- For every occurrence of $s_j$ in the *first* or *third* position of some triple $c_i = (s_i^1, s_i^2, s_i^3)$, the graph $G_C$ contains a copy of the *Source-Source* gadget; the contact points are identified with vertex $x_j$, and with vertex $y_i^1$ (first position) or $y_i^3$ (third position), respectively.
- For every occurrence of $s_j$ in the *second* position of some triple $c_i = (s_i^1, s_i^2, s_i^3)$, the graph $G_C$ contains a copy of the *Source-Sink* gadget; the contact point $b$ is identified with vertex $x_j$, and the contact point $a$ is identified with vertex $y_i^2$.

This completes the definition of the graph $G_C$. We argue that $G_C$ is a comparability graph by considering the following orientation: All *Source-Sink* and *Source-Source* gadgets are oriented as shown in Figure 2. The edges of the paths $y_i^1, y_i^2, y_i^3$ are directed towards the middle vertices $y_i^2$. In the resulting orientation, all vertices $x_j \in X$ and all end vertices of paths are sources, and all middle vertices of paths are sinks. Hence, arcs incident to these vertices can not violate transitivity, and the remaining arcs are within the gadgets.

We claim that $G_C$ has a 2-subcoloring if and only if the corresponding instance of SET SPLITTING has answer YES.

Assume that $G_C$ has a 2-subcoloring. We construct the following set splitting: If $x_j$ is colored 1, then $s_j \in S_1$. If $x_j$ is colored 2, then $s_j \in S_2$. Consider a triple $c_i = (s_i^1, s_i^2, s_i^3)$ in $C$. If it is contained in $S_1$ or $S_2$, then the three vertices that correspond to $s_i^1, s_i^2, s_i^3$ must all have the same color, and the three vertices $y_i^1, y_i^2, y_i^3$ on the $P_3$ corresponding to $C_i$ must all have the opposite color. But then this $P_3$ would be monochromatic, and the coloring would not be a subcoloring.

Next assume that $C$ possesses a set splitting of $S$ into $S_1$ and $S_2$. We construct the following coloring: If $s_j \in S_1$, then we color vertex $x_j$ by 1. If $s_j \in S_2$, then we color vertex $x_j$ by 2. Then we extend this coloring to the *Source-Sink* and *Source-Source* gadgets according to Figure 2. It is easily checked that the resulting coloring is a 2-subcoloring of $G_C$. This completes the proof of the theorem. □

The NP-hardness result on 2-subcoloring for comparability graphs can easily be generalized to $r$-subcolorings with $r \geq 3$.

**Theorem 5.3.** *For any $r \geq 2$, the $r$-subcoloring problem on comparability graphs is NP-hard.*

*Proof.* We proceed by induction on $r$. The starting case $r = 2$ has been settled in Theorem 5.2. So assume that we have proved the statement up to $r$, and that we want to prove it for $r + 1$. This will be done as follows: For every comparability graph $G_r$, we construct in polynomial time a comparability graph $G_{r+1}$ such that $G_r$ is $r$-subcolorable if and only if $G_{r+1}$ is $(r + 1)$-subcolorable.

Let $K = K_{r+1,\ldots,r+1}$ be the complete $(r + 1)$-partite graph containing $r+1$ classes of $r+1$ vertices. Recall that by Proposition 1.2, $\chi_{sub}(K) = r+1$. We put $G_{r+1} = G_r \bowtie K$ where the new vertex is called $v$. Observe that $G_{r+1}$ is a comparability graph: $G_r$ and $K$ are comparability graphs; we take their transitive orientations, and we orient all edges that are incident with the new vertex $v$ away from $v$.

Assume that $G_{r+1}$ is subcolorable in $r + 1$ colors. By Proposition 1.2 the vertices of the graph $K$ must use all $r + 1$ colors; in particular, the color $c$ of the new vertex $v$ is used in $K$. But this implies that color $c$ cannot be used for the vertices of $G_r$, since this would yield a monochromatic $P_3$ in color $c$. Hence, the graph $G_r$ is subcolorable in $r$ colors.

Now assume that $G_r$ is subcolorable in $r$ colors. Take this $r$-subcoloring, and color the new vertex $v$ by a new color. Color $K$ by $r + 1$ colors in such a way that every independent class of $r + 1$ vertices receives all $r + 1$ colors;

in other words, every color class induces a clique of size $r+1$ in $K$. The resulting $(r+1)$-coloring of $G_{r+1}$ is a subcoloring. $\square$

## 5.2 Interval graphs

In this section we design for every fixed $r \geq 2$ a polynomial time algorithm for the $r$-subcoloring problem on interval graphs. Let $G = (V, E)$ be an interval graph with $|V| = n$. Without loss of generality we may assume that the left endpoints of the intervals $I_1, \ldots, I_n$ that represent $G$ are the integers $1, 2, \ldots, n$. For $k = 1, \ldots, n$ we denote by $G_k$ the subgraph that is induced by the first $k$ intervals $I_1, \ldots, I_k$. For a clique $Cl$ in $G$, we denote by $\text{inter}(Cl) \neq \emptyset$ the intersection of all intervals in $Cl$ and by $\text{union}(Cl)$ the union of all these intervals. Note that $\text{inter}(Cl)$ and $\text{union}(Cl)$ are also intervals.

Consider an arbitrary color class $C$ in an arbitrary $r$-subcoloring of $G_k$. This color class $C$ is the union of a number $q$ of disjoint cliques $Cl_1, \ldots, Cl_q$; without loss of generality we assume that $\text{union}(Cl_i)$ always lies completely to the left of $\text{union}(Cl_{i+1})$. Now assume that we would like to extend the subcoloring to the graph $G_{k+1}$ by adding interval $I_{k+1}$ (with left endpoint $k+1$) to color class $C$. There are only two possibilities for doing this:

(a) If the point $k+1$ lies to the right of $\text{union}(Cl_q)$, then interval $I_{k+1}$ may start a new clique in $C$.

(b) If the point $k+1$ lies within $\text{inter}(Cl_q)$, then interval $I_{k+1}$ may be added to the rightmost clique $Cl_q$ in $C$.

Note furthermore that the *left* endpoints of $\text{inter}(Cl_q)$ and $\text{union}(Cl_q)$ do not exceed $k$. Hence, for deciding whether case (a) or case (b) holds, we only need to know the *right* endpoints of $\text{inter}(Cl_q)$ and $\text{union}(Cl_q)$. These observations suggest the following dynamic programming formulation.

Every state is specified by a $(2r+1)$-tuple $[k; i_1, i_2, \ldots, i_r; u_1, \ldots, u_r]$. Here $1 \leq k \leq n$, and the variables $i_1, \ldots, i_r$ and $u_1, \ldots, u_r$ either specify right endpoints of some of the intervals $I_1, \ldots, I_k$, or they take the dummy value '$*$'. Hence, altogether there are $O(n^{2r+1})$ states. For every state, we compute a Boolean value $B[k; i_1, \ldots, i_r; u_1, \ldots, u_r]$. This Boolean value is TRUE, if and only if there exists a subcoloring of $G_k$ with color classes $C_1, \ldots, C_r$ with the following properties for $j = 1, \ldots, r$: If $C_j$ is empty, then $i_j = u_j = *$. And if $C_j$ is non-empty, then $i_j$ is the right endpoint of inter($Cl$) and $u_j$ is the right endpoint of union($Cl$) of the rightmost clique $Cl$ in color class $C_j$.

The values $B[k; i_1, \ldots, i_r; u_1, \ldots, u_r]$ are computed first for level $k = 1$, then for level $k = 2$, and so on up to level $k = n$. Since in any subcoloring for $G_k$ the interval $I_{k+1}$ can be added in at most two possible ways (a) and (b) to at most $r$ color classes, every TRUE value at level $k$ generates at most $2r$ TRUE values at level $k + 1$. The graph $G$ is $r$-subcolorable, if and only if there exists a TRUE value at level $n$. Summarizing, we get the following theorem.

**Theorem 5.4.** *For any fixed $r$, the $r$-subcoloring problem for an interval graph with $n$ vertices can be solved in $O(r \cdot n^{2r+1})$ time.*

## 5.3 Permutation graphs

In this section we design for every fixed $r \geq 2$ a polynomial time algorithm for the $r$-subcoloring problem on permutation graphs. We use a dynamic programming approach that is quite similar to that designed for interval graphs in Section 5.2.

Let $\sigma = \langle \sigma(1), \sigma(2), \ldots, \sigma(n) \rangle$ be a permutation, and let $G = G[\sigma]$ be the associated permutation graph; see Figure 3 for an illustration. The vertices in $G$ are denoted by $1, \ldots, n$ as in the ordering on the upper line of the permutation diagram. By $G_k$ we denote the subgraph that is induced by the first $k$ vertices $1, \ldots, k$. For any clique $Cl$ in $G$ with vertices $x_1 < x_2 < \cdots < x_\ell$ ordered along the upper line, these numbers $x_i$ must show up in the reverse ordering $x_\ell, x_{\ell-1}, \ldots, x_2, x_1$ along the lower line. For a clique $Cl$, we denote by min($Cl$) and by max($Cl$) its smallest and its largest vertex, respectively.

Consider an arbitrary color class $C$ in an arbitrary $r$-subcoloring of $G_k$. This color class $C$ is the union of a number $q$ of disjoint cliques $Cl_1, \ldots, Cl_q$; without loss of generality we assume that max($Cl_i$) is always strictly smaller than min($Cl_{i+1}$). Now assume that we would like to extend the subcoloring to the graph $G_{k+1}$ by adding vertex $k + 1$ to color class $C$. There are only two possibilities for doing this:

(a) If on the lower line, the number $k + 1$ lies to the right of min($Cl_q$), then vertex $k + 1$ may start a new clique in $C$.

(b) If on the lower line, the number $k + 1$ lies to the left of max($Cl_q$) and min($Cl_{q-1}$) doesn't lie to the right of $k + 1$, then vertex $k + 1$ may be added to the rightmost clique $Cl_q$ in $C$.

Hence, for deciding whether case (a) or case (b) holds, we only need to know $\min(Cl_q)$, $\max(Cl_q)$ and $\min(Cl_{q-1})$, where $q$ is the index of the rightmost clique in $C$. This leads to the following dynamic programming formulation.

Every state is specified by a $(3r + 1)$-tuple $[k; \min'_1, \ldots, \min'_r; \min_1, \ldots, \min_r; \max_1, \ldots, \max_r]$. Here $1 \leq k \leq n$, and the variables $\min'_1, \ldots, \min'_r$, $\min_1, \ldots, \min_r$ and $\max_1, \ldots, \max_r$ either specify vertices, or take the dummy value '$*$'. Hence, altogether there are $O(n^{3r+1})$ states. For every state, we compute a Boolean value $B[k; \min'_1, \ldots, \min'_r; \min_1, \ldots, \min_r; \max_1, \ldots, \max_r]$. This Boolean value is TRUE, if and only if there exists a subcoloring of $G_k$ with color classes $C_1, \ldots, C_r$ with the following properties for $j = 1, \ldots, r$: If $C_j$ is empty, then $\min'_j = \min_j = \max_j = *$. If $C_j$ consists from one clique then $\min'_j = *$. And if $C_j$ is non-empty, then $\min_j$ is the smallest vertex and $\max_j$ is the largest vertex of the rightmost clique $Cl$ in color class $C_j$. If $C_j$ contains more than one clique then $\min'_j$ is the smallest vertex of the rightmost clique $Cl'$ in $C_j - Cl$.

The values $B[k; \min'_1, \ldots, \min'_r; \min_1, \ldots, \min_r; \max_1, \ldots, \max_r]$ are computed first for level $k = 1$, then for level $k = 2$, and so on up to level $k = n$. Since in any subcoloring for $G_k$ the vertex $k + 1$ can be added in at most $2r$ possible ways, every TRUE value at level $k$ generates at most $2r$ TRUE values at level $k + 1$. The graph $G$ is $r$-subcolorable, if and only if there exists a TRUE value at level $n$. Summarizing, we get the following theorem.

**Theorem 5.5.** *For any fixed $r$, the $r$-subcoloring problem for a permutation graph with $n$ vertices can be solved in $O(r \cdot n^{3r+1})$ time.*

# 6 Concluding remarks and questions

- What is the computational complexity of $r$-subcoloring for cocomparability graphs?
- What is the computational complexity of $r$-subcoloring for chordal graphs?
- What is the computational complexity of 2-subcoloring for AT-free graphs? (In Section 3, we have proved that 3-subcoloring of AT-free graphs is NP-hard).

- What is the computational complexity of $r$-subcoloring for interval graphs and permutation graphs, if $r$ is part of the input? (In Section 5, we have proved that these problems are polynomially solvable if $r$ is fixed and not part of the input).

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# A Graph classes

In this appendix, we collect definitions and simple properties of the graph classes as considered in this paper. We refer the reader to the book of

Golumbic [11] for broader information on perfect graphs, and to the book by Brandstädt, Le & Spinrad [3] for additional references.

**Perfect graphs.** A graph $G$ is *perfect* if for every induced subgraph $G'$ of $G$, the chromatic number $\chi(G')$ equals the size $\omega(G')$ of the largest clique.

**Comparability graphs.** An orientation $H = (V, D)$ of an undirected graph $G = (V, E)$ assigns one of the two possible directions to each edge $e \in E$. The orientation is *transitive* if $(a, b) \in D$ and $(b, c) \in D$ implies $(a, c) \in D$. A graph $G = (V, E)$ is a *comparability graph* if there exists a transitive orientation $H = (V, D)$ of $G$.

**Cocomparability graphs.** A graph is a *cocomparability graph* if it is the complement of a comparability graph.

**Permutation graphs.** Let $\sigma$ be a permutation of the set $\{1, 2, \dots, n\}$. We think of $\sigma$ as the sequence $\sigma = \langle\sigma(1), \sigma(2), \dots, \sigma(n)\rangle$, and we associate with it the following permutation diagram: Draw two parallel horizontal lines. Write the numbers $1, 2, \dots, n$ from left to right in the upper line. Write the numbers $\sigma(1), \sigma(2), \dots, \sigma(n)$ from left to right in the lower line. For each $i \in \{1, 2, \dots, n\}$, draw the straight segment from $i$ in the upper line to $i$ in the lower line. See Figure 3 for an illustration.

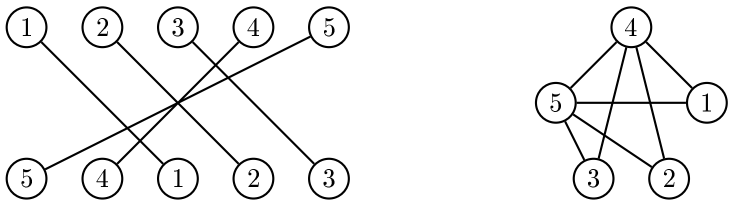


Figure 3: The permutation diagram and the graph $G[\sigma]$ for $\sigma = (5, 4, 1, 2, 3)$.

The *inversion graph* $G[\sigma] = (V, E)$ associated with $\sigma$ has vertex set $V = \{1, 2, \dots, n\}$. There is an edge between $i$ and $j$ if and only if in the permutation diagram the two line segments $[i, \sigma(i)]$ and $[j, \sigma(j)]$ intersect.

Equivalently we may say that there is an edge between $i$ and $j$ if and only if $(i-j)(\sigma^{-1}(i)-\sigma^{-1}(j))<0$ holds.

An undirected graph $G$ is a *permutation graph* if it is isomorphic to $G[\sigma]$ for some permutation $\sigma$. The permutation graphs form a subclass of the comparability graphs. Moreover, it is known that a graph $G$ is a permutation graph if and only if it is both, a comparability graph and a cocomparability graph.

**Cographs.** *Cographs* are the graphs that do not contain an induced $P_4$. An equivalent inductive definition of cographs is the following: A graph $G$ is a cograph, if (i) it consists of a single vertex, or if (ii) there are cographs $G_1$ and $G_2$ such that $G = G_1 \dot{\cup} G_2$, or if (iii) there are cographs $G_1$ and $G_2$ such that $G = G_1 \vee G_2$. The cographs form a subclass of the permutation graphs.

**Interval graphs.** A graph $G = V, E)$ is an *interval graph* if one can assign to each $v \in V$ an interval $I_v$ on the real line such that $(u, v) \in E$ if and only if $I_v \cap I_u \neq \emptyset$. Interval graphs form a subclass of the cocomparability graphs.

**AT-free graphs.** An independent set of three vertices is called an *asteroidal triple (AT)* if every two of them are connected by a path avoiding the neighborhood of the third one. A graph is *AT-free* if it does not contain an asteroidal triple. AT-free graphs were introduced to generalize interval graphs and cocomparability graphs. The class of AT-free graphs is *not* a subclass of the perfect graphs.