Separoids and characterization of linear uniform oriented matroids

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Abstract

In this paper the *geometric dimension* of an oriented matroid is introduced. It is the minimal euclidian dimension where its *separoid* (to be defined) can be realized as a family of convex sets. We show that in the *uniform* case, it is enough to know this invariant to decide if the oriented matroid is *linear*.

1 Introduction

"Oriented matroids can be thought of as a combinatorial abstraction of point configurations over the reals" –so reads the opening remark of Björner et al. (1993) [2] basic reference book. However, they are more general than that, and one of the basic problems in the area is to give meaningful characterizations of those oriented matroids that do arise from point configurations; they are called *linear* or *coordinatizable*. They can also be thought of, by polarity, as a combinatorial abstraction of configurations of oriented hyperplanes, or of oriented (codimension 1) subspheres arrangements on a sphere. From this point of view, the Topological Representation Theorem due to Folkman & Lawrence (1978) [8] settles that, all oriented matroids can be realized if the spheres are let to "wiggle" a bit, that is, if they are not asked to be geometrically flat but only that they keep the topological behavior of spheres, they are then called "pseudospheres arrangements".

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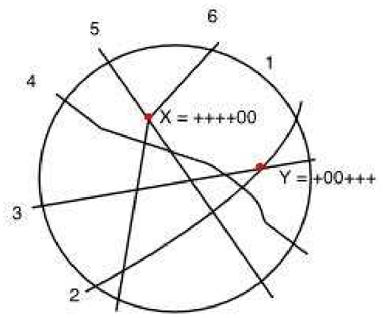

Figure 0. *An oriented matroid.*

Thinking again in terms of points, there should be an analogue of the extra freedom that comes from "wiggling" hyperplanes... it is "fattening" points to convex sets. This is, convex sets play the role of "pseudopoints". But then the natural combinatorial abstraction becomes more general — becomes a *separoid* [1,3,10,12,13,14,15].

Separoids are combinatorial objects that capture the structure arising from a family of convex sets, where some subfamilies are naturally separated from others. Namely, two subfamilies are said to be *separated* if there exists a hyperplane that leaves them on opposite sides of it —the axioms of a separoid are simply the obvious properties of this relation. Theorem 1 settles that all separoids can be represented this way, this is, given an abstract separoid we will construct a family of convex set in $\mathbb{E}^d$ whose separation structure by hyperplanes is that of the given separoid. This will aloud us to introduce the *geometric dimension* invariant as the minimum dimension where this realization can be done.

Separoids run into their first application in the context of *geometric transversal theory* in an attempt to answer the question: *how does the space of hyperplanes transversal to a family of convex sets in $\mathbb{E}^d$ looks like?* As already pointed out by Goodman, Polack & Wenger (1993) [9] the existence of a transversal hyperplane depends on the existence of a suitable oriented matroid. We found that *the space of all such hyperplanes is essential (as a subset of $\mathbb{P}^{d-1}$) if the geometric dimension of the underlying (acyclic) separoid is less than $d-1$* (cf. Arocha et al. 1999 [1]). Further research lead us to an equivalent version of the Basic Sphericity Theorem; the first step to reach the characterization of the *cocircuit graphs* for uniform oriented matroids (cf. Montellano & Strausz 2001 [10]).

In this paper, also in terms of the geometric dimension, those separoids that arises from a configuration of points in *general position* will be characterized: *a general position separaoid is a point separoid if and only if its combinatorial dimension and its geometric dimension are equal.*

We think on oriented matroids as separoids in orther to isolate those axioms from which the main theorem depends. In Section 2 the basic notions and examples of separoids are introduced. Section 3 proves that the geometric dimension is well defined to *all* —not necessarly acyclic— separoids (Theorem 1) and Section 4 is devoted to the characterization of uniform point separoids (Theorem 2). Its proof uses a couple of technical propositions which are left to the end (Section 6). In Section 5 a formal cryptomorphism (Proposition 1) is exhibited to show that the category of oriented matroids is embedded naturally in that of separoids and enounce the application of the two theorem to oriented matroids as Theorem 3.

2 Separoids

The theory of *separoids* forms a broad setting to describe those combinatorial properties that arise from *families of convex sets* and the separations they define. Mathematical objects which appear to be totally different, such as *configurations of points, arrangements of affine subspaces, directed and undirected graphs, oriented matroids, convex polytopes* and *separation axioms of topological spaces,* find a common generalization in the language of separoids.

A *separoid* $S = (X, |)$ over the base set $X \neq \phi$ is a relation $|\subseteq 2^X \times 2^X$ on the subsets of X with the following properties: if $A, B \subseteq X$, then

$$\begin{array}{cc} \circ & A \mid B \Longrightarrow B \mid A, \\ \circ\circ & A \mid A \Longrightarrow A = \phi, \\ \circ\circ\circ & A \mid B \text{ and } A' \subset A \Longrightarrow A' \mid B. \end{array}$$

So we say that a separoid is a *symmetric, quasi-antireflexive, ideal* relation on a family of subsets. The elements of $|$ are called *separations* and, when speaking of a separation $A \mid B$, it is said that "A is *separated from* B". A separoid is *acyclic* if the empty set is separated from the base one, i.e. if $\phi \mid X$. The separations with the empty set are called *trivial* separations and, in the sequel, the base set X is finite. Observe that it is enough to know *maximal separations* to reconstruct the separoid —they encode the whole information.

A pair of disjoint subsets A and B which are not separated are called a

Radon partition and denoted $A \dagger B$, this is,

$$A \dagger B \iff A \cap B = \phi \text{ and } A \not| B.$$

Each part, A and B, is known as a *(Radon) component* and the union $A \cup B$ will be called, following oriented matroid terminology, the *support* of the partition. A *minimal Radon partition* is a Radon partition $A \dagger B$ where each component is minimal by contention, i.e.

$$A' \subset A \Longrightarrow A' \mid B \quad \text{and} \quad B' \subset B \Longrightarrow A \mid B'.$$

Clearly, the set of all minimal Radon partitions encode the whole separoid and it will be denoted by MRP, i.e., $A \dagger B \in MRP$ means that $A \dagger B$ is a minimal Radon partition.

Now, let S and T be two separoids over the base sets X and Y respectively. A *separoid morphism* $S \longrightarrow T$ is a function $\varphi: X \to Y$ with the property that for all $A, B \subset Y$,

$$A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

A *separoids category* is defined with such morphisms between separoids. Two separoids are *isomorphic* if there exists a bijective morphism from one onto the other whose inverse function is also a morphism. Equivalently, S is isomorphic to T if there exists a bijective morphism $\varphi: S \longrightarrow T$ such that

$$A \dagger B \Longrightarrow \varphi^{-1}(A) \dagger \varphi^{-1}(B).$$

Examples:

1. Consider a subset $X \subset \mathbb{E}^d$ of the d-dimensional euclidian space and define the following relation

$$A \mid B \iff \langle A \rangle \cap \langle B \rangle = \phi,$$

where $\langle A \rangle$ denotes de convex hull of A. If X is finite, the pair $\mathcal{P} = (X, |)$ is an acyclic separoid and will be called a *point separoid.* In fact, the name of separoids arises as a generalization of the fact that $A \mid B$ is a non-trivial separation if and only if there exists a hyperplane strictly separating $\langle A \rangle$ from $\langle B \rangle$. Theorem 2 characterizes an important class of point separoids.

2. Consider a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ and define the separoid $S(\mathcal{F})$ as above, this is, two subsets of the family $A, B \subset \mathcal{F}$ are separated if there exists a hyperplane that leaves all members of A on one side of it and those of B on the other. If $\mathcal{F}$ is finite and the elements of $\mathcal{F}$ are compact, then $S(\mathcal{F}) = (\mathcal{F}, |)$ is an acyclic separoid and will be called a *separoid of convex bodies.* The Representation Theorem of Arocha et al. (1999) [1] proves that every finite acyclic separoid is isomorphic to a separoid of convex bodies. Theorem 1 is a generalization of this result for the non-acyclic case —and therefore for *open* convex sets.

3. Consider an oriented matroid $\mathcal{M} = (E, \mathcal{L})$ and identify it with the subset $\mathcal{L} \subseteq \{-, 0, +\}^E$ of its *covectors* in the usual manner. Let $\mathcal{T} = \mathcal{T}(\mathcal{L})$ be the set of *topes,* maximal covectors, and define the following relation $| \subseteq 2^E \times 2^E$ on the subsets of E: $A, B \subset E$ are separated, $A \mid B$, if and only if there exist a tope $T \in \mathcal{T}$ such that $A \subseteq T^+ := \{e \in E : T_e = +\}$, and $B \subseteq T^- := \{e \in E : T_e = -\}$. The pair $S(\mathcal{M}) = (E, |)$ is a separoid. In Section 5 this example will be studied in more detail, in particular it will be shown that the oriented matroid can be reconstructed from its separoid, and hence that separoids generalize oriented matroids (cf. Proposition 1).

4. Edelman (1984) [7] has defined a complex which encodes the separoid of an oriented matroid. He consider the set

$$\Gamma(\mathcal{T}) := \{X \in \{-, 0, +\}^E : X \leq T \text{ and } T \in \mathcal{T}\},$$

where $\mathcal{T}$ denotes the topes of an oriented matroid and $\leq$ denotes the *conformal relation*, i.e., $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. Clearly a signed vector $X \in \Gamma$ is in Edelman's complex if and only if $X^+ \mid X^-$. He uses the Basic Sphericity Theorem to prove that such a complex has the homotopy type of a sphere. Theorem 1 of Montellano & Strausz (2001) [10] is a direct consequence of this result —it is some how the dual version of it— and leads to the characterization of the *cocircuit graphs* of uniform oriented matroids.

5. All acyclic separoids on three elements arise from one of the eight families of convex bodies in Figure 1. Those labeled **a**, **b**, **e** and **h** are the point separoids of order 3; in fact, they come from the four essentially different oriented matroids with three elements.

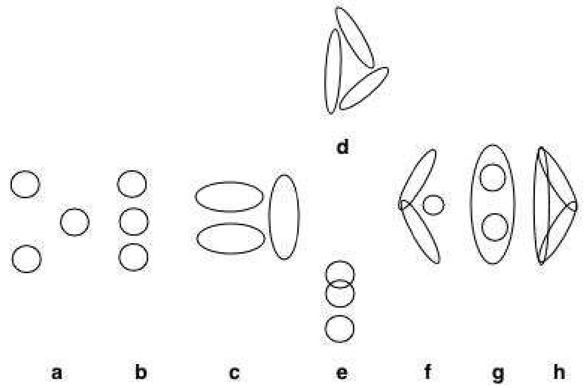

Figure 1. *The acyclic separoids of order 3*

6. Consider a family of convex sets $\mathcal{F}$, choose a point in each of its elements to construct a point separoid $\mathcal{P}$ and define the obvious bijection $\varphi: \mathcal{P} \to \mathcal{F}$. This is a morphism since every hyperplane that separates A from B, subsets of $\mathcal{F}$, also separates their respective points $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$.

7. Consider a family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ and let $\pi: \mathbb{R}^d \to \mathbb{R}^e$ be an affine projection. The obvious bijection $\hat{\pi}: \mathcal{F} \to \pi(\mathcal{F})$ is a morphism between their separoids $S(\mathcal{F}) \longrightarrow S(\pi(\mathcal{F}))$.

8. *Strong* and *weak maps* of oriented matroids are both examples of morphisms between their respective separoids (see Björner et al. (1993) [2] sec. **7.7**).

9. In Figure 1, bijective morphisms go from left to right between every pair of separoids. Observe that there is no a bijective morphism between those separoids labeled **d** and **e**.

Given a subset $X' \subseteq X$ of the base set of a separoid S, the *induced separoid* is defined as the restriction of $|$ to X'. An *embedding* is an injective morphism between separoids such that it is an isomorphism between the domain and the induced separoid of the image. The *order* is the number of elements in X and the *size* is the number of separations defined there.

There is also a notion of dimension on separoids which is easily and intrinsically determined. The *d-dimensional simploid* $\sigma = \sigma^d$ is a separoid of order $d+1$ such that every subset is separated from its complement, which yields $A \mid B \iff A \cap B = \phi$. The simploid can be realized with the vertex set of the simplex, hence its name —Figure 1.**a** represents σ^2.

The *(combinatorial) dimension* of a separoid, denoted by $\mathrm{d}(S)$, is the maximum dimension of its induced simploids.

With the definition of dimension at hand, it is quite easy to translate to separoid terms the classic Radon's theorem; they capture the combinatorial essence of it (cf. Danzer et al. 1963).

Lemma 1 *(Radon). Let $S = (X, |)$ be a d-dimensional separoid, then every subset $Y \subseteq X$ of cardinality greater or equal to $d+2$ contains two disjoint subsets $A, B \subset Y$ such that they are not separated from each other $A \dagger B$.*

Proof. Follows immediately from the fact that Y is not a simploid. $\square$

There have been many authors that observe that the Radon's theorem can be settled in a more precise way (cf. Eckhoff 1993): *Let X be a set of $d+2$ points in $\mathbb{R}^d$ in general position. Then X contains a unique partition in two disjoint subsets whose convex hulls have a common point. Moreover, this point is also unique.* This motivates the next definitions.

A *Radon* separoid is a separoid with the property that for all $A \dagger B, C \dagger D \in MRP$ such that $A \cup B \subseteq C \cup D$ follows that $\{A, B\} = \{C, D\}$, i.e., the elements of MRP are incomparable.

A separoid is said to be in *general position* if every subset $A \subset X$ of cardinality $d+1$ is an induced simploid.

Lemma 2 *(general position). Let S be a d-dimensional separoid in general position. If $A \dagger B \in MRP$ is a minimal Radon partition, then the cardinality of the support $A \cup B$ is at least $d+2$.*

Proof. The cardinality of the support cannot be smaller because every subset $\sigma \subset S$ of cardinality $d+1$ or less is an induced simploid. $\square$

3 The Geometric Dimension

This section introduces a basic invariant in separoids theory. It will be show that Example 2 is in fact the most general example, i.e. when thinking in separoids, we can always have in the mind a family of convex sets and use all the intuition that comes from this picture with out loss of generality. Let us start this section with some general facts of the separoids category.

Given two separoids S and T over the sets X and Y respectively, their *product* $S \times T$ is defined as a separoid over the set $X \times Y$ with its two canonical projections π_X, π_Y and two subsets of it $A, B \subseteq X \times Y$ are separated iff at least one projection is, i.e.,

$$A \mid B \iff \pi_X(A) \mid \pi_X(B) \text{ or } \pi_Y(A) \mid \pi_Y(B).$$

Clearly, this definition implies that the projections π_X, π_Y are separoid morphisms. A straight forward argument proves that it is the categorical product, this is

$$P \longrightarrow S \times T \iff P \longrightarrow S \text{ and } P \longrightarrow T.$$

Once the product has been defined for two separoids, the definition for a finite number of separoids $\prod_{i=1}^{m} S_i$ is obvious.

This product has a geometric counterpart. Let $\mathcal{S}$ and $\mathcal{T}$ be separoids of convex sets in $I\!R^s$ and $I\!R^t$, respectively. The *geometric product* $\mathcal{S} \otimes \mathcal{T}$ is a family of convex set in $I\!R^s \times I\!R^t$ whose elements are of the form $\mathcal{K}_s \times \mathcal{K}_t$, where $\mathcal{K}_s \in \mathcal{S}$ and $\mathcal{K}_t \in \mathcal{T}$. In general, it is not the case that the separoid of $\mathcal{S} \otimes \mathcal{T}$ is isomorphic to its combinatorial counterpart $\mathcal{S} \times \mathcal{T}$ however, in some special cases, if the convex sets are "big enough", $\mathcal{S} \otimes \mathcal{T}$ is a realization of $\mathcal{S} \times \mathcal{T}$.

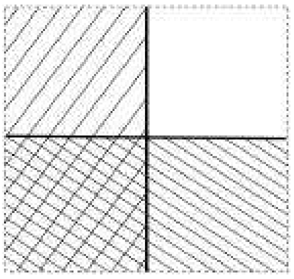

Figure 2. $\mathcal{B} \times \mathcal{B}$

Theorem 1 *(Representation Theorem) Every (finite) separoid of size m can be realized with a family of convex sets in $I\!R^m$.*

Proof. Given a separoid S and a separation in it $A \mid B$, a *characteristic morphism* $\chi_{A|B}: S \to \mathcal{B}$ exists

$$\chi_{A|B}(x) = \begin{cases} +, & \text{if } x \in A \\ -, & \text{if } x \in B \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{B}$ denotes the separoid defined in the set $\{-, 0, +\}$ with the unique maximal separation $- \mid +$. It is not hard to prove that S can be embedded into the product of as many copies of $\mathcal{B}$ as separations S has

$$\chi : S \longrightarrow \prod_{A|B \in S} \mathcal{B}.$$

The existence of such a morphism is guarantee and determined by all the characteristics morphisms.

In order to see that χ can be made injective, take two different elements $x \neq y \in S$. If there exists a separation $A \mid B$ with $x \in A$, since S is an ideal relation, we have a separation of the form $x \mid B$. This leads to the inequality $- = \chi_{x|B}(x) \neq \chi_{x|B}(y) \in \{0, +\}$ and therefore to $\chi(x) \neq \chi(y)$.

Now we may suppose that neither x nor y appears in any separation and then they are mapped to the element $(0, \ldots, 0) \in \prod \mathcal{B}$. We can identify all elements with this property in order make χ and injection; after the realization of $\prod \mathcal{B}$ bellow, we can reconsider these elements and use as many copies of the ambient space $I\!R^m$, the convex set that represents $(0, \ldots, 0)$, to realize the original separoid S. Observe that this identification does not change the separoids size.

Finally, to see that χ is an embedding observe that, if $A \mid B$, there is a projection $\pi : \prod \mathcal{B} \longrightarrow \mathcal{B}$ such that $\chi_{A|B} = \pi\chi$ and therefore $\chi(A) \mid \chi(B)$.

The end of the proof is to show how to realize $\prod_{i=1}^{m} \mathcal{B}$ as a family of convex sets. In the real line, let $\mathcal{B}$ mapped as follows:

$$\begin{array}{ccc} - & \mapsto & I\!R^-, \\ 0 & \mapsto & I\!R, \\ + & \mapsto & I\!R^+, \end{array}$$

Where $I\!R^- = \{r \in I\!R : r < 0\}$ denotes the set of negative real numbers and analogously with $I\!R^+$. Clearly this realizes the separoid $\mathcal{B}$ and the product of m copies of it can be realized in $I\!R^m$ by the geometric product of these family convex sets (see Figure 2). □

The *geometric dimension* of a separoid can be defined as the minimum dimension of the Euclidian space where the given separoid S can be realized as a separoid of convex sets; we denote it by $\mathrm{gd}(S)$. There are not known algorithms to calculate this invariant and it is conjectured that it is, at least, an NP-hard problem. It is important to give better upper bounds of $\mathrm{gd}(S)$ than that implicitly given in the theorem; in particular, we believe it should grow at most linearly with respect to the order (not the size).

We end this section showing how to prove that the combinatorial dimension bounds the geometric dimension.

Lemma 3 *For any separoid S, its combinatorial dimension is not greater than its geometric dimension, i.e., $d(S) \leq gd(S)$.*

Proof. Let S be d-dimensional with geometric dimension $g = \mathrm{gd}(S)$, and suppose that $g < d$. Let $\mathcal{S}$ be a family of convex sets in $\mathbb{R}^g$ that realizes S. Since S is d-dimensional, it contains a d-dimensional simploid $\sigma \subseteq \mathcal{S}$ of order $d+1$. Choose a point for each convex set of σ. This set of points consists of $g+2$ or more points in $\mathbb{R}^g$ and, by the Radon's lemma, there exists a partition of them in two subsets whose convex hulls intersect. Therefore they are not separated. This contradicts the fact that σ was a simploid. ▫

4 Uniform Point Separoids

In this section we will concentrate in a very specific class of separoids. *Point separoids* are those separoids which can be realized by a configuration of points $\mathcal{P} \subset \mathbb{E}^d$ in some Euclidian space. They are extremely difficult to characterize from a purely combinatorial point of view. In fact, it is known that the *stretchability problem* —a polar version in dimension 2— is NP-hard (cf. Shor 1991 [11]). However, from the geometric point of view intrinsic to separoids, we can characterize those point separoids in general position.

Theorem 2 *Let $\mathcal{S}$ be a separoid in general position. $\mathcal{S}$ is a point separoid if and only if its dimension and its geometric dimension are equal.*

Proof. The necessity is clear. For the sufficiency, consider $\mathcal{S}$ as a separoid of convex sets in $\mathbb{R}^d$, where $d = \mathrm{d}(\mathcal{S})$. Choose a point in each convex set, denote by $\mathcal{P}$ the point separoid that they define, and let

$$\varphi: \mathcal{P} \longrightarrow \mathcal{S}$$

be the obvious morphism (see Example 6). We will show that, in fact, this is an isomorphism of separoids.

In one hand, by the construction, we have that for every $A, B \subset \mathcal{S}$,

$$A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

On the other hand, let $A \dagger B \in MRP$ be a minimal Radon partition of $\mathcal{S}$. Since $\mathcal{S}$ is a separoid in general position, the cardinality of the support is $\#(A \cup B) \geq d+2$. Then the preimage of this union consists of $d+2$ or more points in $\mathbb{R}^d$ and by the Radon's lemma there exists a partition

$D \dagger E$ of $\varphi^{-1}(A \cup B)$ in $\mathcal{P}$. Since φ is a bijective morphism, $\varphi(D) \dagger \varphi(E)$ is a Radon partition of $A \cup B$. Finally, due to Proposition 3 (Section 6), $\mathcal{S}$ is a Radon separoid and $\{A, B\} = \{\varphi(D), \varphi(E)\}$. Therefore $\varphi^{-1}(A) \dagger \varphi^{-1}(B)$. Since the set MRP generates all Radon partitions, it follows that for every $A, B \subset S$,

$$A \dagger B \Longrightarrow \varphi^{-1}(A) \dagger \varphi^{-1}(B).$$

Thus, φ is an isomorphism of separoids and $\mathcal{S}$ is a point separoid. □

This result is sharp. The hypothesis of general position cannot be droped without adding a new ingridient. The separoid $\mathcal{B}$ used in the proof of the Representation Theorem is a 1-dimentional Radon separoid, it can be realized in the line but it is not a point separoid. Even if we restricts to acyclic separoids this can be done (see Figure 1.**g**). However, the small examples of non-stretchable pseudolines arrengments suggest the following (cf. Theorem 3)

Conjecture. *An oriented matroid is coordinatizable if and only if its rank minus one equals its geometric dimension.*

5 Oriented Matroids

Oriented matroid theory was introduced in the 1960's when J. Folkman and J. Lawrence proved that every oriented matroid can be thought of as a family of oriented pseudospheres. In particular, they proved that the natural partial order associated to an oriented matroid is a sphere. One of the main bricks of the theory is the classic Radon's theorem (1921). Oriented matroids encode minimal Radon partitions in terms of *circuits*, a symmetric antichain of signed vectors $\mathcal{C} \subset \{-, 0, +\}^E$ with a couple of properties; as we will see, the family of all signed vectors has associated a natural partial order and a separoid is not other thing but a symmetric ideal of such an order. We will translate all the theory developed in the previous section. In particular, after Proposition 1, it will be clear that

Theorem 3

- *Oriented matroids can be represented with families of convex sets,*
- *An uniform oriented matroid is coordinatizable if and only if its rank equals its geometric dimension plus one.*

In this chapter oriented matroids, and separoids, will be handled as families of signed vectors. Thus some notation and definitions have to be introduced.

Let E be any set with n elements and denote by $\mathcal{B}^E = \{-,0,+\}^E$ the set of (signed) vectors with n entries in $\{-,0,+\}$. Given a signed vector $X = (X_e)_{e\in E}$, the set $X^\pm := \{e \in E : X_e \neq 0\}$ is called the *support* of X. The *zero set* of X is the complement of its support, $X^0 := E \setminus X^\pm = \{e \in E : X_e = 0\}$. Its *positive* and *negative* sets are $X^+ := \{e \in E : X_e = +\}$ and $X^- := \{e \in E : X_e = -\}$, respectively. The *opposite* $-X$ is defined by $(-X)_e = -(X_e)$.

In the family of signed vectors $\mathcal{B}^E$ a partial order can be defined as

$$X \leq Y \iff X^+ \subseteq Y^+ \text{ and } X^- \subseteq Y^-.$$

If $X \leq Y$, it will be said that X *conforms to* Y.

With all this at hand, a separoid $S = (E, |)$ can be encoded with signed vectors as follows: $\mathcal{S} \subseteq \mathcal{B}^E$ is a separoid if

$$\begin{array}{lcr} (S1) & X \in \mathcal{S} \Longrightarrow -X \in \mathcal{S}, & \text{(symmetry)} \\ (S3) & X \in \mathcal{S} \text{ and } X' \leq X \Longrightarrow X' \in \mathcal{S}. & \text{(it is an ideal)} \end{array}$$

The separations can be reconstructed with the obvious definition:

$$X \in \mathcal{S} \iff X^+ \mid X^-.$$

Recall that it suffices to know maximal separations to reconstruct the whole separoid —they encode the whole information of it.

To define separoid morphisms in this context, the set $\mathcal{B}^E$ can be interpreted as the family of functions of the form $\alpha: E \to \{-,0,+\}$ where, given one such a function $\alpha = X \in \mathcal{B}^E$, its applications are denoted by $\alpha(e) = X_e$. Also, if F is any other m-set —together with its family $\mathcal{B}^F$— and $\varphi: E \to F$ is any function, the *cofunction* $\varphi^*: \mathcal{B}^F \to \mathcal{B}^E$ can be defined in the usal way: if $\beta \in \mathcal{B}^F$ then $\varphi^*\beta \in \mathcal{B}^E$ is defined as

$$(\varphi^*\beta)(e) = \beta(\varphi(e)).$$

Now, given two separoids $\mathcal{S} \subseteq \mathcal{B}^E$ and $\mathcal{F} \subseteq \mathcal{B}^F$, a *separoid morphism,* denoted by $\mathcal{S} \longrightarrow \mathcal{F}$, is a function $\varphi: E \to F$ such that

$$\beta \in \mathcal{F} \Longrightarrow \varphi^*\beta \in \mathcal{S}.$$

Analogously to the former definition of a separoid, the Radon partitions of a separoid $S = (E, \dagger)$ can be encoded with signed vectors: $\mathcal{S} \subseteq \mathcal{B}^E$ are the Radon partitions of a separoid if

$$\begin{array}{lcr} (R1) & X \in \mathcal{S} \Longrightarrow -X \in \mathcal{S}, & \text{(symmetry)} \\ (R3) & X \in \mathcal{S} \text{ and } X \leq X' \Longrightarrow X' \in \mathcal{S}. & \text{(it is a filter)} \end{array}$$

Once again, recall that the minimal Radon partitons ecode the whole information of the separoid.

An *oriented matroid* $\mathcal{M} = (E, \mathcal{C})$ of order $n = |E|$ is a set of signed vectors, $\mathcal{C} \subseteq \mathcal{B}^E$, with the following properties (cf. Björner et al. (1993) [2] p.103):

$$\begin{array}{ll}
(C1) & \mathbf{0} \notin \mathcal{C} \\
(C2) & X \in \mathcal{C} \Longrightarrow -X \in \mathcal{C} \\
(C3) & X, Y \in \mathcal{C} \text{ and } X^{\pm} \subseteq Y^{\pm} \Longrightarrow X = \pm Y \\
(C4) & X, Y \in \mathcal{C} \text{ and } X_e = -Y_e \neq 0 \Longrightarrow \text{ there exists } Z \in \mathcal{C} \\
 & \text{such that } Z^{+} \subseteq X^{+} \cup Y^{+}, Z^{-} \subseteq X^{-} \cup Y^{-} \text{ and } Z_e = 0
\end{array}$$

The elements of $\mathcal{C}$ are known as the *circuits* of the matroid.

Given an oriented matroid $\mathcal{M}$ the set of its circuits $\mathcal{C}$ can be identified, in a one to one fashion, with the set of minimal Radon partitions of a separoid on the same base set E. We have the following obvious cryptomorphism.

Proposition 1 *The minimal Radon partitions MRP of a separoid $\mathcal{S}$ are the circuits of an oriented matroid if and only if*

$$\begin{array}{ll}
(M1) & \phi \dagger \phi \notin MRP, \\
(M3) & \mathcal{S} \textit{ is a Radon separoid,} \\
(M4) & A \dagger B, A' \dagger B' \in MRP \textit{ and } x \in A \cap B' \Longrightarrow \\
 & \exists A'' \dagger B'' \in MRP : A'' \subseteq A' \cup A \setminus x \textit{ and } B'' \subseteq B \cup B' \setminus x.
\end{array}$$

□

The *topes* $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{B}^E$ of the oriented matroid are the maximal separations and its *covectors* $\mathcal{L}(\mathcal{M}) \subseteq \mathcal{B}^E$ are those separations which composed with topes give topes, i.e.,

$$X \in \mathcal{L} \iff \forall T \in \mathcal{T} : X \circ T \in \mathcal{T} \quad \text{where} \quad (X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise,} \end{cases} .$$

Observe that not every separation is a covector, this is, there are more separations than covectors in an oriented matroid.

6 The Technical Lemmas

In this section we settle some general results on separoids that will be needed some where else. We start with a new "convex version" of the well known Carathéodory's theorem (cf. Danzer et al. 1963 [5] and see also Eckhoff 1993 [6]):

Lemma 4 *(Carathéodory). Let $X = \bigcup_{i \in I} \mathcal{K}_i \subseteq \mathbb{R}^d$, be the union of some convex sets $\mathcal{K}_i$. If $\mathbf{x} \in \langle X \rangle$ is a point in the convex hull of X, then there exists a subset $J \subseteq I$ with $|J| \leq d+1$ and, for every $j \in J$, a point $\mathbf{x}_j \in \mathcal{K}_j$ such that $\mathbf{x}$ is a convex combination of the points $\mathbf{x}_j$.*

Proof. By Carathéodory's theorem, we need at most $d+1$ points of X to express $\mathbf{x}$ as a convex combination of them. It is easy to see that, if two (or more) of these are on the same convex set $\mathcal{K}_j$, they can be replaced by a single point $\mathbf{x}_j \in \mathcal{K}_j$ which is a convex combination of them. Therefore we need at most one point in each convex. □

With this lemma at hand, is easy to see how to "realize" each minimal Radon partition of a separoid.

Proposition 2 *Let $\mathcal{S}$ be a separoid of convex sets. Given a minimal Radon partition $A \dagger B$, there exists a point on each convex set of the support, $\mathbf{a}_i \in \mathcal{K}_i \in A$ and $\mathbf{b}_j \in \mathcal{K}_j \in B$, such that*

$$\langle \mathbf{a}_i : \mathcal{K}_i \in A \rangle \cap \langle \mathbf{b}_j : \mathcal{K}_j \in B \rangle \neq \phi.$$

Proof. If $\mathbf{x} \in \langle A \rangle \cap \langle B \rangle \neq \phi$, by Carathéodory's lemma, we need at most $d+1$ elements of A, $\mathcal{K}_i \in A$, and at most one point in each of them $\mathbf{a}_i \in \mathcal{K}_i$ to express $\mathbf{x}$ as a convex combination of them. By the minimality of the partition, it is clear that we need at least one point in each convex of A. The same argument works for B and we are done. □

We will use also a "continuous version" of Radon's original proof.

Lemma 5 *(continuous Radon). Let $\mathbf{z}_i(t) = (1-t)\mathbf{x}_i + t\mathbf{y}_i$ with $t \in [0,1]$, be $d+2$ segments in $\mathbb{R}^d$. If their respective extreme points, $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, are different point separoids in general position, there exists a $t \in (0,1)$ such that the separoid $\{\mathbf{z}_i(t)\}$ is not in general position.*

Proof. It is easy to see that, for every $t \in [0,1]$ there exist a solution of the following equations

$$\sum \lambda_i(t)\mathbf{z}_i(t) = 0, \quad \sum \lambda_i(t) = 0, \quad \sum |\lambda_i(t)| = 2,$$

and moreover the $\lambda_i(t)$ can be choosen to be continuous. Since the points $\mathbf{x}_i = \mathbf{z}_i(0)$ are in general position, such a solution for $t = 0$ is unique and every $\lambda_i(0)$ is non-zero. Such a solution leads to a unique Radon partition, the positives vs. the negatives

$$\sum_{\lambda_i(0)>0} \lambda_i(0)\mathbf{x}_i = -\sum_{\lambda_i(0)<0} \lambda_i(0)\mathbf{x}_i,$$

$$\sum_{\lambda_i(0)>0} \lambda_i(0) = -\sum_{\lambda_i(0)<0} \lambda_i(0) = 1$$

or, in separoid notation

$$\{\mathbf{x}_i : \lambda_i(0) > 0\} \dagger \{\mathbf{x}_i : \lambda_i(0) < 0\}.$$

The same argument works for $t = 1$, but by hypothesis it yields a "different" partition

$$\{\mathbf{y}_i : \lambda_i(1) > 0\} \dagger \{\mathbf{y}_i : \lambda_i(1) < 0\}.$$

To be different partitions means that there is a j such that $\lambda_j(0)$ and $\lambda_j(1)$ have different signs (while others have the same), then there exists a $t \in (0,1)$ such that $\lambda_j(t) = 0$. For that t, $\{\mathbf{z}_i(t)\}$ is not in general position. □

We close this with a beautiful theorem of separoids that was used to characterize point separoids in general position.

Proposition 3 *If a separoid is in general position and its geometric dimension is equal to its dimension, then it is a Radon separoid.*

Proof. Let S be a d-dimensional separoid in general position. If its geometric dimension is equal to its dimension, it can be realized as a family $\mathcal{S}$ of convex sets in $\mathbb{R}^d$. Suppose that S **is not** a Radon separoid. Then there are subsets of $A, B, C, D \subseteq \mathcal{S}$ such that $A \dagger B$, $C \dagger D \in MRP$, $A \cup B \subseteq C \cup D$ and $\{A, B\} \neq \{C, D\}$. Since S is in general position, the support $\mathcal{S}' := A \cup B$ has at least $d + 2$ elements. Since $C \dagger D$ is minimal, applying Proposition 2 and the classic Radon's theorem, its support has at most $d + 2$ elements. Then, with out loss of generality, we may suppose that $|S| = |A \cup B| = |C \cup D| = d + 2$.

Using again Proposition 2, two configuration of points can be defined, two points on each convex set, in such a way that they realize the two Radon partitions. Considering the line segment that join each couple —inside each convex set— and applying the continuous Radon lemma, we conclude that $\mathcal{S}$ **is not** in general position. □

This is a very surprising statement which needs some time to digest (especially when one realize that we may assume that the separoid is not a Radon separoid, a meaningless assumtion inside oriented matroid theory). The geometric dimension, in the uniform case, encodes axiom (C3).

7 References.

1. J. L. Arocha, J. Bracho, L. Montejano, D. Oliveros and R. Strausz, Separoids: their categories and a Borsuck-Ulam type theorem for transversals, *Discrete & Computational Geometry* **27** (2002) pp. 377–385.
2. A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, Oriented Matroids, *Encyclopedia of Mathematics and its Applications*, **43**, Cambridge University Press, Cambridge, U.K., 1993.
3. J. Bracho, L. Montejano and D. Oliveros, The topology of the space of transversals through the space of configurations, *Topology and Applications* (to appear).
4. J. Bokowski, Oriented Matroids, *Handbook of Convex Geometry*, Ed. P. M. Gruber and J. M. Wills, Elsevier Science Publishers, Amsterdam, The Netherlands 1993, pp. 555–602.
5. L. Danzer, B. Grünbaum and V. Klee, Helly's theorem and its relative, *Convexity*, Proc. Sypos. Pure Math, **VII**, American Math. Soc., Providence, RI 1963, pp. 101–180.
6. J. Eckhoff, Helly, Radon and Carathéodory type theorems, *Handbook of Convex Geometry*, Ed. P. M. Gruber and J. M. Wills, Elsevier Science Publishers, Amsterdam, The Netherlands 1993, pp. 389–448.
7. P. H. Edelman, The acyclic sets of an oriented matroid, *Journal of Combinatorial Theory, Ser B*, **36**, (1984) 36–31.
8. J. Folkman and J. Lawrence, Oriented Matroids, *Jurnal of Combinatorial Theory, Ser. B*, **25**, (1978) 199–236.
9. J. Goodman, R. Pollack and R. Wenger, Geometric Transversal Theory, *New Trends in Discrete and Combinatorial Geometry*, Ed. J. Pach, Algorithms and Combinatorics, **10**, Springer-Verlag, Heidelberg, Germany, 1993, pp. 163–198.
10. J. J. Montellano-Ballesteros and R. Strausz, A characterization of cocircuit graphs of uniform oriented matroids, *KAM-Series*, **2002-565**.
11. P. Shor, Stretchability of pseudolines is NP-hard, *Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift*, P. Gritzmann, B.

Sturmfels, eds., DIMACS Series in Discrete Mathematics and Theoretical Computer Science, **4**, American Math. Soc., Providence, RI 1991, pp. 531–554.
12. R. Strausz, Separoides, *Situs, serie B*, **5**, Universidad Nacional Autónoma de México, México D.F. (1998) pp. 36–41.
13. R. Strausz, Separoides: el complejo de Radon, *M. Sc. Thesis,* Universidad Nacional Autónoma de México, México D.F., 2001.
14. R. Strausz, Density of Separoids Homomorphisms, 2002 (manuscript).
15. R. Strausz, On Separoids, *Ph. D. Thesis,* Universidad Nacional Autónoma de México (in preparation).

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