A characterization of cocircuit graphs of uniform oriented matroids

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Abstract

In this paper the *Radon complex* of an oriented matroid is introduced. It is a natural poset associated to the Radon partitions of a *separoid* —a natural generalization of an oriented matroid— which, in their turn, will be defined. We show that, in the *uniform* case, the 1-skeleton of the Radon complex' dual is the oriented matroid's *cocircuit graph*. This leads to a characterization of such graphs via a natural metric defined in the *k-dual* of the n-cube.

1 Introduction

The oriented matroid theory was introduced in the 60's when J. Folkman and J. Lawrence proved that every oriented matroid can be thought of as a family of oriented pseudospheres [11]. In particular, they proved that the natural partial order associated to an oriented matroid is a sphere —this last result is known as the Basic Sphericity Theorem [3,6]. This result has many applications and we will use, as our intuition source, a slightly different version of it: *The Radon complex of an oriented matroid is a sphere.* As it will be shown later this is a direct consequence of Edelman's theorem (1984) [9] and Alexander duality.

A cornerstone of the theory is the classic Radon's theorem (1921) [7,8]: *Given a family of* $n \geq d+2$ *points in* $\mathbb{R}^d$*, there exists a pair of disjoint subsets of it whose convex hulls intersect.* Oriented matroids encode such pairs in terms of its *circuits*, a set of signed vectors $\mathcal{C} \subset \{-,0,+\}^E$. The family of all signed vectors has a natural poset associated which turns out to be the face lattice of the n-cube. The Radon complex of an oriented matroid is a "cubic" complex (an ideal in the face lattice of the n-cube) whose vertices are identified with those subsets which are"non-separated" of their respective complements —it will be formally defined in Section 2. This complex contains the structure of the oriented matroid. Theorem 2 shows that the dual poset of such a complex can be characterized via the natural combinatorial metric associated to its 1-skeleton.

We follow ideas explored by K. Fukuda and K. Handa (1993) [12] but in the more general context of *separoids* [1,4,5,16,17,18,19,20]: a symmetric ideal (or filter) defined by an antichain in the face lattice of the n-cube (or the n-crosspolytope). Fukuda and Handa characterized every *tope graph* $\mathcal{T} = \mathcal{T}(\mathcal{M})$ of an oriented matroid of rank 3 showing that they are those antipodal planar graphs which can be embedded in the n-cube preserving their graph distance. The planarity of $\mathcal{T}$ induces a dual graph $\mathcal{G} = \mathcal{T}^*$ which can be proved to be the *cocircuit graph* of the oriented matroid. No characterization is known of the cocircuit (or tope) graph in the general case (cf. [2]), but Theorem 2 gives necessary and sufficient conditions for *uniform* oriented matroids with arbitrary rank. We basically prove that *a graph* $\mathcal{G}$ *is the cocircuit graph of a uniform oriented matroid of order* n *and rank* r *if and only if it is of order* $2\binom{n}{r+1}$*, antipodal and can be embedded "metrically" in the* $(n-r-1)$*-dual of the* n*-cube.*

On the other hand, separoids have their origin in the study of Geometric Transversal Theory [4,13,16]. They are general enough to embrace, at the same time, the combinatorial structure of a family of convex sets in euclidian space [1] and that of oriented matroids [5]. In fact, *every separoid arises from a configuration of convex sets.* This result plays the role of "the polar version" of the Topological Representation Theorem (cf. [3] p. 225) however, the combinatorial structure that is preserved turns out to be more general. In other words, when the hyperplanes "wiggle" to be pseudohyperplanes, their polar points "fatten" to become convex sets, but not every configuration of convex sets can be exhibited in this way.

In the following, oriented matroids are thought of as separoids in order to concentrate in those axioms from which the main theorem depends. In particular, the Radon complex is more naturally defined in the separoid than in the oriented matroid. Separoids introduce new elements to be considered (we call them *generalized cotopes*) which are identified with the vertices of the Radon complex. Edelman [9] has used separoids (with no name) to prove that

$$\Gamma(V(\mathcal{T})) := \{X \in \{-,0,+\}^E : X \leq T \text{ and } T \in V(\mathcal{T})\}$$

has the homotopy type of a sphere, where $V(\mathcal{T})$ denotes the set of topes of an oriented matroid (cf [3] p. 180). The Radon complex is just the complement (in $\mathcal{Q}_n = \{-, 0, +\}^E$) of Edelman's complex, but it must be observed that not every vertex of it is a *cotope*. The basic properties of separoids are shown in Subsection 2.1 while Subsection 2.2 introduces the Radon complex formally.

On Subsection 3.1 oriented matroids are defined (in terms of its circuits) and it is shown how separoids generalize them (Proposition 2). Also, a new version of the Basic Sphericity Theorem is presented there as Theorem 1. Subsection 3.2 introduces the *k-dual of the n-cube* and Subsection 3.3 is devoted to the main result (Theorem 2). Section 3 contains, as well, a technical lemma which is proved in Section 4.

2 Separoids

Separoids are combinatorial objects that embrace the structure arising from a family of convex sets, where some subfamilies are naturally separated from others. Namely, two subfamilies are said to be *separated* if there exists a hyperplane that leaves them on opposite sides of it —the axioms of a separoid are simply the obvious properties of this relation.

2.1 Basic Notions.

A *separoid* $\mathcal{S} = (E, |)$ over the base set E is a relation $| \subseteq 2^E \times 2^E$ on the subsets of E with the following properties: if $A, B \subseteq E$,

$$\begin{array}{lc} (S1) & A \mid B \Rightarrow B \mid A, \\ (S2) & A \mid B \Rightarrow A \cap B = \phi, \\ (S3) & A \mid B \text{ and } A' \subset A \Rightarrow A' \mid B. \end{array}$$

The elements of $|$ are called *separations* and, when speaking of a separation $A \mid B$, it is said that "A is *separated from* B". A separoid is *acyclic* if the empty set is separated from the base one, i.e., if $\phi \mid E$. The separations with the empty set are called *trivial* separations and, in this paper, almost all separoids are finite and acyclic. Observe that it is enough to know *maximal separations* to reconstruct the separoid —they encode the whole information of it.

Examples.

1. Consider a family F of convex sets in the d-dimensional euclidian space $I\!R^d$ and define the following relation $|\subseteq 2^F \times 2^F$ on the subsets of F: $A \mid B$ if and only if there exists a hyperplane H that leaves the elements of A in one open semispace (bounded by H) and those of B in the other. Clearly, the pair $(F, |)$ is a separoid. Actually, every separoid can be realized with a family of convex set [5]. More over, the separoid is acyclic if and only if such realization can be done with convex bodies [1,20]. When each convex is a point, we are dealing with a linear oriented matroid (see below) and we call it a *point separoid* (for more on point separoids look for [5,18]).

2. Consider an oriented matroid $\mathcal{M} = (E, \mathcal{L})$ and identify it with the subset $\mathcal{L} \subseteq \{-,0,+\}^E$ of its covectors in the usual manner. Let $\mathcal{T} = \mathcal{T}(\mathcal{L})$ be the set of *topes*, maximal covectors, and define the following relation $|\subseteq 2^E \times 2^E$ on the subsets of E: $A, B \subset E$ are separated, $A \mid B$, if and only if there exists a tope $T \in \mathcal{T}$ such that $A \subseteq T^+ := \{e \in E : T_e = +\}$, and $B \subseteq T^- := \{e \in E : T_e = -\}$. The pair $S(\mathcal{M}) = (E, |)$ is a separoid. In Section 3 this example will be studied in more detail; in particular, it will be shown that the oriented matroid can be reconstructed from its separoid, and hence that separoids generalize oriented matroids. A formal cryptomorphism shows how to interpret *circuits* as *minimal Radon partitions* (and topes as maximal separations) of a separoid.

3. Edelman has defined a complex which encodes the separoid of an oriented matroid. He considers the set $\Gamma(\mathcal{T}) := \{X \in \{-,0,+\}^E : X \leq T \text{ and } T \in \mathcal{T}\}$, where $\mathcal{T}$ denotes the topes of an oriented matroid and $\leq$ denotes the *conformal relation*, i.e., $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. Clearly, a signed vector $X \in \Gamma$ belongs to Edelman's complex if and only if $X^+ \mid X^-$. He uses the Basic Sphericity Theorem to prove that such a complex has the homotopy type of a sphere. Theorem 1 is a direct consequence of this result —it is somehow its dual.

4. All acyclic separoids of order three arise from one of the eight families of convex sets shown in Figure 1. Only those in Figures 1.a, 1.b, 1.e and 1.h are point separoids —they come from the four essentially different acyclic oriented matroids with three elements.

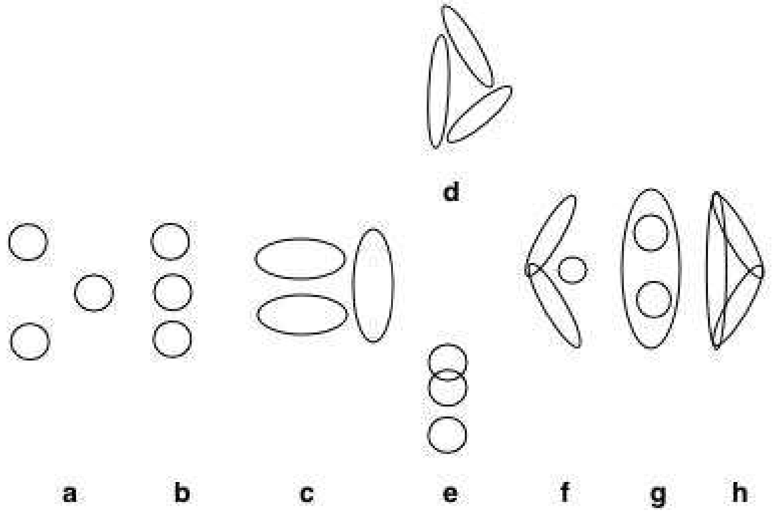

Figure 1. *The acyclic separoids of order three.*

Given a subset $F \subseteq E$ of the base set of a separoid $\mathcal{S}$, the *induced separoid* is defined as the restriction of $\mid$ to F. The *order* is the number of elements in E.

There is a notion of dimension on separoids which is easily and intrinsically determined. The *d-dimensional simploid* $\sigma = \sigma^d$ is a separoid of order $d+1$ such that every subset is separated from its complement, which by the third condition (S3) yields $A \mid B \iff A \cap B = \phi$. The simploid can be realized with the vertex set of the simplex, hence its name —Figure 1.a represents σ^2. The *dimension* of a separoid is the maximum dimension of its induced simploids. It is easy to see that the dimension of the separoid of an oriented matroid is its *rank* minus 1 (see Section 3).

With the notion of dimension the classic Radon's theorem (see [7]) can be translated to separoids terms; they capture the combinatorial essence of it.

Lemma 1 *(Radon) Let $\mathcal{S} = (E, \mid)$ be a d-dimensional separoid, then every subset $F \subseteq E$ of cardinality at least $d+2$ contains two disjoint subsets $A, B \subset F$ such that they are not separated from each other.*

Proof. Clearly follows from the fact that F is not a simploid. □

A *Radon partition* consists of two non-separated disjoint sets, and it will be denoted by $A \dagger B$. Considering the Radon partitions of a separoid

$\mathcal{S} = (E, |)$ as a relation $\dagger \subset 2^E \times 2^E$, it satisfies the following properties:

$$\begin{array}{ll} (R1) & A \dagger B \Rightarrow B \dagger A, \\ (R2) & A \dagger B \Rightarrow A \cap B = \phi, \\ (R3) & A \dagger B \text{ and } C \subseteq E \setminus A \Rightarrow A \dagger B \cup C. \end{array}$$

This leads to an equivalent definition of a separoid. The separations can be reconstructed with the obvious definition; $A \mid B$ if and only if $A \cap B = \phi$ and there are no subsets $A' \subseteq A$ and $B' \subseteq B$ such that $A' \dagger B'$.

Each part, A and B, is known as a *Radon component* and the union $A \cup B$ will be called, following oriented matroid terminology, the *support* of the partition.

A *minimal Radon partition* is a Radon partition $A \dagger B$ where each component is minimal by contention, i.e.,

$$A' \subset A \Rightarrow A' \mid B \qquad \text{and} \qquad B' \subset B \Rightarrow A \mid B'.$$

The set of all minimal Radon partitions of a given separoid determines the separoid and will be denoted by MRP, so $A \dagger B \in MRP$ means that $A \dagger B$ is a minimal Radon partition.

Some authors have observed that the Radon's theorem can be stated in a more precise way (cf. [6]): *Let E be a set of $d+2$ points in $\mathbb{R}^d$ in general position. Then E contains a unique partition in two disjoint subsets whose convex hulls have a common point. Moreover, this point is also unique.* This takes us to the next definitions.

A *Radon* separoid is a separoid with the property that for every $A \dagger B, C \dagger D \in MRP$ with the same support ($A \cup B = C \cup D$), $\{A, B\} = \{C, D\}$. That is to say, the elements of MRP are unique on a fixed support. Therefore, point separoids are Radon separoids.

A d-dimensional separoid is said to be in *general position* or to be *uniform*, following oriented matroid terminology, if every subset $F \subseteq E$ of cardinality $d+1$ is an induced simploid.

In relation to what has been said, we have the following

Lemma 2 *(general position) Let $\mathcal{S}$ be a uniform d-dimensional separoid. If $A \dagger B \in MRP$ is a minimal Radon partition, then the cardinality of the support $A \cup B$ is at least $d+2$.*

2.2 The Radon Complex.

We will associate to each separoid of order n a "cubic" complex which is contained in the n-cube. The Radon components of a separoid can be identified with some vertices of the n-cube and the *Radon complex* of the separoid will be defined as the induced complex of those vertices. It was proved in [17] that, if the separoid is a point separoid, such set of vertices induces a topological sphere (see Proposition 1).

Let $\mathcal{Q}_n$ denote the n-cube (see Figure 2). Its vertices $V(\mathcal{Q}_n)$ will be identified with the family of subsets 2^E of the n-set E. Its faces are intervals of the natural contention partial order defined in 2^E, i.e., each face is of the form

$$[A,B] := \{X \subseteq E : A \subseteq X \subseteq B\}.$$

This definition leads to an n-ball but in the sequel the n-cube will be thought of as an $(n-1)$-sphere so the face $[\phi, E]$ is dropped out.

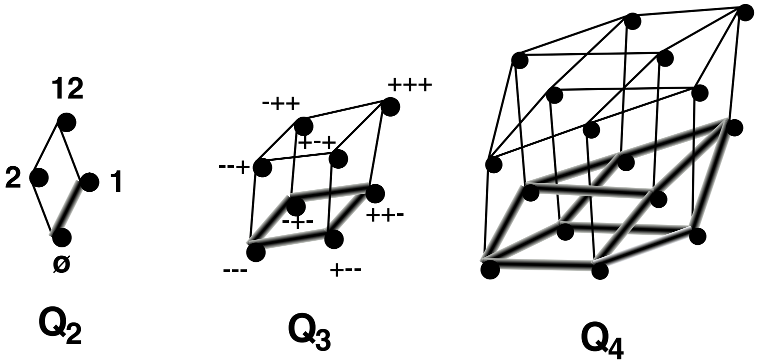

Figure 2. *The n-cube, for $n = 2, 3, 4$.*

We call *generalized cotopes* the Radon partitions of the form $A \dagger (E \setminus A)$. Given a separoid $\mathcal{S}$, for each generalized cotope $A \dagger (E \setminus A)$, take the vertex $A \in V(\mathcal{Q}_n)$; the complex induced by all those vertices is what we call the *Radon complex* of the separoid and we denote it by $\mathcal{R} = \mathcal{R}(\mathcal{S})$. Here, by *induced* we mean that a face of $\mathcal{Q}_n$ is in the complex if and only if all of its vertices are in such complex. Some small Radon complexes are shown in Figure 3.

It follows from the definition and (R3) that

Lemma 3 *If $A \dagger B$ is a Radon partition of $\mathcal{S}$ then $[A, E \setminus B]$ is a face of $\mathcal{R}(\mathcal{S})$.*

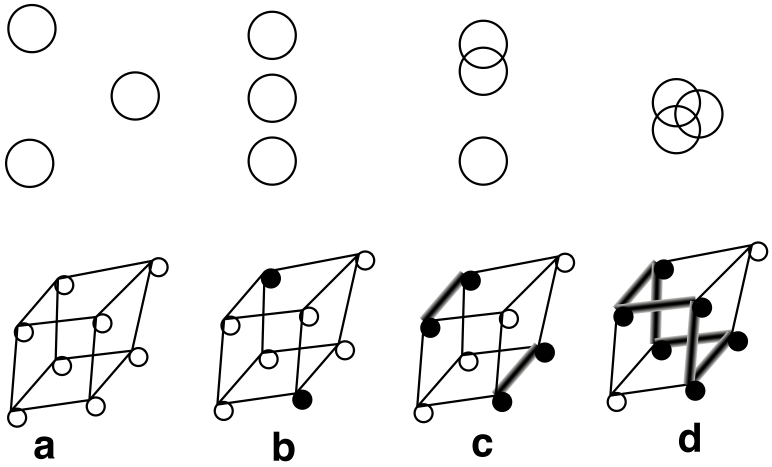

Figure 3. *The four point separoids of order three and their Radon complex.*

Proposition 1 *The Radon complex of a point separoid of order n and dimension d is homotopically equivalent to the $(n-d-2)$-sphere. Moreover, if the separoid is in general position, they are homeomorphic.*

The proof of Proposition 1 is in [17] but, in order to be self contained, its basic ideas will be exposed:

Let $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset I\!\!R^d$ be a point separoid. Define a linear function $\varphi\colon I\!\!R^n \to I\!\!R^d$ as $\varphi(\mathbf{x}) = \sum x_i \mathbf{p}_i$ where x_i denotes the ith coordinate of $\mathbf{x} \in I\!\!R^n$. Then, $A \dagger B$ is a Radon partition of $\mathcal{P}$ if and only if there exists $\lambda_i \in I\!\!R$ such that

$$\sum_{i \in A} \lambda_i \mathbf{p}_i = \sum_{i \in B} \lambda_i \mathbf{p}_i, \text{ and } \quad \sum_{i \in A} \lambda_i = \sum_{i \in B} \lambda_i = 1, \text{ and } \quad \lambda_i \geq 0.$$

Clearly, this is equivalent to

$$\sum_{i \in A \cup B} \lambda_i \mathbf{p}_i = 0, \text{ and } \quad \sum_{i \in A \cup B} \lambda_i = 0, \text{ and } \quad \sum_{i \in A \cup B} |\lambda_i| = 2.$$

If we denote by $K = \varphi^{-1}(0)$, by $\Pi = (1, \ldots, 1)^{\perp} \subset I\!\!R^n$ and by $\mathcal{O}_n = \{\mathbf{x} \in I\!\!R^n : \|\mathbf{x}\| = 2\}$ (where $\|.\|$ denotes the Manhattan norm) then the solution space of the three previous equations is

$$\mathcal{R}^*(\mathcal{P}) = K \cap \Pi \cap \mathcal{O}_n.$$

The proof ends showing that the duality from the n-crosspolytope to the n-cube sends $\mathcal{R}^*(\mathcal{P})$ to $\mathcal{R}(\mathcal{P})$, i.e., these two complexes are dual one from the other. ◻

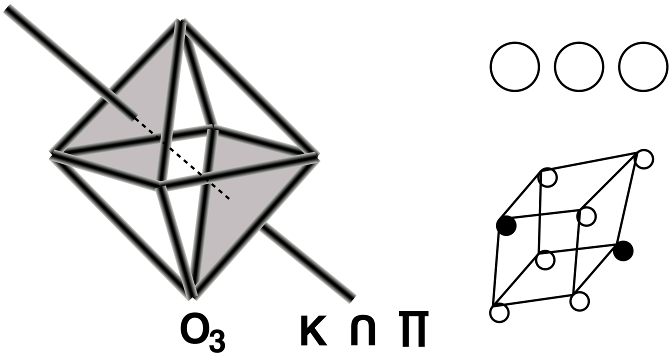

Figure 4. *The proof of Proposition 1.*

Notice that Proposition 1, for $n = d{+}2$ points in general position, implies that $\mathcal{R}$ is a sphere of dimension 0 (two antipodal vertices) which represents the unique Radon partition. Therefore, Proposition 1 generalizes Radon's theorem —hence the name of the complex.

Also, it is clear that a separoid $\mathcal{S}$ (thought of as a subcomplex of $\mathcal{O}_n$) is a point separoid if and only if there exists a flat K such that $\mathcal{O}_n \cap \Pi \cap K$ is a retraction of $\mathcal{S}$, i.e., $\mathcal{S} \searrow \mathcal{O}_n \cap \Pi \cap K$. It is well known that the "Stretchability Problem" is NP-hard [14].

3 Oriented Matroids

Proposition 1 can be extended to the more general class of separoids that comes from oriented matroids. Edelman proved in 1984 that if the set of topes of an oriented matroid $\mathcal{T}(\mathcal{M})$ is closed by the *conformal relation* $\leq$ (cf. Example 3), the homotopy type of its *covectors* $\mathcal{L} = \{X \in \{-,0,+\}^E : T \in \mathcal{T} \Rightarrow X \circ T \in \mathcal{T}\}$ is preserved. Therefore, by the Basic Sphericity Theorem, $\Gamma(\mathcal{T})$ has the homotopy type of a sphere. Since the Radon complex of $\mathcal{M}$ is the complement (in $\mathcal{Q}_n = \{-,0,+\}^E$) of Γ, by the Alexander duality (see e.g. Spanier [15]), $\mathcal{R}(\mathcal{M})$ has also the homotopy type of a sphere. Hence it follows that

Theorem 1 *The Radon complex of an oriented matroid is homotopic to a sphere.*

This result did guide our intuition to prove Theorem 2 below.

In the sequel, separoids, as usually in oriented matroids, will be handled as families of signed vectors. Therefore some notation has to be introduced.

3.1 Cryptomorphism.

Let E be any set with n elements and denote the set of (signed) vectors with n entries in $\{-,0,+\}$ as $\{-,0,+\}^E$. Given a signed vector $X = (X_e)_{e\in E}$, the set $X^\pm := \{e \in E : X_e \neq 0\}$ is called the *support* of X. The *zero set* of X is the complement of its support, $X^0 := E \setminus X^\pm = \{e \in E : X_e = 0\}$. Its *positive* and *negative* sets are $X^+ := \{e \in E : X_e = +\}$ and $X^- := \{e \in E : X_e = -\}$, respectively. The *opposite* $-X$ is defined by $(-X)_e = -(X_e)$. Additionally, in the family of signed vectors $\{-,0,+\}^E$ a partial order can be defined as

$$X \leq Y \iff X^+ \subseteq Y^+ \text{ and } X^- \subseteq Y^-.$$

If $X \leq Y$, it will be said that X *conforms to* Y. Actually, this poset is the face lattice of the n-crosspolytope and dual of the n-cube.

Given two signed vectors X and Y, the *separator* of X and Y is the set

$$S(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}.$$

Two signed vectors X, Y with the same support size ($|X^\pm| = |Y^\pm| < n$) will be said to be *adjacent* if there exist $i, j \in E$ such that $X_k = Y_k$ for all $k \notin \{i,j\}$, $X_i = 0 \neq Y_i$ and $Y_j = 0 \neq X_j$.

This notion of adjacency defines a graph G_n whose vertex set is the family of all signed vectors. It naturally leads to the definition of *moving a zero* from one place to another (non-zero place) which is a step of a walk in the graph. Therefore, the distance in G_n from one vector to another is the minimum number of moves of zeros needed to reach the destination vector (cf Lemma 5). This motivates the following definition: the *traversen* of two signed vectors X, Y is

$$T(X,Y) = \{e \in E : X_e = 0 \neq Y_e \text{ or } Y_e = 0 \neq X_e\}.$$

Remark. *X and Y are adjacent in G_n if and only if $S(X,Y) = \phi$ and $|T(X,Y)| = 2$.*

This characterization of the adjacency will be used in three different settings: as adjacency on the circuit graph of an oriented matroid; as adjacency in the k-dual graph of the n-cube; and as adjacency of k-subcubes of the n-cube.

With all this at hand, the Radon partitions of a separoid $(E, \dagger)$ can be encoded with signed vectors as follows: $\mathcal{S} \subseteq \{-,0,+\}^E$ are the Radon partitions of a separoid if and only if

$$\begin{array}{lcr} (R1) & X \in \mathcal{S} \Rightarrow -X \in \mathcal{S}, & \text{(symmetry)} \\ (R3) & X \in \mathcal{S} \text{ and } X \leq X' \Rightarrow X' \in \mathcal{S}. & \text{(it is a filter)} \end{array}$$

Thus, the Radon partitions are reconstructed with this definition

$$A \dagger B \iff \exists X \in \mathcal{S} : A = X^+ \text{ and } B = X^-.$$

Recall that it suffices to know minimal Radon partitions to reconstruct the separoid —they encode the whole information of it.

An *oriented matroid* $\mathcal{M} = (E, \mathcal{C})$ of order $n = |E|$ is a set of signed vectors,

$$\mathcal{C} \subseteq \{-,0,+\}^E,$$

with the following properties (cf. [3] p. 103):

$$\begin{array}{lcr} (C1) & \mathbf{0} \notin \mathcal{C} & (\phi \mid \phi) \\ (C2) & X \in \mathcal{C} \Rightarrow -X \in \mathcal{C} & \text{(symmetry)} \\ (C3) & X, Y \in \mathcal{C} \text{ and } X^{\pm} \subseteq Y^{\pm} \Rightarrow X = \pm Y & \text{(uniqueness)} \\ (C4) & X, Y \in \mathcal{C}, X \neq \pm Y \text{ and } X_e = -Y_e \neq 0 \Rightarrow \text{ there exists } Z \in \mathcal{C} & \\ & \text{such that } Z^+ \subseteq X^+ \cup Y^+, Z^- \subseteq X^- \cup Y^- \text{ and } Z_e = 0 & \text{(weak elimination)} \end{array}$$

The elements of $\mathcal{C}$ are known as the *circuits* of the matroid.

Given an oriented matroid $\mathcal{M}$ the set of its circuits $\mathcal{C}$ can be identified, in a one to one fashion, with the MRP set of a separoid on the same base set E. As a consequence, we have the following cryptomorphism:

Proposition 2 *The minimal Radon partitions MRP of a separoid $\mathcal{S}$ are the circuits of an oriented matroid if and only if*

$$\begin{array}{lc} (M1) & \phi \dagger \phi \notin MRP, \\ (M3) & \mathcal{S} \textit{ is a Radon separoid}, \\ (M4) & A \dagger B, A' \dagger B' \in MRP \textit{ and } x \in A \cap B' \Rightarrow \exists A'' \dagger B'' \in MRP \\ & A'' \subseteq A' \cup A \setminus x \textit{ and } B'' \subseteq B \cup B' \setminus x. \end{array}$$

Given the circuits of an oriented matroid, its *vectors*, $\mathcal{V} = \mathcal{V}(\mathcal{M})$, can be reconstructed by an operation known as *composition*, defined as

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise,} \end{cases}$$

via the following: $\mathcal{V} \supseteq \mathcal{C}$ is the minimal superset of $\mathcal{C}$ closed by composition, i.e., $X, Y \in \mathcal{V} \Rightarrow X \circ Y \in \mathcal{V}$. Observe that $\mathcal{V} \subset \mathcal{S}$, i.e., since vectors close circuits by composition and separoids by conformal relation, in general there are more Radon partitions in the separoid than vectors in the oriented matroid. Therefore, generalized cotopes (maximal Radon partitions) effectively generalize *cotopes*, maximal vectors.

As an example, consider the point separoid (realizable oriented matroid) of Figure 5; it contains only two circuits $1 \dagger 23 = (+, -, -, 0)$ and its opposite. From the oriented matroid point of view, these two circuits are all the vectors, but for the separoid there are four more Radon partitions $1 \dagger 234, 14 \dagger 23$ and its opposites.

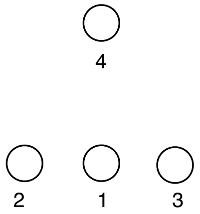

Figure 5. *A point separoid of order 4.*

Finally, in the proof of Theorem 2 we will use the following definitions (cf. [3] Lemma 4.1.8): If $\mathcal{V}$ denotes the set of vectors of an oriented matroid $\mathcal{M}$ on E and $A \subset E$, then the set of vectors of the *deletion* $\mathcal{M} \setminus A$ equals

$$\mathcal{V} \setminus A = \{X_{E \setminus A} : X \in \mathcal{V}\}$$

and the set of vectors of the *contraction* $\mathcal{M}/A$ equals

$$\mathcal{V}/A = \{X_{E \setminus A} : X \in \mathcal{V} \text{ and } A \subset X^0\},$$

where $X_{E \setminus A} \in \{-, 0, +\}^{E \setminus A}$ denotes the restriction.

3.2 The k-dual of the n-cube.

Every oriented matroid $\mathcal{M} = (E, \mathcal{C})$ has a graph $\mathcal{G} = \mathcal{G}(\mathcal{M})$ associated whose vertices are the circuits of the matroid. Two of those vertices $X, Y \in$

$\mathcal{C}$ are adjacent if $X \circ Y = Y \circ X$ while for every $Z \leq X \circ Y$ it happens that $Z \in \{X, Y\}$. This graph is what we call the *circuit graph* of the oriented matroid —in the literature [2,10,12] this graph is studied via the dual oriented matroid so it is better known as *cocircuit graph*. It is well known that the cocircuit graph of an oriented matroid is the 1-skeleton of the cell decomposition induced by the pseudospheres that realize the oriented matroid via the Topological Representation Theorem of Folkman and Lawrence [11], therefore, two (co)circuits X, Y are adjacent if and only if $|S(X,Y)| = 0$ and $|T(X,Y)| = 2$ (recall the definition of "moving a zero").

Let $\mathcal{Q}_n^k$ denote the *k-dual of* $\mathcal{Q}_n$ defined as follows ($k > 0$): the vertices of $\mathcal{Q}_n^k$ are the k-subcubes of $\mathcal{Q}_n$ and two of them are adjacent if their respective subcubes intersect in a $(k-1)$-subcube. From now on, we will denote the faces of $\mathcal{Q}_n$ by the standard signed vectors; in other words, each face $[A, B]$ is denoted by $X \in \{-, 0, +\}^E$ where

$$X_i = \begin{cases} + & \text{if } i \in A, \\ 0 & \text{if } i \in B \setminus A, \\ - & \text{otherwise.} \end{cases}$$

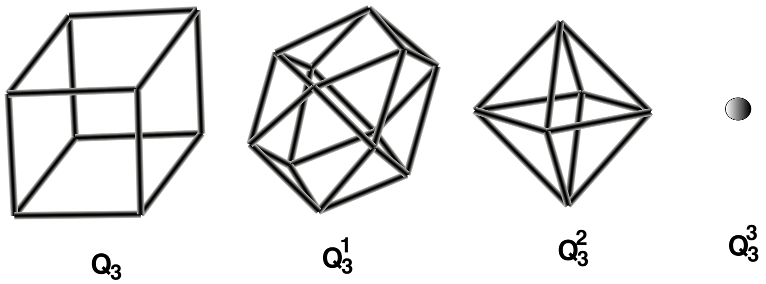

Figure 6. *The three k-duals of* $\mathcal{Q}_3$.

We call *antipodal automorphism* the function which sends each vector X to its opposite $-X$. Observe that if every circuit of a separoid has the same support size (say $n - k$), the 1-skeleton of the dual poset of the Radon complex of such a separoid is a subgraph of $\mathcal{Q}_n^k$ and it is closed by the antipodal automorphism.

Lemma 4 *The circuit graph of a uniform oriented matroid is the 1-skeleton of the dual of its Radon Complex.*

Proof. Let $\mathcal{M}$ be a uniform oriented matroid of order n and dimension d, $\mathcal{S}$ its separoid and $\mathcal{R}$ its Radon complex. Since $\mathcal{M}$ is uniform ($\mathcal{S}$ is in general position), all faces of $\mathcal{R}$ are of dimension $k = n - d - 2$ and therefore the dual is well defined. Each facet $[A, B]$ of $\mathcal{R}$ can be identified with a circuit of the form $A \dagger E \setminus B$ (and viceversa). Now, by the Remark, two k-subcubes X, Y of $\mathcal{Q}_n$ are "adjacent" (they intersect in a $(k-1)$-subcube) if and only if $|S(X,Y)| = 0$ and $|T(X,Y)| = 2$. □

This subsection ends with the statement of Lemma 5 which will be used below but, since its proof is too technical, it will be postpone until Section 4.

Lemma 5 *The graph distance in $\mathcal{Q}_n^k$ $(k > 0)$ is, for $X \neq Y$*

$$d_{\mathcal{Q}_n^k}(X,Y) = \begin{cases} |S(X,Y)| + 1 & \textit{if } X^{\pm} = Y^{\pm}, \\ |S(X,Y)| + \frac{1}{2}|T(X,Y)| & \textit{otherwise.} \end{cases}$$

3.3 Main theorem

By a graph *embedding* $G \hookrightarrow H$ it is meant an injective function $i\colon V(G) \to V(H)$ of its vertices that sends edges to edges. Besides, in such a case we will identify the vertices of the domain with those of its image and we will refer to the vertices of the domain with the name of their respective images. In particular, if a graph is embedded in $\mathcal{Q}_n^k$, each vertex of the graph will be denoted by the signed vector of its image. As usual, an embedding is said to be *isometric* if the graph distance of the domain is preserved by its image. An embedding $G \hookrightarrow \mathcal{Q}_n^k$ is said to be *antipodal* if it is closed by the antipodal automorphism of $\mathcal{Q}_n^k$, i.e. if $X \in V(G)$ then $-X \in V(G)$.

In order to isolate the problem, let us summarize our observations in the following

Lemma 6
I. *If $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d+2$ then there exists an antipodal embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ such that*

$$X \neq \pm Y \in V(\mathcal{G}) \Rightarrow d_{\mathcal{Q}_n^{n-d-2}}(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)| \qquad (*)$$

II. *If $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ is an antipodal embedding that satisfies $(*)$, then $V(\mathcal{G})$ fulfils the axioms $(C1)$, $(C2)$ and $(C3)$ of an oriented matroid.*

Proof.
I. The existence of the embedding results from Lemma 4. The embedding must by antipodal because of axiom $(C2)$; and axiom $(C3)$ together with

Lemma 5 implies the metric condition $(*)$.

II. The antipodality of the embedding implies that the vertices of $\mathcal{G}$ are the set of circuits of a separoid, and then $(C1)$ and $(C2)$ are fulfilled. It follows from Lemma 5 and condition $(*)$ that such separoid is a Radon separoid and therefore axiom $(C3)$ is fulfilled. □

Considering Lemma 6, we can see that to make a characterization of the circuit graphs of the uniform oriented matroids, in terms of an embedding in $\mathcal{Q}_n^{n-d-2}$, we need to focus our attention in translating the weak elimination axiom $(C4)$ into some aditional properties of the embedding different from the antipodality and the condition $(*)$. As we will see, Theorem 2 implies that to require of the embedding to be also isometric is a sufficient condition for a graph to be a circuit graph of a uniform oriented matroid, however, it is not a necessary condition, as the following picture (Figure 7) shows. In it, the vertices X and Y are non-antipodal (in fact we are depicting only the projective half of the oriented matroid); $|S(X,Y)| + \frac{1}{2}|T(X,Y)| = 2$ but $d_{\mathcal{G}}(X,Y) = 3$.

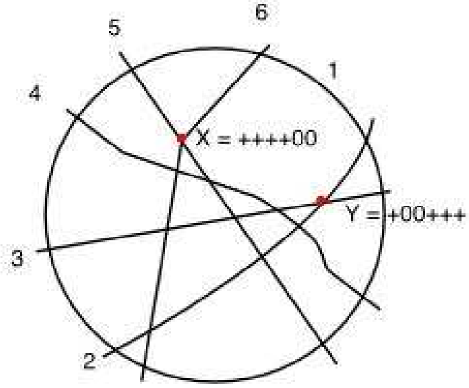

Figure 7. *An non-isometric embedding.*

A problem with the isometric embeddings is that we are requiring of every pair $X \neq \pm Y \in V(\mathcal{G})$ that there must be a XY-path $P \subset \mathcal{G}$ such that for every $Z \in V(P)$ it results that $d_{\mathcal{Q}_n^k}(Z,Y) < d_{\mathcal{Q}_n^k}(X,Y)$, and therefore

$$|S(Z,Y)| < |S(X,Y)| \quad \text{or} \quad |T(Z,Y)| < |T(X,Y)|.$$

Furthermore, if $P = (X, Z^1, ..., Z^m = Y)$, for every $\ell < m$,

$$d_{\mathcal{Q}_n^{n-d-2}}(X, Z^\ell) < d_{\mathcal{Q}_n^{n-d-2}}(X, Z^{\ell+1}) \quad \text{and} \quad d_{\mathcal{Q}_n^{n-d-2}}(Y, Z^{\ell+1}) < d_{\mathcal{Q}_n^{n-d-2}}(Y, Z^\ell).$$

In other words, from the point of view of $\mathcal{Q}_n^{n-d-2}$, at each and every step along P, its elements must get away from X and closer to Y. But oriented matroids are more flexible than that.

Let us introduce a weaker (hence more general) concept of metric embeddings. An embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ is said to be *i-metric* if for every pair of vertices $X \neq \pm Y \in V(\mathcal{G})$ there exists a XY-path $P = (X, Z^1, ..., Z^m = Y)$ in $\mathcal{G}$ such that for every pair of successive vertices $Z^\ell, Z^{\ell+1}$ of P, it happens that $(Z^{\ell+1})_i \in \{X_i, Y_i\}$ if $(Z^\ell)_i = (Z^{\ell+1})_j = 0 \neq (Z^\ell)_j$ (i.e., if we are moving a zero from i to j). In a certain way, every step takes "the right direction" in $\mathcal{Q}_n^k$. Then, for every $Z \in V(P)$ we have that

$$S(Z,Y) \subseteq S(X,Y) \quad \text{and} \quad |T(Z,Y)| \leq |T(X,Y)|,$$

and, as a consequence, $d_{\mathcal{Q}_n^k}(Z,Y) \leq d_{\mathcal{Q}_n^k}(X,Y)$. Such a path will be called an i-metric path. Clearly, an isometric embedding is i-metric.

Lemma 7 *Let $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ be an embedding. If $P \subset \mathcal{G}$ is an i-metric path from X to Y then for every $e \in S(X,Y)$ there exists a $Z \in V(P)$ such that $e \in Z^0$, $Z^- \subseteq X^- \cup Y^-$ and $Z^+ \subseteq X^+ \cup Y^+$.*

Proof. Since changing an element of the separator $S(X,Y)$ while walking in an XY-path into $\mathcal{Q}_n^k$, from one sign to the other requires to move the sign to zero and, after that, to the other sign, for every element in the separator there exists a vertex Z in any XY-path with a zero in that position. To see that such a vertex satisfies the extra sign conditions if we are in an i-metric path, let $e \in S(X,Y)$ and $P = (X = Z^0, Z^1, \ldots, Z^{m-1}, Z^m = Y)$ be an i-metric XY-path. Now, let us suppose that $(Z^s)^+ \not\subseteq X^+ \cup Y^+$, let $i \in (Z^s)^+ \setminus (X^+ \cup Y^+)$ and $\ell = \min\left\{j : (Z^j)_i = (Z^s)_i = +\right\}$. Since $X_i \neq (Z^\ell)_i$, clearly $\ell > 0$ thus, again by the Remark, it follows that $(Z^{\ell-1})_i = 0$. But then, the step $(Z^{\ell-1}, Z^\ell)$ is not allowed in an i-metric XY-path, and this is a contradiction. □

Theorem 2 *A graph $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d+2$ if and only if its order is $2\binom{n}{d+2}$ and there exists an antipodal i-metric embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ such that for every pair $X \neq \pm Y \in V(\mathcal{G})$, $d_{\mathcal{Q}_n^{n-d-2}}(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)|$.*

Proof. The sufficiency of Theorem 2 results from the order of $\mathcal{G}$ and Lemas 6 and 7. That is to say, given a graph $\mathcal{G}$ with the properties stated by

Theorem 2, it follows that $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d+2$.

For the necessity of Theorem 2, let $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ be the natural embedding of the circuit graph $\mathcal{G}$ of an uniform oriented matroid $\mathcal{M} = (E, \mathcal{C})$. By Lemma 6, it just remains to show that this embedding is i-metric. To do this, we will consider here three cases. For $X \neq \pm Y \in V(\mathcal{G})$,

Case 1 ($S(X,Y) = \phi = X^0 \cap Y^0$).

In this case their composition $\tau = X \circ Y$ is a tope of the oriented matroid. In the topological representation this tope is the ball that results of intersecting a number of closed semispaces (cf. [3] Proposition 4.3.6) and we may suppose, without loose of generality (via reorientation) —and with a little abuse of notation— that

$$\tau = \bigcap H_i^+ = + + \cdots +,$$

where $H_i^+ = \{V \in \mathcal{V}(\mathcal{M}) : V_i \in \{0,+\}\}$. Since the boundary of such a ball $\partial\tau$ is connected and it contains both X and Y, there exists a geodesic XY-path $P \subset \partial\tau$ in it. As we walk along this path from X to Y, the signs of the elements of the path are those of the tope. As a consequence, it follows that P is an i-metric path.

For the remaining two cases, we will proceed by induction. Clearly if $|E| = 3$ the embedding is i-metric (see Figure 3). Now, let us suppose that for every oriented matroid of order less than n such an embedding is i-metric and let $|E| = n$.

Case 2 ($X^0 \cap Y^0 \neq \phi$).

Let $j \in X^0 \cap Y^0$. Consider the matroid $\mathcal{M}' = \mathcal{M}/j$ and denote by $\mathcal{G}'$ its circuit graph. Also let X', Y' denote the contraction of the circuits X and Y. Clearly X' and Y' are two non-antipodal vertices and, by the induction hypothesis, there exists an i-metric path $P' = (X', Z^1, \ldots, Z^m = Y')$ in $\mathcal{Q}_n^{n-d-3}$. Now, if we restore the j-th coordinate to each vector Z^ℓ —i.e., if we consider the vectors $\hat{Z}^\ell \in \mathcal{C}$ for which $(\hat{Z}^\ell)_k = (Z^\ell)_k$ (for all $k \neq j$)—, each vector receives a zero in that coordinate. Therefore, if Z^ℓ and $Z^{\ell+1}$ are adjacents in $\mathcal{G}'$ then $\hat{Z}^\ell$ and $\hat{Z}^{\ell+1}$ are adjacents in $\mathcal{G}$. So, the path $P = (X, \hat{Z}^1, \ldots, \hat{Z}^m = Y)$ is an XY-path in $\mathcal{Q}_n^{n-d-2}$ and since P' is an i-metric path, it is not difficult to see that P is also an i-metric path.

Case 3 ($X^0 \cap Y^0 = \phi$ and $S(X,Y) \neq \phi$).

Let $k \in S(X,Y)$. Consider the matroid $\mathcal{M}' = \mathcal{M} \setminus k$ and denote by $\mathcal{G}'$ its circuit graph. Also let X', Y' denote the restriction of the circuits X and Y. Once again, X' and Y' are two non-antipodal vertices and, by the induction hypothesis, there exists an i-metric path $P' = (X', Z^1, \ldots, Z^m = Y')$. Let us restore the k-th coordinate to each vector Z^ℓ, and consider the resultant succesion $(X, \hat{Z}^1, \ldots, \hat{Z}^m = Y)$ in $\mathcal{G}$. If $\hat{Z}^\ell$ and $\hat{Z}^{\ell+1}$ are not adjacents in $\mathcal{G}$, since Z^ℓ and $Z^{\ell+1}$ are adjacent in $\mathcal{G}'$ (i.e., $S(Z^\ell, Z^{\ell+1}) = \phi$ and $|T(Z^\ell, Z^{\ell+1})| = 2$), we must have that

$$S(\hat{Z}^\ell, \hat{Z}^{\ell+1}) = \{k\} \quad \text{and} \quad |T(\hat{Z}^\ell, \hat{Z}^{\ell+1})| = 2.$$

Let $T(\hat{Z}^\ell, \hat{Z}^{\ell+1}) = \{r, s\}$. Now, due to the week elimination axiom, there exists a circuit $W \in \mathcal{C}$ such that $k \in W^0$, $W^+ \subseteq (\hat{Z}^\ell)^+ \cup (\hat{Z}^{\ell+1})^+$ and $W^- \subseteq (\hat{Z}^\ell)^- \cup (\hat{Z}^{\ell+1})^-$. Observe then that for every $i \notin \{r, s, k\}$, $W_i = (\hat{Z}^\ell)_i = (\hat{Z}^{\ell+1})_i$ and so, without loose of generality, let $0 = (\hat{Z}^\ell)_r \neq W_r = (\hat{Z}^{\ell+1})_r$ and $0 = (\hat{Z}^{\ell+1})_s \neq W_s = (\hat{Z}^\ell)_s$.

From here it is not difficult to notice the following three facts:

(i) W is adjacent to both $\hat{Z}^\ell$ and $\hat{Z}^{\ell+1}$, and therefore $(\hat{Z}^\ell, W, \hat{Z}^{\ell+1})$ is a path in $\mathcal{G}$.

(ii) If $(\hat{Z}^\ell)_i = W_j = 0 \neq (\hat{Z}^\ell)_j$, then $j = k$, $i \neq s$ and as a result $W_i = (Z^{\ell+1})_i \in \{X_i, Y_i\}$.

(iii) If $W_i = (Z^{\ell+1})_j = 0 \neq W_j$, then $j = s$ and $i \neq r$. Since $k \in S(X, Y)$, if $i = k$ then $(Z^{\ell+1})_i \in \{X_i, Y_i\}$; otherwise $W_i = (Z^\ell)_i$ and $W_j = (Z^\ell)_j$. Accordingly, we have that $(Z^\ell)_i = (Z^{\ell+1})_j = 0 \neq (Z^\ell)_j$ which on its turn implies that $(Z^{\ell+1})_i \in \{X_i, Y_i\}$ since P' is i-metric.

In this way we can construct a XY-path in $\mathcal{G}$ from the succesion $(X, \hat{Z}^1, \ldots, \hat{Z}^m = Y)$ which is i-metric since P' is i-metric. □

This theorem leads to a new axiomatization of uniform oriented matroids. However, the hypothesis of uniformity cannot be dropped without adding a new ingredient because otherwise the circuit graph of a non-uniform oriented matroid may not be embeddable in $\mathcal{Q}_n^k$. We believe that there should be a notion of distance related to the first baricentric subdivision of the n-cube that leads to a similar theorem but for the general (non-uniform) case.

4 The Distance Lemma

Proof of Lemma 5. Let $X, Y \in V(\mathcal{Q}_n^k)$. First of all, we exhibit a XY-path with the desired length —this will show that the distance in $\mathcal{Q}_n^k$ is at most that of the statement. There are four cases:

Case 1 ($S(X,Y) = \phi$ and $T(X,Y) = \phi$).

This condition is equivalent to $X = Y$.

Case 2 ($S(X,Y)=\phi$ and $T(X,Y)\neq\phi$).

Let $T_0(X,Y)=\{i\in E : X_i=0\neq Y_i\}$ (analogously $T_0(Y,X)=\{i\in E : Y_i=0\neq X_i\}$). Clearly $T(X,Y)=T_0(X,Y)\cup T_0(Y,X)$ and, since they have the same support size, $|T_0(X,Y)|=|T_0(Y,X)|$. Let us assign an arbitrary (but fixed) linear order to both previously defined sets: $T_0(X,Y)=(\tau_1,\ldots,\tau_{|T_0(X,Y)|})$ and $T_0(Y,X)=(\pi_1,\ldots,\pi_{|T_0(Y,X)|})$. Now, let $\{Z^1,Z^2,\ldots,Z^{\frac{1}{2}|T(X,Y)|}\}$ be defined as follows:

$$(Z^m)_i=\begin{cases} Y_i & \text{if } i\in\{\tau_1,\ldots,\tau_m,\pi_1,\ldots,\pi_m\},\\ X_i & \text{otherwise.}\end{cases}$$

Observe that

$$S(X,Z^1)=S(Z^1,Z^2)=\ldots=S(Z^{\frac{1}{2}|T(X,Y)|-1},Y)=\phi,$$

$$|T(X,Z^1)|=|T(Z^1,Z^2)|=\ldots=|T(Z^{\frac{1}{2}|T(X,Y)|-1},Y)|=2,$$

and $Z^{\frac{1}{2}|T(X,Y)|}=Y$. Therefore, by the Remark, $(X,Z^1,Z^2,\ldots,Z^{\frac{1}{2}|T(X,Y)|}=Y)$ is a XY-path and its length is $\frac{1}{2}|T(X,Y)|$.

Case 3 ($S(X,Y)\neq\phi$ and $T(X,Y)\neq\phi$).

Let us assign an arbitrary (but fixed) linear order to the separator: $S(X,Y)=(\sigma_1,\ldots,\sigma_{|S(X,Y)|})$, and let $\{Z^1,Z^2,\ldots,Z^{|S(X,Y)|}\}$ be defined as follows:

$$(Z^m)_i=\begin{cases} Y_i & \text{if } i\in\{\tau_1,\sigma_1,\ldots,\sigma_{m-1}\},\\ 0 & \text{if } i=\sigma_m,\\ X_i & \text{otherwise.}\end{cases}$$

Observe that,

$$S(X,Z^1)=S(Z^1,Z^2)=\ldots=S(Z^{|S(X,Y)|-1},Z^{|S(X,Y)|})=\phi,$$

$$|T(X,Z^1)|=|T(Z^1,Z^2)|=\ldots=|T(Z^{|S(X,Y)|-1},Z^{|S(X,Y)|})|=2.$$

Furthermore, $S(Z^{|S(X,Y)|},Y)=\phi$ and

$$|T(Z^{|S(X,Y)|},Y)|=|T(X,Y)\setminus\{\tau_1\}\cup\{\sigma_{S(X,Y)}\}|=|T(X,Y)|.$$

Now, let us construct a $Z^{|S(X,Y)|}Y$-path as in the previous case. As it can be seen, both paths combined become an XY-path of the desired length.

Case 4 ($S(X,Y)\neq\phi$ and $T(X,Y)=\phi$).

Let $i_0\in X^0=Y^0$ be arbitrary (but fixed) and let Z^1 be defined as follows

$$(Z^1)_i=\begin{cases} 0 & \text{if } i=\sigma_1,\\ + & \text{if } i=i_0,\\ X_i & \text{otherwise.}\end{cases}$$

Observe that $S(Z^1, Y) = S(X, Y) \setminus \{\sigma_1\}$ and $T(Z^1, Y) = \{\sigma_1, i_0\}$, therefore the previous cases apply.

To finish the proof, we have to show that the distance in $\mathcal{Q}_n^k$ is at least that one stated by the Lemma. We do it by induction.

Let $d\colon V(\mathcal{Q}_n^k) \times V(\mathcal{Q}_n^k) \to I\!N$ be the following function

$$d(X,Y) = \begin{cases} |S(X,Y)| + 1 & \text{if } X^{\pm} = Y^{\pm}, \\ |S(X,Y)| + \frac{1}{2}|T(X,Y)| & \text{otherwise.} \end{cases}$$

By the Remark, it follows that $d(X, Y) = 1$ if and only if $d_{\mathcal{Q}_n^k}(X, Y) = 1$. Let suppose that for every X, Y and for every $m < m_0$, we have that $d(X, Y) = m$ if and only if $d_{\mathcal{Q}_n^k}(X, Y) = m$. Let $(X, Z^1, \ldots, Z^{m_0} = Y)$ be a geodesic XY-path (of length $d_{\mathcal{Q}_n^k}(X, Y)$). We want to prove that $d(X, Y) \leq m_0$ so, suppose that $d(X, Y) > m_0$.

Since the path is geodesic, it follows that $d_{\mathcal{Q}_n^k}(X, Y) = d_{\mathcal{Q}_n^k}(X, Z^1) + d_{\mathcal{Q}_n^k}(Z^1, Y)$ which by hypothesis implies that $m_0 = 1 + d(Z^1, Y)$, hence $d(X, Y) > 1 + d(Z^1, Y)$.

Let

$$\delta_{XY} = \begin{cases} 1 & \text{if } X^{\pm} = Y^{\pm}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can denote $d(X, Y) = |S(X, Y)| + \frac{1}{2}|T(X, Y)| + \delta_{XY}$ including in one equation both cases of $d(X, Y)$ definition. Recall that $X^{\pm} = Y^{\pm}$ if and only if $T(X, Y) = \phi$.

With this notation at hand we have that

$$|S(X,Y)| + \frac{1}{2}|T(X,Y)| + \delta_{XY} > 1 + |S(Z^1,Y)| + \frac{1}{2}|T(Z^1,Y)| + \delta_{Z^1Y}.$$

Since X is adjacent to Z^1, there exist $i, j \in E$ such that for every $\ell \notin \{i, j\}$; $X_\ell = (Z^1)_\ell$, $X_i = 0 \neq (Z^1)_i$ and $X_j \neq 0 = (Z^1)_j$. Then $S(X, Y)$ and $S(Z^1, Y)$, and respectively $T(X, Y)$ and $T(Z^1, Y)$, differ only in the i-th and j-th coordinates. This motivates the following notation: Given $F \subseteq E$, let $S_F(X, Y) = F \cap S(X, Y)$ and $T_F(X, Y) = F \cap T(X, Y)$. Thus, we have that

$$|S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| + \delta_{XY} > 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)| + \delta_{Z^1Y}.$$

We consider two cases:

Case 1 ($T_{ij}(X, Y) = \phi$).

Since $X_i = 0 \neq (Z^1)_i$ and $X_j \neq 0 = (Z^1)_j$, then $Y_i = 0$ and $Y_j \neq 0$ therefore, $\{i, j\} \subset T_{ij}(Z^1, Y)$ and $i \notin S_{ij}(X, Y)$. But

$$2 \geq |S_{ij}(X,Y)|+\frac{1}{2}|T_{ij}(X,Y)|+\delta_{XY} > 1+|S_{ij}(Z^1,Y)|+\frac{1}{2}|T_{ij}(Z^1,Y)|+\delta_{Z^1Y} \geq 2,$$

an obvious contradiction.

Case 2 ($T_{ij}(X,Y) \neq \phi$).

Clearly, in this case, $\delta_{XY} = 0$. Then

$$|S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| > 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)|.$$

Since $X_i = 0$ then $i \notin S_{ij}(X,Y)$, in consequence $j \in S_{ij}(X,Y)$. Therefore $j \in T_{ij}(Z^1,Y)$, which implies that

$$1+\frac{1}{2} \geq |S_{ij}(X,Y)|+\frac{1}{2}|T_{ij}(X,Y)| > 1+|S_{ij}(Z^1,Y)|+\frac{1}{2}|T_{ij}(Z^1,Y)| \geq 1+\frac{1}{2},$$

a new contradiction.

This concludes the proof. $\square$

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