Midsummer Combinatorial Workshop VIII

Prague
July 30 – August 3, 2001

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Preface

The Eighth Prague Midsummer Combinatorial Workshop was held from July 30 to August 3, 2001 at Malostranské náměstí building of Charles University which is depicted on the cover of this publication. The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA centre. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Colin McDiarmid and Imre Bárány among us.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students, and as in the last two years, 5 selected undergraduate students from the USA and 3 students from Charles University took part in the workshop, together with student guides Ondřej Pangrác and Clifford Smyth, in the framework of the joint DIMATIA-DIMACS REU program.

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest from these proceedings some of the atmosphere at the workshop and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

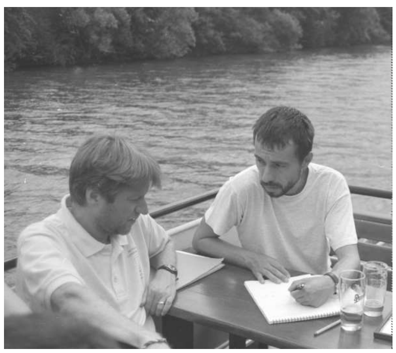
This volume was edited by Petra Smolíková. Most of the problems described here were supplied by the authors in electronic form; in a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

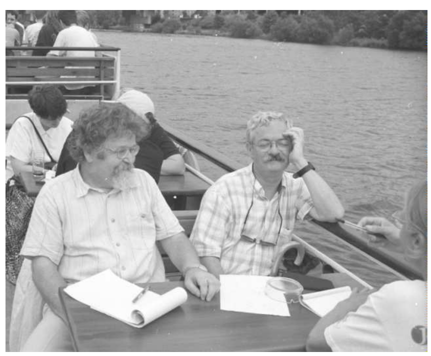
The conference photos were taken on the river boat which took participants to Zbraslav. We have seen there the new instalation of the internationally known collection of oriental art of National Gallery. Also, the conference dinner was held in a nearby restaurant.

This summer workshop was partly supported by a Charles University grant GAUK 158, and Kontakt 337. The support of our new institute ITI (financed by the Ministry of Education of the Czech Republic as project LN00A056) was instrumental for the success of the workshop.

Based on our past experience and being encouraged by several participants, we hope to organize the Ninth Prague Combinatorial Workshop in the summer of 2002. We hope to meet you all there!

Jaroslav Nešetřil





The minimum area convex lattice n-gon

Imre Bárány

Let $A(n)$ be the minimum area a convex lattice n-gon can have. It is known that

$$\frac{1}{8\pi^2} < \frac{A(n)}{n^3} \le \frac{1}{48}(1 + o(1)).$$

We prove here that $\lim A(n)/n^3$ exists. This is done in a series of steps the first of which is to reformulate the problem considering the edges of the minimizing polygon. This reformulation makes it possible to use number theoretic tools and symmetrization of convex bodies. As a corollary we get that

$$\frac{A(n)}{n^3} \ge \left(\frac{1}{54} - D\frac{\log b}{b}\right)$$

where D is a constant and $b = b(n)$ is the lattice width of the edge set of the minimizing polygon. This shows that the limit in question is $1/54 = 0.0185185...$ if $b(n)$ goes to infinity, in which case the minimizing polygon would be very close to a disk. But we show that this is not the case. We do so by finding an infinite sequence of convex lattice n-gons, P_n, such that their areas are asymptotically $0.0185068n^3$. We also prove that the limit of $A(n)/n^3$ is equal to the minimum of finitely many minimization problems. (But finitely many is about 10^{10}, too many to solve.) It follows that the true minimizer is of oblong shape: it is c_1n wide in its lattice width direction, and it is c_2n^2 long in the direction, perpendicular to its lattice width. This is joint work with N. Tokushige.

Sets of natural numbers with bounded gap size

Tom C. Brown

Let $\mathbb{N}$ denote the set of natural numbers. Let A be a finite subset of $\mathbb{N}$. We say that the *gap size* of A is d, or $gs(A) = d$, if $A = \{a_1 < a_2 < \dots a_n\}$ and $d = \max\{a_{j+1} - a_j; 1 \le j \le n-1\}$. (If $|A| = 1$, set $gs(A) = 1$.)

It is well known that if $\mathbb{N}$ is finitely colored, then there exist a fixed $d \ge 1$ and arbitrarily large (finite) monochromatic sets A with $gs(A) = d$. (This result has applications in semigroup theory and ergodic theory. There is a natural multi–dimensional generalization.)

The talk had two parts. In the first part, the above result is generalized to trees and forests of finite subsets of $\mathbb{N}$, by combining it with the infinite Ramsey theorem.

In the second part, the still open problem of finding a "canonical" version of the above result is discussed. The problem is to find a minimal family of "patterns" of colorings such that whenever an arbitrary coloring of $\mathbb{N}$ is given (using a finite or infinite number of colors), there exist a fixed $d \geq 1$ and arbitrarily large (finite) sets A with $gs(A) = d$ such that the restriction of the given coloring to A belongs to this family of patterns.

Clearly the 1–1 coloring and constant coloring are necessary patterns. An example is given showing that these are not sufficient. That is, a coloring $f : \mathbb{N} \to \mathbb{N}$ is described, such that whenever $d \geq 1$ is given, there do not exist arbitrarily large sets A, with $gs(A) = d$, such that f restricted to A is either 1–1 or constant.

The interlace polynomial

Zdeněk Dvořák

A new graph polynomial, called the interlace polynomial is defined in [1] for circle graphs based on reflections about the number of Euler circuits in 2-in, 2-out digraphs. This definition surprisingly extendeds to any undirected graph. We showed several properties of this polynomial and revealed meaning of coefficients of this polynomial for general graphs. We presented several questions (most of them coming from [1]) – algorithmic (What is the complexity of counting coefficients of the polynomial, and of evaluating it?) and structural (Is sequence of coeficients of interlace polynomial of any graph unimodal?).

References

[1] ARRATIA R., BOLLOBÁS B., SORKIN G. B., *The Interlace Polynomial: A New Graph Polynomial*, SODA 2000, 237–245.

Results and conjectures on a paint shop problem for words

Thomas Epping

Motivated by an application in the automobile industry, we introduce a new combinatorial problem.

We are given a word $w = (w_1, w_2, \ldots w_n)$ and a color vector $f = (f_1, f_2 \ldots f_n)$ with f_i denoting the color of w_i. Our objective is to find a permutation σ:$\{1, 2 \ldots n\}$ $\rightarrow \{1, 2 \ldots n\}$ so that $w_{\sigma(i)} = w_i$ for all i, and the number of color changes within $\sigma(f) = (f_{\sigma(1)}, \ldots f_{\sigma(n)})$ is minimized.

We show that this problem is polynomially (but not efficiently) solvable if we bound both the number of different letters in w and the number of different colors, and is NP–complete otherwise.

We then focus on k–regular instances, where each letter is available in each color k times (with $k \geq 1$ being a fixed integer), and present some results and conjectures on upper bounds for the minimal number of color changes.

We end with a list of open questions, including the correctness of the proposed conjectures and the approximation (or the exact and efficient computation) of an optimal coloring for k–regular instances.

Connectivity and minors

Gašper Fijavž

We call a graph G *minor-minimal k-connected* if G is k-(vertex)connected yet G does not contain a proper k-connected minor. It is well known [Tutte; Halin, Jung] that K_4 is the only minor-minimal 3-connected graph and K_5 and K_{222} are the only minor-minimal 4-connected graphs. The class of minor-minimal 5-connected graphs is, however, not known. Dirac has shown that icosahedron is the only planar minor-minimal 5-connected graph. We determine projective-planar minor-minimal 5-connected graphs. Namely, every 5-connected projective planar graph which embeds into projective plane with face-width at least 3 contains K_6 as a minor. And if a 5-connected graph embeds into projective plane with face-width at most 2 then it contains icosahedron, G_0 or G_1 as a minor.

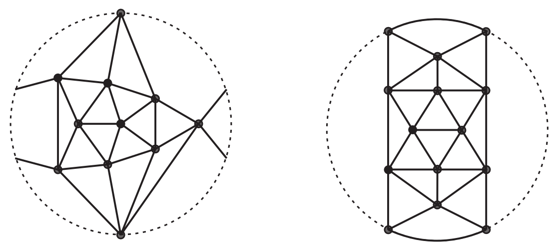


Figure 1: The graphs G_0 and G_1.

Erdős–Szekeres theorem with forbidden order types

Gyula Károlyi

The classical Erdős–Szekeres theorem states that *for every integer $n \geq 3$ there is an N_0 such that, among any set of $N \geq N_0$ points in general position in the plane, there is the vertex set of a convex n-gon.* Reversing the question one may ask, for any $n \geq 3$, for the largest number $f(n)$ such that any set of n points in general position in the plane contains the vertex set of a convex $f(n)$-gon. It is known that $f(n) = \Theta(\log n)$.

The relation between the Erdős–Szekeres problem and Ramsey's theorem has a wide literature. From the viewpoint of Ramsey theory, the vertex set of a convex polygon (which we will simply refer to as a convex polygon) can be called as a *homogeneous* subset of the underlying set. The Erdős–Szekeres theorem claims that no sufficiently large set can avoid 'large' homogeneous subsets. In classical graph-Ramsey theory, homogeneous subsets of the vertex set of a graph are those sets which induce either complete or empty subgraphs, that is, cliques and independent sets. According to a conjecture of Erdős and Hajnal, for every graph H there is a positive constant $\varepsilon = \varepsilon(H)$ such that every graph on n vertices which does not contain H as an induced subgraph contains a large homogeneous set whose size is at least n^{ε}. This conjecture is verified only for certain classes of graphs, and such graphs are said to possess the *Erdős–Hajnal property*. In this talk we extend these notions in the context of the Erdős–Szekeres problem.

Throughout this paper we will always assume that every point set is in general position in the plane, that is, no three points of the configuration are collinear.

Two such configurations are said to be of the same *order type* if there is a one-to-one correspondence between them which preserves the orientation of each triple. Thus, order types are equivalence classes of configurations. We will say that the configuration P contains the order type $\mathcal{T}$ if there is a subset of P which belongs to $\mathcal{T}$. Ramsey theoretic aspects of order types have already been studied by Nešetřil and Valtr.

Now let $\mathcal{T}$ be a fixed order type which is not in convex position, or more generally, let $\mathcal{T}$ be an arbitrary family of such order types. Define, for $n \geq 3$, $f_{\mathcal{T}}(n)$ as the largest number such that any set of n points in general position in the plane, which does not contain $\mathcal{T}$, contains the vertex set of a convex $f(n)$-gon. Clearly $f_{\mathcal{T}}(n) \geq f(n)$, and $f_{\mathcal{T}}$ is a monotone increasing function.

We say that $\mathcal{T}$ has the *Erdős–Hajnal property* if there exists a positive constant ε, depending only on $\mathcal{T}$, such that $f(n) > n^{\varepsilon}$. If in addition there exists a positive constant c such that $f_{\mathcal{T}}(n) > cn$, then we say that $\mathcal{T}$ possesses the *strong Erdős–Hajnal property*. The aim of this talk is to study which order types admit the strong Erdős–Hajnal property, and to see if every order type has the Erdős–Hajnal property, a question raised also by Gil Kalai.

Let $k \geq 3$ and $E = \{a, b_1, b_2, \ldots, b_k\}$ be a configuration such that $b_1 b_2 \ldots b_k$ is a convex k-gon inside the triangle ab_1b_k. Then E belongs to a unique order type that we denote by $\mathcal{E}_k$. If, for example, $\mathcal{T} = \mathcal{E}_3$, it is clear that $f_{\mathcal{T}}(n) = n$ for every $n \geq 3$.

Theorem *Let $\mathcal{T}$ be any order type whose convex hull is a triangle. $\mathcal{T}$ has the strong Erdős–Hajnal property if and only if $\mathcal{T} = \mathcal{E}_k$ for some integer $k \geq 3$.*

Corollary *If an order type $\mathcal{T}$ has the strong Erdős–Hajnal property then for every order type $\mathcal{S} \subseteq \mathcal{T}$ whose convex hull is a triangle there is an integer $k \geq 3$ such that $\mathcal{S} = \mathcal{E}_k$.*

In fact, we give a somewhat stronger necessary condition and also show that $f_{\mathcal{S}}(n) = O(\sqrt{n})$ holds for every order type $\mathcal{S}$ that does not satisfy this condition.

A class of order types which have the Erdős–Hajnal property is what we call half-moons. Let $k, \ell \geq 3$ and $F = \{a_1, a_2, \ldots, a_\ell, b_2, \ldots, b_{k-1}\}$ be a configuration such that $a_1 a_2 \ldots a_\ell$ is a convex ℓ-gon containing $b_2, \ldots, b_{k-1}$ and $a_1 b_2 \ldots b_{k-1} a_\ell$ is a convex k-gon with lines $a_1 b_2$ and $a_\ell b_{k-1}$ intersecting segments $a_\ell a_{\ell-1}$ and $a_1 a_2$, respectively. In particular, $\{a_1, a_i, a_\ell, b_2, \ldots, b_{k-1}\}$ is of type $\mathcal{E}_k$ for every $1 < i < \ell$. Then F belongs to a unique order type that we denote by $\mathcal{F}_{k,\ell}$. Thus, $\mathcal{F}_{k,3} = \mathcal{E}_k$.

Theorem *Every order type $\mathcal{F}_{k,\ell}$ ($k, \ell \geq 3$) has the Erdős–Hajnal property.*

There are, however, order types which do not have the Erdős–Hajnal property. The following result indicates that the analogue of the Erdős–Hajnal conjecture is not true for order types in general. It may even be the case that $f_{\mathcal{T}} = f$ for the configuration $\mathcal{T}$ in the following theorem.

Theorem *There is an order type $\mathcal{T}$ such that $f_{\mathcal{T}}(n) < \log n + 2$. More precisely, for every integer $n \geq 4$ there is a configuration of 2^{n-2} points in general position in the plane which contains neither $\mathcal{T}$ nor a convex n-gon.*

Open Problem *Characterize those configurations wich possess the strong Erdős–Hajnal property. In particular, does $\mathcal{F}_{4,4}$ have this property?*

This talk is based on a joint work with József Solymosi.

Growth rates of hereditary classes of unordered and ordered graphs

Martin Klazar

A *hereditary class of graphs* or a *hereditary graph property* is a set $\mathcal{P}$ of finite simple undirected graphs such that $V(G) \subset \mathbf{N}$ ($\mathbf{N} = \{1, 2, \ldots\}$) for every $G \in \mathcal{P}$ and

- $\mathcal{P}$ is closed to induced subgraphs: If $G \in \mathcal{P}$ and H is a graph such that $V(H) \subset \mathbf{N}$ and an injection $f : V(H) \to V(G)$ exists with $\{x, y\} \in E(H) \Leftrightarrow \{f(x), f(y)\} \in E(G)$, then $H \in \mathcal{P}$.

In particular, $\mathcal{P}$ is closed to isomorphism. For a hereditary class of graphs $\mathcal{P}$ and $n \in \mathbf{N}$, let $\mathcal{P}_n = \{G \in \mathcal{P} : V(G) = \{1, 2, \ldots, n\}\}$. How fast the counting function $n \mapsto |\mathcal{P}_n|$ can grow? Intuitively, $|\mathcal{P}_n|$ is the number of labelled graphs on n vertices with property $\mathcal{P}$. In this generality the question was addressed only in 1994 by Scheinerman and Zito ([3]) who proved that either (i) (constant growth) $|\mathcal{P}_n|$ is for $n > n_0$ constantly $0, 1$ or 2 or (ii) (polynomial growth) $c_1 n^k < |\mathcal{P}_n| < c_2 n^k$ with constants $0 < c_1 < c_2$ and $k \in \mathbf{N}$ or (iii) (exponential growth) $k^n / p_1(n) < |\mathcal{P}_n| < k^n \cdot p_2(n)$ with some polynomials p_i and a constant $k \in \mathbf{N}$, $k > 1$, or (iv) (factorial growth) $n^{c_1 n} < |\mathcal{P}_n| < n^{c_2 n}$ with constants $0 < c_1 < c_2$ or, finally, (v) (superfactorial growth) $|\mathcal{P}_n| > n^{cn}$ for

$n > n_0$ for every constant $c > 0$. These results were made much more precise by Balogh, Bollobás and Weinreich ([1]) who proved that either (i) (at most exponential growth) $|\mathcal{P}_n| = \sum_{i=1}^{k} p_i(n) i^n$ for $n > n_0$ with some k (possibly zero) polynomials p_i or (ii) (factorial growth) $|\mathcal{P}_n| = n^{(1-1/k+o(1))n}$ for a constant $k \in \mathbf{N}$, $k > 1$, or (iii) (the penultimate rate of growth) $n^{(1+o(1))n} < |\mathcal{P}_n| < 2^{o(n^2)}$ or, finally, (iv) (k-partite growth) $|\mathcal{P}_n| = 2^{(1-1/k+o(1))n^2/2}$ for a constant $k \in \mathbf{N}$, $k > 1$. Notice that, remarkably, in the case (i) they give no estimate but an exact formula. They investigate region (iii) further in [2].

Problem. Determine growth rates of hereditary classes of *ordered* graphs.

"Ordered" means this: Everything is as before, "only" the injection $f : V(H) \to V(G)$ is now required to be increasing with respect to the standard linear order of $\mathbf{N}$. As an example, consider the set $\mathcal{Q}$ of all graphs which have as an ordered induced subgraph none of the following six ordered graphs: $(\{1,2\},\{1,3\})$, $(\{1,3\},\{2,3\})$, $(\{1,2\},\{2,3\})$, $(\{1,3\},\{1,2\},\{2,3\})$, $(\{1,3\},\{2,4\})$, and $(\{1,4\},\{2,3\})$. Equivalently, $G \in \mathcal{Q}$ iff G is composed of some components $C_1 < C_2 < \ldots < C_k$ where each C_i is either a single vertex or an edge. Clearly, $\mathcal{Q}$ is a hereditary class of ordered graphs. By induction, $|\mathcal{Q}_n| = F_n$, the nth Fibonacci number. Fibonacci numbers $(F_n)_{n\geq 1} = (1, 2, 3, 5, 8, 13, 21, \ldots)$ grow like $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$. Thus the base of the exponential is now nonintegral. For exponential growth rates of ordered graphs subtler picture is to be expected.

References

[1] J. Balogh, B. Bollobás and D. Weinreich, The speed of hereditary properties of graphs, *J. Comb. Theory, Ser. B* **79** (2000), 131–156.

[2] J. Balogh, B. Bollobás and D. Weinreich, The penultimate rate of growth for graph properties, *Eur. J. Comb.* **22** (2001), 277–289.

[3] E. R. Scheinerman and J. Zito, On the size of hereditary classes of graphs, *J. Comb. Theory, Ser. B* **61** (1994), 16–39.

Bipartite subgraphs of planar graphs

Daniel Kráľ

It follows from the four colour theorem that it is enough to remove $1/3$ of the edges of a planar graph in order to make it bipartite. This fact can be actually proved without the four colour theorem by considering the dual graph of a planar triangulation with weighted edges and observing that the dual graph contains a perfect matching with a sum of weights at most $1/3$ of the total sum of its edge weights. This suggests a polynomial time algorithm for finding a small set of edges whose removal makes a planar graph bipartite. We posted the following problem: Is there a linear time algorithm for planar graphs which finds a set of edges of size at most $m/3$ (where m is the number of the edges) such that the removal of these edges makes the graph bipartite?

On convex polytopes in the plane "containing" and "avoiding" zero

Daniel Krasner
(with Alexander Kelmans and Ricardo Collado)

The main motivation for considering our problem is the study of monotone Boolean functions. It is known and easy to show that a monotone Boolean function f is uniquely defined by the family T_f of so–called minimal true sets as well as by the family F_f of so–called maximal false sets of variables of the function. One natural problem is to generate "efficiently" all minimal true sets (the elements of T_f) or all maximal false sets (the elements of F_f) or both (the elements of $F_f \cup T_F$).

Due to Gurvich and Khachiyan (1995) and Fredman and Khachiyan (1996) there exists a quasi–polynomial "time" algorithm to generate incrementally all minimal true and maximal false sets of a monotone Boolean function, i.e. the elements of $F_f \cup T_f$. It is not known whether or not there exists a polynomial or quasi–polynomial time algorithm to generate the elements T_f (the same is true for F_f).

Suppose that a monotone Boolean function satisfies the condition:

(C) there exists a polynomial function p such that $p(|T_f|) \geq |F_f|$.

In other words, the number T_f of minimal true sets grows at least as a polynomial function of the number T_f of maximal false sets. Then the above mentioned algorithm allows also to generate all minimal true sets (the elements T_f) in quasi–polynomial time. For that reason it is interesting to describe some natural classes of monotone Boolean functions satisfying condition (C).

A set X of points in the plane is called *convex* if for every two points a, b in X the set of all points on the line segment, joining a and b, belongs to X. Let S be a nonempty finite set of points in the plane. The *convex hull* of S is a convex set H containing S and minimal by inclusion (i.e. H has no proper convex subset containing S). The notions of a convex set and the convex hull of a set is defined similarly for a space of any given dimension. Let S be a finite set of points in the plane, $X \subseteq S$, and z be a point not in S.

- X is a z–containing set if z is in the interior of the convex hull of X.
- A z–containing set X is minimal in S if X has no proper z–containing subset.
- X is a z–avoiding set if z is not contained in the interior of the convex hull of X.
- A z–avoiding set X is maximal in S if there is no z–avoiding set containing X properly.

Let $C(S)$ and $A(S)$ denote the sets of minimal z–containing and maximal z–avoiding subsets of S, respectively. The families $C(S)$ and $A(S)$ can be interpreted as the families of minimal true sets and maximal false sets, respectively, of some monotone Boolean function f_S.

E. Boros and V. Gurvich raised the following question:

Question 1. *Is it true that $|A(S)| \leq 2d \cdot |C(S)|$, where d is the dimension of the space?*

If the above inequality holds then the function f_S satisfies condition (C). In the case when $d = 2$, i.e. for the plane, we proved the following:

Theorem 2. *Let S be a finite set of points in the plane and z be a point not in S. Suppose that z is in the interior of the convex hull of S. Then*

$$|A(S)| \leq 3 \cdot |C(S)| + 1$$

and the equality holds if and only if $|S| = 4$, the convex hull of S is a quadrilateral Q, and z is the point of intersection of the two diagonals of Q.

Using a different approach, A. Kelmans and A. Rubinov proved the following more general result.

Theorem 3. [A. Kelmans, A. Rubinov] *Let S be a finite set of points in the d–dimensional space and z be a point that belongs to no hyperplane containing at least two points from S. Suppose that z is in the interior of the convex hull of S. Then*

$$|A(S)| \leq d \cdot |C(S)| + 1$$

and the equality holds if and only if $|S| = d + 1$ and the convex hull of S is a $(d+1)$–dimensional simplex.

Theorem 3 implies $|A(S)| \leq 2d \cdot |C(S)|$, giving a positive answer to the question raised by E. Boros and V. Gurvich for points in general position.

Complexity of partial covers of small graphs

Jan Kratochvíl

Conjecture [Kratochvíl, Fiala]: For every three distinct odd nonnegative integers a, b, c such that $GCD(a, b, c) = 1$, there exists an integer m such that the equation

$$xa + yb + zc = m$$

with x, y, z being nonnegative integers

1. has at least one solution such that $x = y + z + 1$,
2. has at least one solution such that $y = x + z + 1$,
3. has at least one solution such that $z = x + y + 1$, and
4. has no solution satisfying $2\max\{x, y, z\} \leq x + y + z$.

Let $P(a, b, c)$ denote the graph consisting of two vertices of degree three connected by paths of lengths a, b and c (all the inner vertices of each path have degrees two). If the above conjecture is true then we can prove that for every three distinct odd integers a, b, c the question if a given graph allows a locally injective homomorphism into $P(a, b, c)$ is NP-complete. This question is of interest because of connection to generalized channel assignment problem and distance constrained labellings of graphs. E.g., locally injective homomorphism onto $P(1, 2, 3)$ is equivalent to an $L(2, 1)$-labelling of the input graph (a labelling by integers such that labels of adjacent vertices differ by at least two and vertices at graph distance two receive distinct labels).

References

[1] J. FIALA, T. KLOKS AND J. KRATOCHVÍL, *Fixed-parameter complexity of λ-labelings*, Discrete Applied Math. 113 (2001), 59-72.

[2] J. FIALA AND J. KRATOCHVÍL, *Complexity of partial covers of graphs*, accepted to ISAAC'01.

On the span in channel assignment problems

Colin McDiarmid

The *channel assignment problem* involves assigning radio channels to transmitters, using a small span of channels but without causing interference. We first describe a standard model for channel assignment, the *constraint matrix model*, which extends ideas of graph colouring.

Given a graph $G = (V, E)$ and a weight or length $l(uv)$ for each edge uv of G, we call an assignment $\phi : V \to \{1, \ldots, t\}$ *feasible* if $|\phi(u) - \phi(v)| \geq l(uv)$ for each edge uv. The nodes correspond to transmitters, and the lengths $l(uv)$ specify minimum channel separations to avoid interference. The least t for which there is a feasible assignment is the *span* of the problem. When each edge length is 1 this is just the chromatic number $\chi(G)$.

We discuss computing the span and bounds on the span. In particular, we consider two topics. The first is an extension of the Gallai-Roy Theorem, which relates the chromatic number of a graph to path lengths in orientations of the graph. The second concerns sequential methods for assigning channels, and we see that the span is at most $1 + \Delta_l(G)$, where the 'weighted maximum degree' $\Delta_l(G)$ is the maximum over all nodes v of the sum of the weights of the edges incident with v. This upper bound of course corresponds to the bound $\chi(G) \leq 1 + \Delta(G)$. (When can we save the '+1'?)

Bounds for graphs

Jaroslav Nešetřil

Several of my problems which were presented at the earlier workshops have been successfully solved. I want to report here on one such example which has been finished during this workshop.

Theorem 1 (P. Ossona de Mendez, J. Nešetřil). *For every $k \geq 3$ and for every proper minor closed class $\mathcal{K}$ of graphs there exists a graph $H(k, \mathcal{K}) = H$ with the following properties:*

(i) H is K_k–free;
(ii) For every K_k–free graph G, $G \in \mathcal{K}$, there exists a homomorphism from G to H.

(This has been asked earlier for the class of planar graphs and $k = 3, 4$; note that for $k = 5$ the graph can be constructed without reference to 4–Colour Theorem. Thus our theorem is sandwiched between 4 and 5 colour theorem.)

This is perhaps a surprising result in the view of the following remarks: First, note that for the class $\mathcal{K}_d$ of all graphs with all degrees bounded by d ($d \geq 1$), a similar result has been proved by Häggkvist and Hell. However, for each of the classes of d–degenerated graphs, the answer is negative (as shown by the graphs K_n^{**} which we obtain by subdividing each edge by 2 vertices). In this context it is natural to consider the following: Let $F_1, \ldots, F_t$ be a finite set of graphs. Denote by $Forb_h(F_1, \ldots, F_t)$ the class of all graphs G for which there is no homomorphism from F_i to G for any $1 \leq i \leq t$. Thus $Forb(K_k)$ is the class of all K_k-free graphs and $Forb(C_{2k+1})$ is the class of all graphs with the odd girth larger than $2k + 1$.

Problem 1. *Is it true that for every proper minor closed class $\mathcal{K}$ of graphs and for every choice of graphs $F_1, \ldots, F_t$ there exists a graph $H(\mathcal{K}, F_1, \ldots, F_t) = H$ with the following properties:*

(i) $H \in Forb_h(F_1, \ldots, F_t)$;

(ii) For every graph $G \in Forb_h(F_1, \ldots, F_t)$ there exists a homomorphism from G to H.

The problem is open even for the class of planar graphs, $t = 1$ and $F_1 = C_5$. I hope to be able to report on a progress at one of the future workshops.

Connected permutations and hypermaps

Patrice Ossona de Mendez
(with Pierre Rosenstiehl)

Abstract

A link is developed between the orbits of a bi-generated permutation group and the components of a permutation σ over an interval of $\mathbb{N}$, these components corresponding to sub-intervals fixed by σ. Several bijections are established between combinatorial families whose equi-cardinality were considered as mysterious by the literature so far. A coding of pointed maps and hypermaps follows.

Consider the language $\mathcal{L} = \{\alpha, \beta\}^\star$. Any word $w \in \mathcal{L}$ has a unique factorization of the form:

$$w = \alpha^{i_1}\beta\alpha^{i_2}\beta\ldots\alpha^{i_k}, \quad (i_1, \ldots, i_k \geq 0) \tag{1}$$

which defines the *signature* of w

$$\mathtt{S}(w) = (k, i_1, \ldots, i_k). \tag{2}$$

We shall accordingly consider the total order on $\mathcal{L}$ defined by the lexicographical on the signatures:

$$\forall w_1, w_2 \in \mathcal{L}, \quad (w_1 < w_2) \iff (\mathtt{S}(w_1) < \mathtt{S}(w_2)) \tag{3}$$

Hence, we have, $\forall w_1, w_2 \in \mathcal{L}, \forall 0 \leq a \leq b:$

$$(w_1\alpha^a \leq w_2 \leq w_1\alpha^b) \iff (\exists a \leq i \leq b, \quad w_2 = w_1\alpha^i) \tag{4}$$

1 θ-minimality and θ-connectivity

Consider any fixed permutation θ of a finite set B and a linear order $<$ on B with minimum $b_0 \in B$.

Definition 1.1. An element $b \in B$ is θ-*minimal* if $\forall b' > b$, we have $\theta(b') > \theta(b)$.

For instance, $\theta^{-1}(b_0)$ and the maximal element of B are θ-minimal.

Lemma 1.1. *Let r be a θ-minimal element, then the greatest element $b < r$ such that $\theta(b) < \theta(r)$ (if it exists) is θ-minimal.*

Proof. For any $b' > b$, either $b' < r$ and $\theta(b) < \theta(r) < \theta(b')$ (by maximality of b), or $b' = r$ and $\theta(b) < \theta(b')$, or $b' > r$ and $\theta(b) < \theta(r) < \theta(b')$ (as r is θ-minimal). Hence, b is θ-minimal. $\square$

Lemma 1.2. *Let r be a θ-minimal element of B. Then, $\theta(r) < r$.*

Proof. As b_0 is not θ-minimal, there exists $b' \geq r$, such that $\theta(b') < r$. As r is θ-minimal, $\theta(b') \geq \theta(r)$. Hence, $r > \theta(r)$. $\square$

Definition 1.2. A linear order $<$ on B with minimum b_0 is θ-*connected* if:

$$\forall b \in B \setminus \{b_0\}, \exists b' \in B, \quad b' \geq b \text{ and } \theta(b') < b. \tag{5}$$

Lemma 1.3. *Let $<$ be a θ-connected linear order on B with minimum b_0. Let b be any non θ-minimal element of B different from b_0 and $r \in B$ be the smallest θ-minimal element strictly greater than b. Then, $\theta(r) < b$.*

Proof. According to Lemma 1.1, for any element $b' \in B$ with $b \leq b' < r$, we have $\theta(b') > \theta(r)$. Moreover, for any element $b' > r$, $\theta(b') > \theta(r)$ are r is θ-minimal. As $<$ is θ-connected, there exists $b' \geq b$ with $\theta(b') < b$. Thus, $\theta(r) < b$. $\square$

As a corollary of Lemma 1.2 and Lemma 1.3, we get:

Corollary 1.4. *Let $<$ be a θ-connected linear order on B with minimum b_0 and let $r_1 < r_2$ be two consecutive θ-minimal elements. Then, $\theta(r_2) \leq r_1$.*

Given a fixed linear order on B with minimum b_0, we shall conversely define a connectivity concept for permutations on B:

Definition 1.3. A permutation θ on B is $<$-connected if and only if $<$ is θ-connected.

2 Hypermaps

Now, consider a transitive group $\langle \sigma, \theta \rangle$ on a set B (that is: a *hypermap*) and any particular element $b_0 \in B$. Moreover, we will assume that b_0 satisfies the following fixed-point property:

$$\sigma\theta^{-1}(b_0) = b_0 \tag{6}$$

Consider the mapping $\phi : \mathcal{L} \to \langle \sigma, \theta \rangle$ recursively defined by:

$$\phi_\emptyset = \text{Id}, \phi_{w\alpha} = \sigma\phi_w \text{ and } \phi_{w\beta} = (\sigma\theta^{-1})\phi_w. \tag{7}$$

As $\left\langle \sigma, (\sigma\theta^{-1}) \right\rangle = \langle \sigma, \theta \rangle$ is transitive on B, any element $b \in B$ may be written as $b = \phi_w(b_0)$, for some $w \in \mathcal{L}$. Thus, we may define the maping $\mathtt{w} : B \to \mathcal{L}$:

$$\forall b \in B, \quad \mathtt{w}(b) = \min\{w \in \mathcal{L}, \quad \phi_w(b_0) = b\} \tag{8}$$

This last mapping induces a total order on B:

$$\forall b_1, b_2 \in B, \quad (b_1 < b_2) \iff (\mathtt{w}(b_1) < \mathtt{w}(b_2)) \tag{9}$$

Lemma 2.1. *Let $b_1 < b_2$ be two elements of B, such that $b_2 = \sigma^a(b_1)$ (with $a \geq 0$ minimal). Then,* $\mathtt{w}(b_2) = \mathtt{w}(b_1)\alpha^a$.

Proof. As $\phi_{\mathtt{w}(b_1)\alpha^a}(b_0) = \sigma^a\phi_{\mathtt{w}(b_1)}(b0) = \sigma^a(b_1) = b_2$, we have $\mathtt{w}(b_2) \leq \mathtt{w}(b_1)\alpha^a$. As $\mathtt{w}(b_2) \geq \mathtt{w}(b_1)$, according to (4), there exists $0 \leq i \leq a$, such that $\mathtt{w}(b_2) = \mathtt{w}(b_1)\alpha^i$ and hence $b_2 = \sigma^i(b_1)$. The result follows from the minimality of a. □

As a consequence, we get that, for any orbit γ of σ:

$$\sigma(\max\gamma) = \min\gamma \tag{10}$$

as $\max\gamma = \sigma^{|\gamma|-1}(\min\gamma)$.

Lemma 2.2. *Let γ be an orbit of σ. Then, either* $\mathtt{w}(\min\gamma) = \emptyset$ *or there exists* $w \in \mathcal{L}$, *such that* $\mathtt{w}(\min\gamma) = w\beta$.

Proof. Otherwise, there exists $w \in \mathcal{L}$ and $a > 0$, such that $\mathtt{w}(\min\gamma) = w\beta\alpha^a$. Then, as $\phi_{w\beta}(b_0) = \sigma^{-a}(\min\gamma)$, we have $\mathtt{w}(\sigma^{-a}(\min\gamma)) \leq w\beta < \mathtt{w}(\min\gamma)$, what contradicts the minimality of $\min\gamma$. □

Lemma 2.3. *Let γ be an orbit of σ. Then* $\min\{\theta(b), b \in \gamma\} = \theta(\max\gamma)$ *and either* $\mathtt{w}(\min\gamma) = \emptyset$ *or* $\mathtt{w}(\min\gamma) = \mathtt{w}(\theta(\max\gamma))\beta$.

Proof. Let $b_1 = \min\{\theta(b), b \in \gamma\}$. Then, $b_1 \leq \theta(\max\gamma)$. Moreover, we have $b_1 = \min\{\theta(b), b \in \gamma\} = \min\{\theta\sigma^{-1}(b), b \in \gamma\} = \min\{(\sigma\theta^{-1})^{-1}(b), b \in \gamma\}$.

- Assume $\mathtt{w}(\min\gamma) \neq \emptyset$. Then, according to Lemma 2.2, there exists $w \in \mathcal{L}$, such that $\mathtt{w}(\min\gamma) = w\beta$. Then,

$$\begin{aligned} \mathtt{w}((\sigma\theta^{-1})(b_1)) &\leq \mathtt{w}(b_1)\beta && \text{as } \phi_{\mathtt{w}(b_1)\beta}(b_0) = \sigma\theta^{-1}(b_1) \\ &\leq \mathtt{w}((\sigma\theta^{-1})^{-1}(b), b \in \gamma) && \text{by definition of } b_1 \\ &\leq w\beta = \mathtt{w}(\min\gamma) && \text{as } \phi_w(b_0) = (\sigma\theta^{-1})^{-1}(\min\gamma) \\ &\leq \mathtt{w}(\sigma\theta^{-1}(b_1)) && \text{by Lemma 2.1, as } \sigma\theta^{-1}(b_1) \in \gamma. \end{aligned}$$

  Thus, we get $\sigma\theta^{-1}(b_1) = \min\gamma$ (and thus $b_1 = \theta(\max\gamma)$) and $\mathtt{w}(\min\gamma) = \mathtt{w}(b_1)\beta$.

- If $\mathtt{w}(\min\gamma) = \emptyset$, then $\theta(\max\gamma) = \theta\sigma^{-1}(b_0) = b_0$ (according to (6)). As $b_0 \leq b_1 \leq \theta(\max\gamma) = b_0$, we get $b_1 = b_0 = \theta(\max\gamma)$.

□

Lemma 2.4. *Let γ_1, γ_2 be orbits of σ, such that $\min\gamma_1 < \min\gamma_2$ and let $b \in \gamma_2$. Then, $\theta(\max\gamma_1) < \theta(b)$.*

Proof. Assume first that $\min\gamma_1 \neq b_0$. Then, according to Lemma 2.2, there exists words $w_1, w_2 \in \mathcal{L}$, such that $\mathtt{w}(\min\gamma_i) = w_i\beta$ (for $i \in \{1,2\}$), with $w_1 < w_2$. According to Lemma 2.3, we further get that $\theta(\max\gamma_i) = w_i$ ($i \in \{1,2\}$) and thus $\theta(\max\gamma_1) < \theta(\max\gamma_2)$ and, according to Lemma 2.3, $\theta(\max\gamma_2) \leq \theta(b)$.

If $\min\gamma_1 = b_0$, then $\mathtt{w}(\max\gamma_1) = \alpha^{|\gamma_1|-1} < \beta$. As b does not belong to γ_1, $\theta(b) \geq \beta$ and the result follows. □

Lemma 2.5. *Let γ be an orbit of σ and let $b \in \gamma$ be one of its elements. Then, b is θ-minimal if and only if $b = \max\gamma$.*

Proof. Assume $b = \max\gamma$. Then, for any $b' > b$, b' belongs to an orbit γ' such that $\min\gamma' > \min\gamma \neq b_0$. Hence, according to Lemma 2.4, $\theta(b') > \theta(\max\gamma) = \theta(b)$ and thus b is θ-minimal.

Conversely: if $b \neq \max\gamma$, then $\theta(\max\gamma) < \theta(b)$, according to Lemma 2.3. Thus b is not θ-minimal. □

Lemma 2.6. *The linear order on B is θ-connected.*

Proof. Let $b \neq b_0$ be an element of B, let γ be its orbit by σ and let $b' = \max \gamma \geq b$. If $\min \gamma = b_0$, then $\theta(b') = b_0 < b$. Otherwise, there exists $w \in \mathcal{L}$, such that $\mathtt{w}(\min \gamma) = w\beta)$ (according to Lemma 2.2) and $\mathtt{w}(\theta(b')) = w$ (according to Lemma 2.3). Thus $\mathtt{w}(\theta(b')) < w\beta = \mathtt{w}(\min \gamma) \leq \mathtt{w}(b)$ and hence $\theta(b') < b$. $\square$

Lemma 2.7. *Let $\lessdot$ be any θ-connected linear order on B with minimum b_0. Then, there exists a unique permutation $\dot\sigma$ such that $\langle \dot\sigma, \theta \rangle$ acts transitively on B, $\dot\sigma\theta^{-1}(b_0) = b_0$ and the linear order defined by the hypermap $\langle \dot\sigma, \theta \rangle$ pointed in b_0 is $\lessdot$.*

Proof. Assume such a permutation $\dot\sigma$ exists the elements of B are ordered as $b_0 \lessdot b_1 \lessdot \ldots b_{|B|-1}$. Let $i_1 < \cdots < i_p$ be the indices of the θ-minimal elements and let $i_0 = -1$. Then, according to Lemma 2.5, $\dot\sigma$ is fully defined by:

$$\forall 0 \leq j < |B|, \sigma(b_j) = \begin{cases} b_{j+1} & \text{if } b_j \text{ is not } \theta\text{-minimal,} \\ b_{i_{k-1}+1} & \text{otherwise, if } j = i_k. \end{cases}$$

Conversely, consider the so defined permutation $\dot\sigma$. By construction, $\dot\sigma(b_{i_1}) = b_0$. As the smallest θ-minimal element of B is obviously $\theta^{-1}(b_0)$, we got $\dot\sigma\theta^{-1}(b_0) = b_0$. Let $B' \subseteq B$ be a minimal subset of B including b_0 and stable for $\dot\sigma$ and θ. Then, consider the hypermap $\left\langle \dot\sigma_{|B'}, \theta_{|B'} \right\rangle$ on B' and the linear order $<$ deduced from the ordering on the words $\mathtt{w}(b)$ of this hypermap. We shall inductively prove that $<$ defines the same linear order as $\lessdot$ and that $B' = B$. First, B' includes the orbit of b_0 by $\dot\sigma$ and hence $B_1 = \{b_0, \ldots, b_{i_1}\} \subseteq B'$. Moreover, for any $x, y \in B_1$, we have $(x \lessdot y) \iff (x < y)$ and, for $x \in B_1$ and $y \in B' \setminus B_1$, we have $x < y$. Thus, let $p > k \geq 1$ and $B_k = \{b_0, \ldots, b_{i_k}\}$. Assume:

$$\begin{aligned} & B_k \subseteq B' \\ & \forall x, y \in B_k, \quad (x < y) \iff (x \lessdot y) \\ & \forall x \in B_k, \forall x \in B' \setminus B_k, \quad x < y \end{aligned}$$

Let $B_{k+1} = \{b_0, \ldots, b_{i_{k+1}}\}$. According to Lemma 1.4, $\theta(b_{i_{k+1}}) \leq b_{i_k}$. Hence, $\theta(b_{i_{k+1}}) \in B_k$ and $B_{k+1} \subseteq B'$. Moreover, the greatest element of $B_{k+1} \setminus B_k$ (with respect to $<$) is $b_{i_{k+1}}$ as it is the element of this orbit with minimal θ-value (according to Lemma 2.3) and we deduce that $<$ and $\lessdot$ coincides on $B_{k+1} \setminus B_k$. Moreover, for any element $b \in B' \setminus B_{k+1}$, we have $\theta(b) > \theta(b_{i_{k+1}})$ as the later is θ-minimal. Hence, $\mathtt{w}(b) > \mathtt{w}(b_{i_{k+1}})$ and $b > b_{i_{k+1}}$. Altogether, the recursion hypothesis holds for $k+1$. $\square$

Theorem 2.8. *Let B be a finite set, $b_0 \in B$ one of its elements and let θ be a permutation on B.*

Then, there exists a natural bijection from the set of the permutations σ such that $\sigma\theta^{-1}(b_0) = b_0$ and $\langle\sigma, \theta\rangle$ acts transitively on B, and the θ-connected linear orders on B having b_0 as minimum.

But also, as two conjugates θ defines isomorphic hypermaps for similarly conjugated linear orders:

Corollary 2.9. *Let B be a finite set, $b_0 \in B$ one of its elements and let $<$ be any linear order on B with minimum b_0.*

Then, there exists a natural bijection from the isomorphism class of the hypermaps $\langle\sigma, \theta\rangle$ pointed at b_0 with $\sigma\theta^{-1}(b_0) = b_0$, and the $<$-connected permutations on B.

Remark that the size of the edges of the hypermap are exactly those of the orbits of θ.

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Closed curves on the sphere

Michal Popule

1. Definition: By a term *curve* we shall always mean an oriented closed cross curve drawn on a sphere in general position.
The curve Ω divides a sphere onto several parts-**discs** locally homeomorph to $\mathbf{R}^2$. More exactly, a disc is a region bordered by a curve, i.e. a set of sphere's points with these properties:
(i) None of them lays on the curve Ω,
(ii) each two of them may be connected by a path on the sphere not intersecting Ω and
(iii) it is maximal set as for inclusion with properties (i) and (ii)

2.1. Definition: Let's overpaint (for example by different colour) onto a curve Ω, starting anywhere on Ω but not in a crossing point, following the Ω's orientation.
Always, when being just a little piece in front of Ω's crossing point we avoid it and move into (still painting, holding a pen on a paper) another part of the curve Ω incident to this crossing point, so that we don't continue overpainting the curve in a "natural" way, we choose the only crossing point's incident flowline having the same direction (entering/leaving) in regard of this crossing point as our entering flowline.
Hence we change the direction of our painting in regard of Ω's orientation. This way we continue till we get to the point where we started. We have painted a contour-topological equivalent of a circle - by getting to the starting point our overpainting is closed curve and as we haven't painted a crossing as we always avoid curve's crossing points. Then we repeat this all - painting contours, starting on any flowline between some neighboring crossing points, that is not (almost) overpainted - while there is such one. In the end we get a contours system, we

shall talk about the (Ω's) **exchanging-contouring**.

2.2. Definition: If we again (like in 2.1.) start somewhere on the curve Ω and overpaint Ω in its direction and again when being just a little piece in front of Ω's crossing point we skip to another Ω's part, but now (on the contrary to 2.1.) such the flowline that we (always) overpaint within the direction of orientation of Ω and the new flowline borders one of two discs bordered by the old one; otherwise everything else stays like in 2.1. We get a contour-system again, we shall call it (Ω's) **straightforward-contouring**. We consider straightforward contours to be oriented in direction induced by direction of Ω.

3. For a contouring we define a natural **colouring** dividing the rest of sphere onto inner and outer points, by choosing some point not laying on any of contours as inner. This colouring naturally induces coluring of Ω-discs, that then make a pseudo-chessboard.

4. Theorem (Popule, 2000/'01): Let Ω be a curve on a sphere, with exchanging- and straightforward- contourings drawn. Choosing some disc D of Ω as inner (both for exchanging- and straightforward- contouring), each Ω-crossing point is inner in exchanging-contours system if and only if it is outer in straightforward-contours system. Moreover, each disc is straightforward-inner if and only if it is exchanging-inner.

6. Theorem (Rosenstiehl): A double-occurence word ω from alphabet C_ω as a set of its symbols is a Gauss code of a curve Ω drawn on a sphere if and only if it satisfies these three "parity" conditions:
(i) The subset of symbols interlaced with each symbol $c \in C_\omega$ of ω is even.
(ii) Whenever two symbols c and d are not interlaced they have an even number of common interlaced symbols.
(iii) There is a subset A of C_ω such that for $B := C_\omega \setminus A$ any two interlaced symbols c, d belong to the same set $(A or B)$ if and only if they have an odd number of common interlaced symbols.

7. In [1] you can find the first **proof** of this theorem, with only a mistake in formulation of condition (iii). But there is another proof by H. de Fraysseix and P.O. de Mendez in [2], that seems to be easier to understand. Finally, another proof by P. Rosenstiehl is to appear.

8. Remark: Set A from 6. is not uniquelly determined, not even up to complement. For a curve there may exist more sets satisfying (iii) since the interlacement graph $G = (C_\omega, S =$set of pairs of interlaced letters) need not be connected. But, given a closed curve Ω on the sphere, Mr. Rosenstiehl's proof [1] leads to a constructive way to find certain such A: We make Ω's exchanging contours. Now, having chosen any continent O as inner, A we get right as the crossing points that are without loss of generality (instead of A we may always take its complement) inner in exchanging-contouring: Of course, they are equivalently (according to 4.) straightforward-(without loss of generality) outer.

9. Theorem (Popule, Loebl 2000/'01): For each closed curve Ω on a sphere with the Gauss code ω from the symbols-set C_ω and with $A \subseteq C_\omega$ satisfying (iii) each symbol $c \in C_\omega$ is interlaced with an even number of symbols -elements of A (and, according to 1.8.1.(i) of non-elements of A as well). **Proof (P.O. de Mendez):** We consider an interlacement graph. We denote the set of neighbors of any vertex p as $N(p)$. Let A be arbitrary set satisfying (iii). Then $S(c) := \sum_{x \in N(c)} |N(c) \cap N(x)| = 2.$number of triangles in an interlacement graph with c as one of vertices, as each such triangle is taken twice exactly ($y \in N(x) \cap N(c)$ iff $x \in N(y) \cap N(c)$). Hence, $S(c)$ is even. But the sum of $S(c)$ may as well be divided into

$$S(c) = \sum_{x \in N(c) \cap A} |N(c) \cap N(x)| + \sum_{x \in N(c) \setminus A} |N(c) \cap N(x)|.$$

First, for $c \in A$, each $|N(r) \cap N(x)|$ in the second sum, i.e. for $x \in N(r) \setminus A$, is even, so the whole sum is even, so the first sum must be even as well. But its each member $|N(r) \cap N(x)|$ is odd according to condition (iii). Hence, the number of the first sum's members, i.e. cardinality of the set $N(r) \cap A$, is even.
Similarly, for $c \notin A$, each $|N(r) \cap N(x)|$ in the first sum, i.e. for $x \in N(r) \cap A$, is even, so the whole sum is even, so the second sum must be even as well. But its each member $|N(r) \cap N(x)|$ is odd. Hence, the number of the the second sum's members, i.e. cardinality of the set $N(r) \setminus A$, is even. So also, according to condition (i), $|N(r) \cap A|$ is even. Q.E.D.

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On star coloring of graphs

André Raspaud

In this talk, we deal with the notion of star coloring of graphs. A *star coloring* of an undirected graph G is a proper vertex coloring of G (i.e., no two neighbors are assigned the same color) such that any path of length 3 in G is not bicolored.

We give the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs and graphs with bounded treewidth and planar graphs. The relationship between acyclic coloring and star coloring is mentioned also.

All those results and proofs can be found in the paper:

FERTIN, G., RASPAUD, A. AND REED, B., *On Star Coloring of Graphs,* 27th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2001), Boltenhagen, June 2001.

Disjointly union-free families

Miklós Ruszinkó

Union-free families were introduced by Kautz and Singleton [10]. They studied binary codewords with the property that the disjunctions (bitwise OR) of any pair of distinct at most r-tuples of codewords have to be different. Later this question has been investigated in several papers on multiple access communication (see, e.g. [1], [9]). The same problem has been posed – in different terms – by Erdős, Frankl and Füredi ([2], [3]) in combinatorics, by Sós [12] in combinatorial number theory, and by Hwang and Sós [8] in group testing. One can find an easy proof of the best known upper bound of these codes in the papers by Füredi [5] and Ruszinkó [11]. In the paper of Füredi and Ruszinkó [6] the connection of these codes to the big distance ones is shown. The geometric version of this problem has been posed by Ericson and Györfi [4] and later investigated by Füredi and Ruszinkó [7].

A family $\mathcal{F}$ of an n-set is $r-$*union-free* if

$$\bigcup_{i=1}^{k} A_i \neq \bigcup_{j=1}^{\ell} B_i$$

for any

$$\{A_1, A_2, \ldots, A_k\} \neq \{B_1, B_2, \ldots, B_\ell\},$$

$1 \leq k, \ell \leq r$; $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_\ell \in \mathcal{F}$. $\mathcal{F}$ is $r-$*cover-free*, if

$$A_0 \not\subseteq A_1 \cup A_2 \cup \cdots \cup A_r$$

holds for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$. Let $f'(n,r)$ and $f(n,r)$ be the maximum size of an $r-$union-free and r-cover-free family, respectively. It is known from the previous works (see some references above) that

$$2^{c_1 n/r^2} \leq f(n,r) \leq f'(n,r) \leq f(n,r-1) \leq 2^{c_2 n \log r/r^2} \tag{1}$$

Recent applications in computer science (Linial, Szegedy-Vishvanathan, Buhrman-Miltersen-Radhakrishnan-Venkatesh, Buhrman-Laplante-Miltersen) raised the attention to this problem. Therefore, I pose the following problem again.

Problem 1. Try to narrow the gap in (1).

Quite recently L. Györfi and the author observed that the bounds are completely different for *disjointly* $r-$*union-free* families. A family $\mathcal{F}$ of an n-set is *disjointly* $r-$*union-free* if

$$\bigcup_{i=1}^{k} A_i \neq \bigcup_{j=1}^{\ell} B_i$$

for any

$$\{A_1, A_2, \ldots, A_k\} \cap \{B_1, B_2, \ldots, B_\ell\} = \emptyset$$

$1 \leq k, \ell \leq r$; $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_\ell \in \mathcal{F}$. Let $f^*(n,r)$ be the maximum size of a *disjointly* $r-$*union-free* family. We could easily prove that

$$2^{c_3 n/r} \leq f^*(n,r) \leq 2^{c_4 n \log r/r} \tag{2}$$

Therefore, I pose the following problem.

Problem 2. Try to narrow the gap in (2).

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Weak pancyclicity of locally connected graphs

Zdeněk Ryjáček

A graph G is called *locally connected* if the neighborhood of every vertex induces a connected graph. A graph G is *weakly pancyclic* if G contains a cycle of length l, for all l between the girth and the circumference of G.

Conjecture 1. Every locally connected graph is weakly pancyclic.

Comments:

1. The conjecture is true for
   - claw–free graphs (it follows from a result of L.Clark from 1980)
   - planar triangulations
   - graphs in which every induced subgraph contains a simplicial vertex (a vertex is *simplicial* if its neighborhood induces a complete graph).
2. It can be proved that if G is locally connected, than every vertex $x \in V(G)$ has at least 2 neighbors x_1, x_2 such that contracting any of the edges xx_1, xx_2 does not affect local connectedness of G (this lemma could possibly help to obtain a proof of the conjecture by induction).

On pseudo-transitive graphs and their geometric applications

Farhad Shahrokhi

Motivated by geometric applications, we define a directed acyclic graph $G = (V, E)$ $n = |V|$ and $m = |E|$, to be pseudo-transitive with respect to a given subset of edges E_1, if $ab \in E_1$ and $bc \in E$ implies that $ac \in E$. When $G = (V, E)$ is pseudo-transitive with respect to E_1, we write $G = (V, E_1, E)$.

Let $G = (V, E_1, E)$ be pseudo-transitive, and let $G_2 = (V, E_2)$, where $E_2 = E - E_1$. We give two exact algorithms for computing longest chains in pseudo transitive graphs. The first algorithm computes a longest chain of any pseudo-transitive graph G, in $O(n^{\omega_2+1}m)$ time, where ω_2 is the length of a longest chain in the graph $G_2 = (V, E_2)$. This algorithms can be applied to different classes of map labeling problems, and its time complexity is better than the square root of the previous algorithms. When E_1, and $E_2 = E - E_1$ are partial orders on V, we present a second algorithm that computes a longest chain of G in $O(\sum_{x\in V} \deg^2(x))$, time where $deg(x)$ is the degree of x. An interesting application of this algorithm is to compute, in low order polynomial time, the largest set of disjoint line segments in the plane, when all segments have exactly one end point on the x-axis, and make acute angels between -90 and $+90$ degrees with it.

We also derive approximate chain-antichain covering results in certain classes of pseudo-transitive graphs. The results imply that the gaps between the chromatic numbers and the largest clique sizes, and the gaps between the clique cover numbers and the independence numbers, are small, in many intersection graphs whose underlying vertices are the closed and bounded sets in R^k. In particular, for circle graphs our general result imply that $\chi = O(\omega . \log(\alpha))$, where χ, ω, and α, denote the chromatic number, the size of a largest clique, and the size of a largest independent set, respectively.

The maximum size of an equilateral set and Kusner's conjecture

Clifford Smyth

We say a subset S of a metric space (X, d) is equilateral if each pair of points of S determines the same distance; i.e. if there is a number $r \geq 0$ such that for all $s, t \in S$ with $s \neq t$, we have $d(s,t) = r$. Let $e(X)$ denote the maximum cardinality of an equilateral subset of X, if this maximum exists. For $p \in [1, \infty]$, let $L^p(n)$ be the metric space induced on $\mathbb{R}^n$ by the L^p norm; i.e. $L^p(n) = (\mathbb{R}^n, d_p)$ where $d_p(x,y) = ||x-y||_p$ for all $x, y \in \mathbb{R}^n$. In 1983, Kusner conjectured that for $1 < p < \infty$, $e(L^p(n)) = n+1$. We show $e(L^p(n)) = O(n^{(p+1)/(p-1)})$. This improves the previously best known upper bound, which was essentially $e(L^p(n)) \leq 2^n$.

The Colin de Verdiere invariant and outerplanar graphs

Jan Vondrák

Definition 1. Let $G = (V, E)$ be a graph on n vertices. The Colin de Verdiere invariant $\mu(G)$ is the maximum corank of a symmetric $n \times n$ matrix satisfying

- (M1) $\forall i \neq j; M_{ij} < 0$ if $\{i, j\} \in E$ and $M_{ij} = 0$ if $\{i, j\} \notin E$.
- (M2) M has exactly one negative eigenvalue.
- (M3) $\forall X; X_{ij} = 0$ if $i = j$ or $\{i, j\} \in E; MX = 0 \Rightarrow X = 0$.

As Colin de Verdiere himself proved, $\mu(G) \leq 2$ if and only if G is an outerplanar graph. Also, $\mu(G) \leq 1$ if and only if G is the subgraph of a path. So, for any 2-connected graph, $\mu(G) \geq 2$. Moreover, if $\mu(G) = 2$, an outerplanar embedding of G can be obtained easily from its Colin de Verdiere matrix.

Definition 2. Let M be any matrix satisfying (M1) and (M2) for a graph G and $x, y \in \mathbf{R^n}$ two linearly independent vectors such that

$$Mx = My = 0.$$

The null space representation corresponding to x, y is a mapping $\mathbf{u} : V \rightarrow \mathbf{R^2}$,

$$\mathbf{u}(i) = (x_i, y_i).$$

The normalized null space representation is a mapping $\mathbf{v} : V \rightarrow \mathbf{R^2}$,

$$\mathbf{v}(i) = \frac{\mathbf{u}(i)}{||\mathbf{u}(i)||}.$$

László Lovász and Alexander Schrijver proved the following.

Theorem 3. If G is 2-connected and $\mu(G) = 2$, then the normalized null space representation corresponding to any Colin de Verdiere matrix gives an outerplanar embedding of G such that the vertices are mapped to points on the unit circle and the edges are non-intersecting line segments.

We give an alternative proof of the same theorem. Our approach is different in that we do not assume that G is outerplanar if $\mu(G) \leq 2$. (Which follows from the characterization of outerplanar graphs by forbidden minors K_4 and $K_{2,3}$.) Instead, we prove the following.

Theorem 4. Let G be 2-connected, without a K_4 or $K_{2,3}$ minor, and M its Colin de Verdiere matrix of corank at least 2. Then the normalized null space representation corresponding to M gives an outerplanar embedding.

This implies the previous theorem, since $\mu(K_4) = \mu(K_{2,3}) = 3$ and the Colin de Verdiere parameter is monotone with respect to taking minors. However, this can also be seen as a new proof that (2-connected) outerplanar graphs are exactly those without a K_4 or $K_{2,3}$ minor.

This is the first step in an attempt to give an analogous proof for planar graphs ($\mu(G) = 3$) and perhaps linklessly embeddable graphs ($\mu(G) = 4$).

Light subgraphs of graphs embedded in compact 2-manifolds

Heinz-Jürgen Voss

A graph H is said to be *light* in a class $\mathcal{C}$ of graphs if infinitely many members of $\mathcal{C}$ contain a copy of H and there is an integer $\phi(H, \mathcal{C})$ such that each member G of $\mathcal{C}$ with a copy of H also has a copy of H with maximum degree $\Delta_G(H) \leq \phi(H, \mathcal{C})$ in G. If such an integer does not exist the graph H is said to be heavy. The *weight* $w(H)$ of a subgraph H of a graph G is the sum of the valencies (in G) of its vertices.

1 Graphs of minimum degree 5 embedded in a fixed compact 2-manifold

Recently, with S. Jendrol' we studied light subgraphs in the class $\mathcal{C}_{\mathcal{M}}$ of all graphs G of minimum degree ≥ 5 embedded in a fixed compact 2-dimensional manifold $\mathcal{M}$ of Euler characteristic $\chi(\mathcal{M}) \leq 0$ with the properties that G has minimum degree $\delta(G) \geq 5$ and G has an order $\geq 2000\,|\chi(\mathcal{M})|$.

Theorem 1. Let G $\in \mathcal{C}_{\mathcal{M}}$.

(i) The graph G has a 3-cycle C_3 of weight $w(C_3) \leq 18$.

(ii) The graph G has a 4-cycle $\overline{C}_4$ with a diagonal of weight $w(\overline{C}_4) \leq 27$.

Moreover, if G has no 12-vertices, or no 12-, and 11-vertices, or no 12-, 11-, and 10-vertices then G has a 4-cycle $\overline{C}_4$ with a diagonal of weight $w(\overline{C}_4) \leq 26$, or $w(\overline{C}_4) \leq 25$, or $w(\overline{C}_4) \leq 24$, respectively.

(iii) The graph G has a 5-cycle $\overline{C}_5$ with a diagonal of weight $w(\overline{C}_5) \leq 32$.

Moreover, if G has no 12-vertices then G has a 5-cycle $\overline{C}_5$ with a diagonal of weight $w(\overline{C}_5) \leq 31$.

All bounds are tight.

There are graphs in $\mathcal{C}_{\mathcal{M}}$ having no 5-cycle $\tilde{C}_5$ with two diagonals meeting in one vertex. But $\tilde{C}_5$ is light in the class $\mathcal{C}_{\mathcal{M}}$. Let $\mathcal{C}^*_{\mathcal{M}}$ be the subclass of all graphs of $\mathcal{C}_{\mathcal{M}}$ containing a 5-cycle $\tilde{C}_5$ with two diagonals meeting in one vertex.

Theorem 2. There are two positive real numbers c_1 and c_2 such that:

(i) each graph $G \in \mathcal{C}^*_{\mathcal{M}}$ has a 5-cycle $\tilde{C}_5$ of weight $w(\tilde{C}_5) \leq c_1 \, |\chi(\mathcal{M})|$,

(ii) there are graphs $\overline{G} \in \mathcal{C}^*_{\mathcal{M}}$ in which each $\tilde{C}_5$ has a weight $w(\tilde{C}_5) \geq c_2 \, \sqrt{|\chi(M)|}$.

We can show that the cycle C_s is heavy for $s \geq 11$. An open question is:

Problem 1. Is the cycle C_s light in $\mathcal{C}_{\mathcal{M}}$ for $6 \leq s \leq 10$?

More about light subgraphs in graphs embedded in $\mathcal{M}$ can be found in [1].

2 Light and heavy paths in plane graphs of minimum degree δ

All results presented here can be found in our survey [2].

A nice result of Fabrici and Jendrol' says that only the paths are light in the class of all 3-connected plane graphs (obviously, $\delta = 3$). By Fabrici, Hexel, Jendrol', and Walther the same is true if $\delta = 4$. Moreover, due to Mohar this is true, if $\delta = 4$ and the graphs under condideration are 4-connected.

The situation is different if arbitrary plane graphs with $\delta = 3$ or $\delta = 4$ are considered. Only paths of small lengths are light.

If $\delta = 2$ then by Euler P_1 is light with $w(P_1) \leq 5$ and P_2 is heavy.

If $\delta = 3$ then by Kotzig P_2 is light with $w(P_2) \leq 13$ and P_3 is heavy.

If $\delta = 4$ then we proved that P_4 is light and P_5 is heavy.

If δ = 4 and the edge weight is at least 9 then by Mohar, Škrekovski and Voss the path P_6 is light and P_8 is heavy.

All the presented bounds are tight. An open question is:

Problem 2. Is the path P_7 light in this class?

If δ = 5 then P_{11} is heavy and the situation for the paths P_s , $s \leq 10$, follows from the results on cycles (see below).

3 Light and heavy cycles in plane graphs of minimum degree δ

The situation with respect to cycles is the following.

If $\delta \leq 4$ then no cycle is light even in the class of all 4-connected plane graphs.

If δ = 4 and the edge weight is at least 9 then by Mohar, Škrekovski, and Voss the cycle C_s is light, if $3 \leq s \leq 6$, and C_s is heavy, if $s \geq 7$. Moreover, $w(C_3) \leq 21$, and this bound is tight.

If δ = 5 then in plane triangulations the cycle C_s is light, if $3 \leq s \leq 10$, and C_s is heavy, if $s \geq 11$. This has been proved by Jendrol', Madaras, Soták, and Tuza, besides the case s = 10 ; this case has been proved by Madaras and Soták alone. From the above result for plane graphs of minimum degree δ = 4 and edge weight at least 9 it follows that C_3 , ... , C_6 are even light in the class of all plane graphs of minimum degree δ = 5 . By Borodin $w(C_3) \leq 17$, and this bound is tight. An open question is:

Problem 3. Are the cycles C_7 , ... , C_{10} light in the class of all plane graphs with minimum degree δ = 5 ?

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