Edge–Disjoint Odd Cycles in Planar Graphs

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Abstract

We prove $\tau_{\text{odd}}(G) \leq 2\nu_{\text{odd}}(G)$ for each planar graph G where $\nu_{\text{odd}}(G)$ is the maximum number of edge–disjoint odd cycles and $\tau_{\text{odd}}(G)$ is the minimum number of edges whose removal makes G to be bipartite, i.e. which meet all the odd cycles. For each k, there is a 3–connected planar graph G_k with $\tau_{\text{odd}}(G) = 2k$ and $\nu_{\text{odd}}(G) = k$.

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1 Introduction

The goal of a feedback vertex (edge) problem in a graph is to find a set of vertices (edges) such that each cycle contains at least one of them. This

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problem is well–studied in the general case where you want to meet all the cycles both from the structural and algorithmic point of view, see [5] for a survey. We study relations among the size $\tau_{\mathrm{odd}}(G)$ of a minimum set of edges which meets all the odd cycles of a graph G and the size $\nu_{\mathrm{odd}}(G)$ of a maximum set of edge–disjoint odd cycles in a graph G. Brass ([17]) conjectured that $\tau_{\mathrm{odd}}(G) \le 2\nu_{\mathrm{odd}}(G)$ for all graphs G, but this conjecture turned out to be false ([1]). In 1999, Reed[1] proved ([9]) that for each positive integer s, there exists a projective–planar graph G with $\tau_{\mathrm{odd}}(G) = s$ and $\nu_{\mathrm{odd}}(G) = 1$, i.e. the graph which does not contain two edge–disjoint odd cycles, but it is necessary to delete at least s of its edges to make it bipartite (Reed considered in [9] the vertex version of this problem, but his graphs are cubic and thus his results translate also to the edge case). In other words, odd cycles in general graphs do not satisfy the Erdös–Pósa property[2], i.e. $\tau_{\mathrm{odd}}(G)$ is not bounded by a function of $\nu_{\mathrm{odd}}(G)$ for general graphs. We remark that it was proved in [8, 12] that each $2000(k+1)$–connected graph contains either $k+1$ vertex–disjoint odd cycles or $2k$ vertices meeting all the odd cycles. On the other hand, further results of [9] directly imply that the number of vertices meeting all the odd cycles is a function of the maximum number of vertex–disjoint odd cycles for planar graphs. In addition, Berge and Reed in 2000 in [2] proved that odd cycles in planar graphs also in the edge version of the problem satisfy the Erdös-Pòsa property, i.e. $\tau_{\mathrm{odd}}(G)$ is bounded by a function of $\nu_{\mathrm{odd}}(G)$ for planar graphs G. However, their function grows very fast. In this paper, we prove that the Brass' conjecture is true for planar graphs:

Theorem 1 *Let G be a plane graph. Then $\tau_{odd}(G) \le 2\nu_{odd}(G)$.*

The inequality is tight for infinitely many planar graphs G as stated in Theorem 2 of Section 4. The precise bound for the corresponding vertex version remains to be an open problem.

We briefly survey related results in the case that one wants to meet all the triangles of a graph (note that triangles in graphs satisfy the Erdös-Pósa property by trivial arguments): Let $\tau_t(G)$ be the size of a minimum set of

[1] Reed wrote in [9] that this particular property of Escher walls was pointed out to him by Lovász and Schrijver.

[2] The name of this property comes from the paper [4] by Erdös and Pósa. They proved by probabilistic arguments for general graphs that the minimum number of vertices whose deletion makes a graph acyclic is bounded by a function of the number of vertex–disjoint cycles. Constructive proofs of this are given in [11, 14]. Other related results can be found in [15, 16].

edges which meets all the triangles and let $\nu_t(G)$ be the size of a maximum set of edge–disjoint triangles of G. Tuza in [13] proved that $\tau_t(G) \leq 2\nu_t(G)$ for planar graphs and the factor 2 is the best possible one for this class of graphs. He conjectures that the inequality $\tau_t(G) \leq 2\nu_t(G)$ should hold for all graphs G. Tuza's proof was extended to a class of graphs which do not contain a subdivision of $K_{3,3}$ by Krivelevich in [7]. However, the original conjecture of Tuza remains still open.

Since we want to minimize the number of edges which meets all the odd cycles, we look for a maximum (in terms of the number of edges) bipartite subgraph of a graph, i.e., a maximum cut. For planar graphs, there is a nice connection between minimum sets of edges which meets all the odd cycles and T–join problems explored by Hadlock in [6]; we describe this connection in Section 2. In particular, it is possible to determine the number $\tau_{\text{odd}}(G)$ for planar graphs in polynomial time ([6]). But for general graphs this problem is NP–complete, see [18].

We use the standard graph notation throughout the paper and we refer the reader to any graph theory textbook if necessary. $N(v)$ denotes a vertex v together with all its neighbours. $G[U]$ is a subgraph of G induced by the vertices of U, $U \subseteq V(G)$. We write $\tau_{\text{odd}}(G)$ for the size of a minimum set of edges which meets all the odd cycles of a graph G and $\nu_{\text{odd}}(G)$ for the size of the maximum set of edge–disjoint odd cycles in a graph G. We mean by a planar graph a graph which can be embedded in the plane and by a plane graph a fixed embedding of a planar graph. We say that a face of a plane graph is odd if the number of edges on its boundary is odd (counting bridges twice), i.e. it contains an odd cycle (cf. Lemma 1).

The paper is structured as follows: We recall basic concepts of linear programming and its relation to the feedback edge set problem for odd cycles in planar graphs in Section 2. We present the bound on $\tau_{\text{odd}}(G)$ in terms of $\nu_{\text{odd}}(G)$ for planar graphs G in Section 3 and we prove that it is attained by infinitely many planar graphs in Section 4. We conclude in Section 5.

2 Relation of Linear Programming to Odd Cycles in Planar Graphs

2.1 Matchings in Complete Graphs

We refer the reader to [3, 10] for a more detailed introduction to linear programming and its connection to perfect matchings. We briefly survey just some concepts in this section. Let K be a complete graph on an even number of vertices and w a function which assigns a weight to each edge of K. Let $O(K)$ be the set of all the odd–cardinality subsets of V of size three and more; we mean by an odd subset a subset with an odd number of vertices throughout the paper. The following linear program can be used to compute the weight of a minimum perfect matching in K:

$$\begin{aligned}\sum_{v\in e\in E(K)} x_e &= 1 \text{ for each } v \in V(K),\\ \sum_{|e\cap o|=1, e\in E(K)} x_e &\geq 1 \text{ for each } o \in O(K),\\ x_e &\geq 0 \text{ for each } e \in E(K),\end{aligned}$$

$$\min \sum_{e\in E(K)} x_e w(e).$$

There exists a solution $x_e \in \{0,1\}$ for $e \in E(K)$ which achieves the optimal value; the edges e with $x_e = 1$ form a perfect matching of K of minimum weight. Hence, the optimal value $\sum_{e\in E(K)} x_e w(e)$ of the primal linear program is equal to the weight of the minimum perfect matching of K

The dual linear program is the following:

$$\begin{aligned}\sum_{v\in e, v\in V(K)} y_v + \sum_{|e\cap o|=1, o\in O(K)} y_o &\leq w(e) \text{ for each } e \in E(K), \quad (1)\\ y_o &\geq 0 \text{ for each } o \in O(K),\end{aligned}$$

$$\max \sum_{v\in V(K)} y_v + \sum_{o\in O(K)} y_o.$$

The optimal value $\max \sum_{v\in V(K)} y_v + \sum_{o\in O(K)} y_o$ is equal to the weight of the minimum perfect matching of K (i.e., to the optimal value of the primal linear program). One can define based on an optimal solution of the dual program reduced edge weights:

$$w^\star(e) := w(e) - \sum_{v\in e, v\in V(K)} y_v - \sum_{|e\cap o|=1, o\in O(K)} y_o. \tag{2}$$

We define also a reduced weight of an edge e with respect to a vertex set W containing both the end–vertices of e:

$$w^\star_W(e) = w(e) - \sum_{v\in e, v\in V(K)} y_v - \sum_{|e\cap o|=1, o\subset W, o\in O(K)} y_o. \tag{3}$$

By (1), the reduced weights of all the edges are non–negative. The complementary slackness conditions imply that if $x_e > 0$ for an optimal primal solution, then $w^\star(e) = 0$. If y is an optimal dual solution, then any perfect matching of K formed by the edges whose reduced weights are equal to zero is a minimum perfect matching of K.

In addition to the above results, we recall also some additional properties which an optimal solution of the dual program may have. An optimal dual solution y is called nested if for any $o_1, o_2 \in O(K)$ the inequalities $y_{o_1} > 0$ and $y_{o_2} > 0$ imply that it holds either $o_1 \cap o_2 = \emptyset$ or $o_1 \subset o_2$ or $o_2 \subset o_1$. There always exists an optimal dual solution which is nested. Moreover, there exists such a solution which besides the fact that it is nested satisfies the following condition: Let $o \in O(K)$ with $y_o > 0$ be inclusion minimal with respect to the property that its dual variable is non–zero, i.e. $y_{o'} = 0$ for all $o' \subset o, o' \in O(K)$. Then:

- The vertex set o induces the subgraph $K[o]$ of K which contains a cycle of length $|o|$ formed by the edges $e = ab$ with zero reduced weight with respect to o, i.e. $w^\star_o(e) = w(e) - y_a - y_b = 0$.
- All the vertices of o except for exactly one of them are matched one to the other in K, i.e. exactly one vertex v of o is matched to a vertex not in o.
- If we contract the vertices of o to a vertex v', set $y_{v'} = y_o$, $w(v'a) = \min_{b\in o}\{w(ba) - y_b\}$, preserve the rest of y (replacing the set o with the vertex v') and match v' to the only vertex to which a vertex of o is matched (this defines the function x for the edges incident to v'), we obtain again an optimal primal and an optimal dual solution for the complete graph K' of the obtained new problem which is nested and has the just introduced properties.

If the weights of the edges are integral, then there exists an optimal dual solution which is nested and half–integral; a vector a is said to be *half–integral* if $2a$ has integral coordinates. If K is a complete graph and the weights of its edges satisfy the triangle inequality, i.e., $w(v_1v_2) \leq w(v_1u) + w(uv_2)$ for all $u, v_1, v_2 \in V(K)$, then there exists an optimal dual solution which is nested, half–integral and $y_v \geq 0$ for all $v \in V(K)$.

2.2 T–join Problem and Maximum Cuts in Planar Graphs

Matchings in complete graphs are closely related to T–join problems. In a T–join problem, you are given a connected graph G and an even–cardinality subset T of vertices of G. The goal is to find a subgraph G' of G with the least number of edges such that the set T consists of precisely the vertices of odd degrees in G'. Each T–join problem has an optimal solution consisting of $|T|/2$ edge–disjoint paths (each of them connecting two different vertices of T). A T–join problem can be reduced to a weighted matching problem in a complete graph: One forms a complete graph K on the vertex set T and the weight of the edge joining the vertices u and v is the length of a shortest path between u and v in G. The weight of a minimum weight perfect matching is precisely the number of edges of an optimal solution of the corresponding T–join problem and the edges included in the matching correspond to the paths in G which form an optimal solution. Note that the weights of the edges in the so constructed complete graph K satisfy the triangle inequality.

In [6] Hadlock explored a nice connection between T-join problems and deleting a minimum set of edges from a planar graph to make it bipartite. Let G be a plane graph, $G^\star$ its dual graph and T the set of all odd–degree vertices in $G^\star$. Let E be the set of all edges contained in the $|T|/2$ edge–disjoint paths of an optimal solution of the T–join problem in $G^\star$. Then $G^\star\setminus$ has only vertices of even degrees, and $G \setminus E$ (as the dual of $G^\star \setminus E$) is bipartite. Thus the optimal solutions of the T–join problem in $G^\star$ one–to–one correspond (through the duality) to the minimum edge sets E of G such that $G \setminus E$ is bipartite. The vector x determines the pairs of vertices between which the edges of shortest paths have to be deleted and the vector y witnesses the optimality of this solution through the "control zone concept" which we introduce in Section 3. This observation led to a polynomial–time algorithm for a maximum cut problem for planar graphs ([6]).

2.3 Concept of "Control Zones"

We introduce the concept of "control zones" in the dual graph corresponding to the optimal dual solution y of the T–join problem. This concept is essentially the same as the concept of "moats" known from the perfect matching problem for points in the plane.

Fix a plane graph G. Let $G^\star$ be the dual graph of G and T the set of all odd–degree vertices of $G^\star$. Let x and y be the optimal primal and dual solutions corresponding to the T–join problem. We assume that y is non–negative, half–integral and nested. We define a *distance of an edge* e *to a vertex* v to be the length of the shortest path starting at v whose last edge is e, i.e. the distance of an edge e incident to a vertex v from v is 1. We assign to each vertex $v \in T$ all edges of $G^\star$ at distance at most $\lfloor y_v \rfloor$ from v in $G^\star$; if y_v is not integral, we assign to the vertex v also the (closer) "halves" of the edges at distance $\lceil y_v \rceil$ from v (Figure 1). Thus, we create a "control zone" for each vertex of T. Note that in case that $y_v = 0$, the control zone of v is empty. In addition, we also create a "control zone" for each $o \in O$ with $y_0 > 0$ (recall O is the set of all the odd–cardinality subsets of T of size three and more). A control zone of o extends control zones of v, $v \in o$, and o', $o' \subset o$: It contains all the edges at distance at most $\lfloor y_v + \sum_{o' \subset o, o' \in O} y_{o'} + y_o \rfloor$ from some vertex v, $v \in o$, which are not included to the control zones of v, $v \in o$, or o', $o' \subset o$; the control zone of o also contains the closer halves of the edges at distance $\lceil y_v + \sum_{o' \subset o, o' \in O} y_{o'} + y_o \rceil$ for some $v \in o$ if $y_v + \sum_{o' \subset o, o' \in O} y_{o'} + y_o$ is not an integer. The union of the control zones of o, of all o', $o' \subset o$, and of all v, $v \in o$, comprises all edges (or their halves) of $G^\star$ whose distance from some vertex v, $v \in o$, is at most $y_v + \sum_{o' \subset o, o' \in O} y_{o'} + y_o$. The important property is that each edge of $G^\star$ is included to at most one control zone or is halved to two different control zones. Hence any two different control zones have an empty intersection. Through each control zone passes exactly one path of the solution of the T–join problem corresponding to x. The path between the pair of the vertices u and v of T in the solution of the T–join problem corresponding to x pass only through the control zones of u, v and o with $|o \cap \{u, v\}| = 1$. It is easy to see that if we are able to construct control zones (which are "balls", i.e., they are expanded to the same distance in all directions in $G^\star$) and to construct paths joining pairs of the vertices of T with the above described properties, then we are able to find a primal solution x and a dual non–negative half–integral solution y such that the constructed system of paths and control zones corresponds to x and y. We

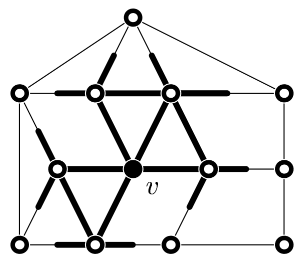

Figure 1: A control zone (bold edges) of a vertex v (drawn by a full circle) with $y_v = 3/2$.

implicitly use this correspondence during the proof without emphasizing this fact.

3 Proof of Theorem 1

We first state an easy lemma which we use during the proof of Theorem 1.

Lemma 1 *A boundary of any odd face of a plane graph contains an odd cycle.*

Proof: Let E be the set of edges on the boundary of the face without the bridges. Since the face is odd and the bridges are counted twice, $|E|$ is odd. The edges of E form an even–degree subgraph H of the plane graph. H can be partitioned into 2–factors, i.e. into cycles. At least one of these cycles contains an odd number of edges and it is actually an odd cycle.

■

We give an intuitive explanation of the proof of Theorem 1 before an accurate proof. Let G be a plane graph, $G^\star$ its dual graph and T the set of odd–degree vertices of $G^\star$. Let x and y be the optimal primal and dual solution corresponding to the T–join problem. We may assume that y is non–negative, half–integral and nested. We find $\lfloor y_v \rfloor$, $v \in T$, edge–disjoint odd cycles formed by the edges of G corresponding to the edges of $G^\star$ joining a vertex at distance i to a vertex at distance $i+1$ from v in $G^\star$ for $0 \le i \le \lfloor y_v \rfloor - 1$. We take additional care when y_v is not integral and two different control zones share the same edge(s). In a fashion similar to finding edge–disjoint odd cycles surrounding vertices v, $v \in T$, we find edge–disjoint odd cycles surrounding sets o, $o \in O$ and $y_o > 0$. In the formal proof of the theorem, we proceed by induction on the number of faces and we find the edge–disjoint cycles one by one.

Proof of Theorem 1: Let G be a plane graph, $G^\star$ its dual graph and T the set of odd–degree vertices of $G^\star$. Let x and y be the optimal primal and dual solution corresponding to the T–join problem such that y is non–negative, half–integral and nested. The proof proceeds by induction on the number of faces of G. We assume that G is bridgeless, but we allow G to be disconnected. Note that $G^\star$ may actually be a multigraph but it is loopless. Let K be the following edge–weighted complete graph on the vertex set T; the weight of an edge uv is the length of a shortest path between u and v in $G^\star$. The complete graph K corresponds to the T–join problem in $G^\star$. Let $x_{uv} \in \{0,1\}$ $(u, v \in T)$ be an optimal solution of the minimum weighted matching problem for the complete graph K. Recall O denotes the set of all odd subsets of T in the rest of the proof. Let y_v $(v \in T)$ and y_o $(o \in O)$ denote the optimal solution of the dual problem. Recall that the dual solution is nested, half–integral and non–negative. Note that the following equality holds (d_{uv} is the length of the shortest path between u and v in $G^\star$):

$$\tau_{\text{odd}}(G) = \sum_{u,v \in T} d_{uv} x_{uv} = \sum_{u \in T} y_u + \sum_{o \in O} y_o$$

Since the reduced weights are non–negative, the following inequality holds:

$$y_u + y_v + \sum_{|o \cap \{u,v\}|=1, o \in O} y_o \le d_{uv}$$

In particular, if $u \in T$ and $v \in T$ are neighbours in $G^\star$, i.e., $d(u,v) = 1$, then the following holds:

$$y_u + y_v + \sum_{|o \cap \{u,v\}|=1, o \in O} y_o \le 1$$

We write C_v for the boundary of the face of G corresponding to the vertex v of $G^\star$.

Let $\mathcal{S}(y)$ be the following expression:

$$\mathcal{S}(y) := \frac{\sum_{u \in T} y_u + \sum_{u \in T} \lfloor y_u \rfloor + \sum_{o \in O} y_o}{2}$$

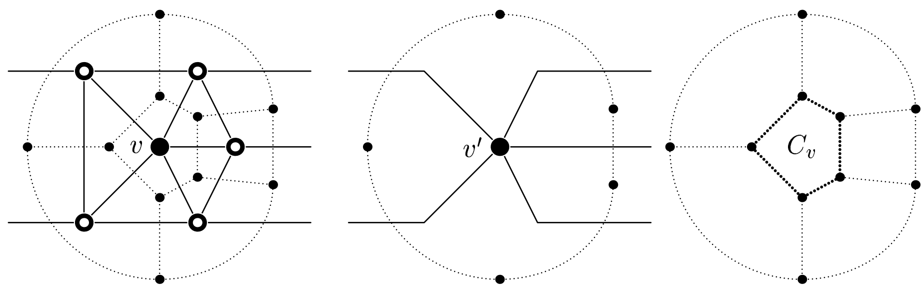

Figure 2: The reduction in the case $y_v > 1$. The vertex v is drawn by a full circle. The edges of $G^\star$ are solid and the edges of G are dotted. The odd cycle of C_v is marked in the left figure.

We construct a set M of at least $\mathcal{S}(y)$ edge–disjoint odd cycles of G. Since $\tau_{\mathrm{odd}}(G) = \sum_{u\in T} y_u + \sum_{o\in O} y_o \le 2\mathcal{S}(y)$, the number of the constructed edge–disjoint odd cycles of G is certainly at least $\tau_{\mathrm{odd}}(G)/2$. Moreover, the cycles in M will satisfy the following condition:

(*) For each $u \in T$ with $y_u \ge 1$, there is exactly one odd cycle in M completely contained in C_u and this cycle is the only one from M which is not edge–disjoint with C_u.

We distinguish three cases which cover all the possibilities:

- **There exists a vertex $v \in T$ with $y_v > 1$.**
  Let E be the union of all C_u, $u \in N(v)$, and let E' be their symmetric difference. Let G' be the graph obtained from G by removing the edges of $E \setminus E'$. This corresponds to a contraction of the set $N(v)$ of $G^\star$ to the vertex v (with simultaneous removal of arising loops) — consult Figure 2. For convenience, let v' denote the vertex after contracting $N(v)$. The boundary of the newly created face is E'. Note that $C_v \subseteq E\setminus E'$. The vertex v is the only vertex of $N(v)$ of odd degree (otherwise the reduced weight of the edge connecting a vertex $v \in T$ and its neighbour in the complete graph K would be negative). Hence $|E'|$ is odd and the newly created face is an odd face. Moreover, all the (edge–disjoint) paths of the T–join except for the path from v are vertex–disjoint with $N(v)$ (at least half of any edge incident with a vertex of $N(v)$ is included in the "control zone" of v since $y_v \ge 3/2$). We set $y'_{v'} := y_v - 1$, $y'_u := y_u$ for $u \neq v$, $u \in T$, and $y'_o := y_o$, $o \in O$

(identifying the vertex v' with v in the sets $o \in O$). We obtain a feasible solution of the dual problem for G'. This decreases $\mathcal{S}(y)$ by 1: $\mathcal{S}(y') = \mathcal{S}(y) - 1$. The primal solution of the original problem is also a primal solution of the new problem (if we identify the vertex v of G with the vertex v' of G') and its weight is decreased by 1. Since the value of the primal solution is equal to the value of the solution of the dual problem, it is an optimal one (thus $\tau_{\text{odd}}(G) = \tau_{\text{odd}}(G') + 1$). Hence we can use induction. We can always extend a set M' of at least $\mathcal{S}(y') = \mathcal{S}(y) - 1$ odd cycles of G' to a set M of at least $\mathcal{S}(y)$ odd cycles of G by adding an odd cycle contained in C_v (it exists due to Lemma 1). If the cycles of M' in G' satisfy condition (*), then the cycles of M in G satisfy condition (*), too.

- **It holds that $y_v \leq 1$ for each $v \in T$ and there exists a set $o \in O$ with $y_o > 0$.**
  Choose a minimal (by inclusion) o with $y_o > 0$. As stated in Section 2, the set o contains a cycle of length $|o|$ (in the complete graph K on the vertices T) consisting of edges e with $w^\star_o(e) = 0$. Let $v_1, v_2, \ldots, v_n$ be this cycle in K (i.e. $n = |o|$ and $o = \{v_1, \ldots, v_n\}$). Since o is inclusion minimal, from (3) it follows that $w(v_i v_{i+1}) = y_{v_i} + y_{v_{i+1}}$, $1 \leq i \leq n$, (indices are taken modulo n). Since w is an integer–valued function and $y_v \leq 1$ for each $v \in T$, either all y_{v_i} are equal to $1/2$ or all y_{v_i} are equal to 0 or 1 for $1 \leq i \leq n$. Hence $d(v_i, v_{i+1}) = w(v_i v_{i+1}) \in \{1, 2\}$ for $1 \leq i \leq n$ (indices are again taken modulo n). By Subsection 2.1, there is exactly one vertex a of o which is matched to a vertex outside o in K.

  In the case that all y_{v_i} for $1 \leq i \leq n$ are integral, we proceed as follows: Let $W = \cup N(v)$ where the union is taken over all vertices $v \in o$ with $y_v = 1$. If $y_v = 0$ for $v \in o$, then both the neighbours v' and v'' of v in the cycle in K have $y_{v'} = 1$ and $y_{v''} = 1$. Then v' and v'' are even neighbours of v in $G^\star$ and $v \in W$. Hence $o \subseteq W$ and the subgraph $G^\star[W]$ is connected. Let E be the union of C_v, taken over all $v \in W$, and let E' be the symmetric difference of C_v, taken over all $v \in W$. Let G' be the graph obtained from G by removing the edges of $E \setminus E'$. Since $G^\star[W]$ is connected, this corresponds to a contraction of the set W in $G^\star$ to the vertex a (with simultaneous removal of arising loops). For convenience, let a' denote the vertex after contracting W. The boundary of the newly created face is E'. Note that $C_{v_i} \subseteq E \setminus E'$ for $y_{v_i} = 1$, $1 \leq i \leq n$ (we do not say anything about C_{v_i} with $y_{v_i} = 0$).

Since the only odd–degree vertices of G contained in W are $v_1, \ldots, v_n$ (the distance between any $u \in T \setminus o$ and $v \in o$ with $y_v = 1$ has to be at least $y_v + y_o \geq 3/2$ due to (1)), the newly created face of G is odd and the degree of a' of $G^\star$ is odd. Moreover, the only (edge–disjoint) paths between vertices of T, which are not vertex–disjoint with W, are the paths from or to the vertices of o. We set $y'_{a'} := y_o$, i. e., $y'_{a'} := y_o + y_a$, if $y_a = 0$, and $y'_{a'} := y_o + (y_a - 1)$, if $y_a = 1$. Further we set $y'_v := y_v$ for $v \in T \setminus o$, and $y'_{o'} := y_{o'}$ for $o' \in O$ (identifying the vertex a' with the set o in the sets $o' \in O$, $o \subset o'$; recall that y is nested; the set o is not in O in G').

We have obtained a feasible solution of the dual problem for G'. This decreases $S(y)$ by at least $\sum_{1 \leq i \leq n} y_{v_i}$, i. e., $\mathcal{S}(y') \geq \mathcal{S}(y) - \sum_{1 \leq i \leq n} y_{v_i}$. The primal solution of the original problem is also a primal solution of the new problem if we replace the edge au with $a'u$ where u is the vertex matched to a in the original solution (its weight is decreased exactly by $\sum_{1 \leq i \leq n} y_{v_i}$). Since in G' the value of the primal solution is equal to the value of the dual solution, it is an optimal one (hence $\tau_{\text{odd}}(G) = \tau_{\text{odd}}(G') + \sum_{1 \leq i \leq n} y_{v_i}$; recall that $y_{o'} = 0$ for $o' \subset o$ due to the choice of o). Thus we may use induction. We can always extend a set M' of at least $\mathcal{S}(y')$ odd edge–disjoint cycles of G' by adding the odd edge–disjoint cycles of C_{v_i} for $y_{v_i} = 1$, $1 \leq i \leq n$, (their existence follows from Lemma 1) to a set M of at least $\mathcal{S}(y') + \sum_{1 \leq i \leq n} y_{v_i} \geq \mathcal{S}(y)$ odd edge–disjoint cycles of G. Note that the conditions $y_{v_i} = y_{v_j} = 1$ for $i \neq j$ imply that C_{v_i} and C_{v_j} are edge–disjoint and fully contained in $E \setminus E'$. If the cycles of M' in G' satisfy the condition (*) then the cycles of M in G also satisfy the condition (*).

Next, we deal with the remaining case that $y_{v_i} = 1/2$ for all $1 \leq i \leq n$ in this paragraph. Let $W := o$ be the set consisting of all v_i for $1 \leq i \leq n$. Let E be the union of C_v, taken over all $v \in W$, and E' the symmetric difference of C_v, taken over all $v \in W$. The graph $G^\star[W]$ is obviously connected. Let G' be the graph obtained from G by removing the edges of $E \setminus E'$. This corresponds to a contraction of the set W to the vertex a (with simultaneous removal of arising loops) — consult Figure 3. For convenience, let a' denote the vertex after the contraction of W. The boundary of the newly created face is E'. Since W consists of n odd-degree vertices of G^*, the newly created face of G is odd and the degree of a' in G^* is odd. We set

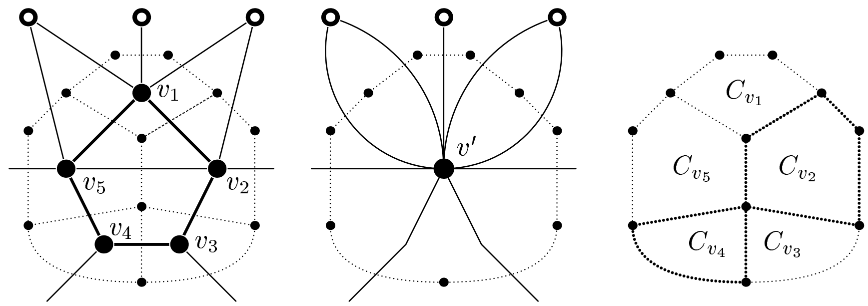

Figure 3: The reduction in the case $y_{v_i} = 1/2$ for all $v_i \in o = \{v_1, v_2, v_3, v_4, v_5\}$, $y_o > 0$. The vertices v_i are drawn by full circles. The edges of $G^\star$ are solid and the edges of G are dotted. The odd cycles of the boundaries C_{v_2} and C_{v_4} are marked in the left figure.

$y_{a'} := y_o + y_a = y_o + 1/2$. We further set $y'_v := y_v$ for $v \in T \setminus o$ and $y'_{o'} := y_{o'}$ for $o' \in O$ (identifying the vertex a' with the set o in the sets $o' \in O$, $o \subset o'$; recall that y is nested; note that the set o is not in O in G'). We obtained a feasible solution of the dual problem for G'. The value of $S(y)$ is decreased by $n/4 + y_o/2$ due to the removal of y_{v_i}, for $1 \leq i \leq n$, and y_o from the sum and it is increased by $1/4 + y_o/2$ due to adding $y_{a'}$ to the sum and additional at least $1/2$ due to the fact that $y_{a'} = y_o + 1/2 \geq 1$. We conclude that $S(y') \geq S(y) - (n-3)/4$. The primal solution of the original problem is also a primal solution of the new problem if we replace the edge au with $a'u$ where u is the vertex matched to a in the original solution (its weight is decreased exactly by $\sum_{1 \leq i \leq n} y_{v_i} - 1/2 = (n-1)/2$). Since its value is equal to the value of the solution of the dual problem, it is an optimal one (hence $\tau_{\text{odd}}(G) = \tau_{\text{odd}}(G') - 1/2 + \sum_{1 \leq i \leq n} y_{v_i} = \tau_{\text{odd}}(G') + (n-1)/2$). Note that $C_{v_i} \subseteq E$, but it might be that $C_{v_i} \cap (E \setminus E') \neq \emptyset$ for all $1 \leq i \leq n$. The boundaries C_{v_i} and C_{v_j} are not disjoint iff v_i and v_j are adjacent in G. Since $G^\star$ is planar, there are at least $n/4$ mutually non–adjacent vertices among $v_1, \ldots, v_n$ due to the four–colour theorem. Hence at least $(n+1)/4$ of the boundaries C_{v_i} (recall that n is odd) are edge–disjoint and each of them contains an odd cycle due to Lemma 1. It is the right time to use induction. G' contains at least $\mathcal{S}(y')$ edge–disjoint odd cycles such that exactly one of them is not disjoint with

$C_{a'}$ in G' (recall that $y'_{a'} \geq 1$). If we replace this odd cycle with at least $(n+1)/4$ edge–disjoint odd cycles mentioned above, we get at least $\mathcal{S}(y') + (n+1)/4 - 1$ edge–disjoint odd cycles of G. A simple calculation gives the following bound:

$$\mathcal{S}(y') + (n+1)/4 - 1 = \mathcal{S}(y') + (n-3)/4 \geq \mathcal{S}(y)$$

Hence, we get at least $\mathcal{S}(y)$ edge–disjoint odd cycles of G which satisfy condition (*).

- **It holds that $y_v \leq 1$ for each $v \in T$ and $y_o = 0$ for each $o \in O$.** Let W be the set of all the vertices v with $y_v > 0$. Recall that $G^\star[W]$ is the subgraph of $G^\star$ induced by the vertices of W. Note that each vertex v, $y_v = 1$, is an isolated vertex of $G^\star[W]$, since the distance between v and any other vertex u with $y_u > 0$ has to be at least $y_v + y_u > 1$. Let n_1 be the number of vertices v with $y_v = 1$ and n_2 the number of vertices v with $y_v = 1/2$. The graph $G^\star[W]$ contains an independent set A of size $n_1 + \lceil n_2/4 \rceil$ (there are n_1 isolated vertices and the rest of $G^\star[W]$ contains an independent set of size at least $\lceil n_2/4 \rceil$ due to the four–color theorem). The boundaries C_v and C_w for $v, w \in A$, $v \neq w$, are edge–disjoint. Each C_v contains an odd cycle due to Lemma 1. The value $\mathcal{S}(y)$ is equal to $(2n_1 + n_2/2)/2 = n_1 + n_2/4$ and hence we have enough cycles. Since we include among these cycles exactly one cycle fully contained in C_v for each v, $y_v \geq 1$, the cycles satisfy also condition (*).

■

4 Tightness of the Bound

Theorem 2 *There is a 3–connected planar graph G with $\tau_{odd}(G) = 2k$ and $\nu_{odd}(G) = k$ for each integer $k \geq 1$.*

Note that if we leave out the assumption that G is 3–connected, the theorem becomes trivial because one can consider a disjoint union of k copies of K_4.

Proof: Let k be the fixed integer from the statement of the theorem. Let G be a 3–connected quadrangulation with at least k vertices of degree 3. We replace k vertices of degree three with the gadget from Figure 4. Let

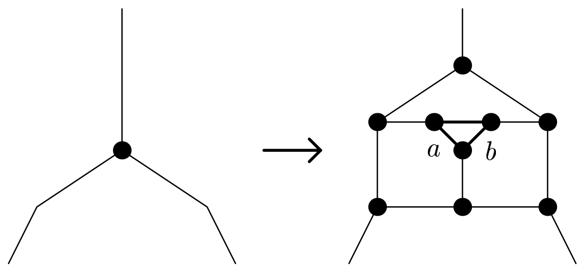

Figure 4: Replacing vertices of degree three in a quadrangulation with the gadget.

G' be the obtained graph. Note that G' is 3–connected. We claim that $\tau_{\text{odd}}(G') = 2k$ and $\nu_{\text{odd}}(G') = k$. In order to destroy all odd cycles fully contained in one gadget, two edges of it have to be deleted. There are k disjoint gadgets in G'. Hence at least $2k$ edges have to be deleted to make G' bipartite. We conclude that $\tau_{\text{odd}}(G') \geq 2k$.

On the other hand: If in each of the k gadgets the edges a and b (cf. Figure 4) are deleted, the obtained graph is bipartite. Hence $\tau_{\text{odd}}(G') \leq 2k$ and $\tau_{\text{odd}}(G') = 2k$. This also implies: Each odd cycle contains the edge a or the edge b from at least one of the gadgets. But then it includes at least two vertices from the inner triangle (bold one in Figure 4). Consequently, each odd cycle contains from at least one gadget two vertices of its inner triangle. Hence G' has no $k+1$ edge–disjoint odd cycles because otherwise by the pigeon–hole principle two of those odd cycles would contain two vertices of the inner triangle of the same gadget and would have a common vertex and, therefore, also a common edge (all the vertices of the inner triangle have degree 3). Thus $\nu_{\text{odd}}(G') \leq k$. Each of the k disjoint gadgets contains an odd cycle and hence $\nu_{\text{odd}}(G') \geq k$. The last inequality also follows from Theorem 1.

■

5 Conclusion

We proved that $\tau_{\text{odd}}(G) \leq 2\nu_{\text{odd}}(G)$ for any planar graph G. On the other hand, Reed's Escher walls from [9] provide an example of projective planar graphs G with $\tau_{\text{odd}}(G)$ arbitrary large and $\nu_{\text{odd}}(G) = 1$. But if $\tau_{\text{odd}}(G)$ is large, these graphs are not embeddable to a torus. Hence the following problem arises:

Problem 1 *What is the relation between $\tau_{odd}(G)$ and $\nu_{odd}(G)$ for graphs G which can be embedded to a torus?*

Or even more generally:

Problem 2 *Describe (if there is any) the relation between the numbers $\tau_{odd}(G)$, $\nu_{odd}(G)$ and $g(G)$ for a graph G which can be embedded to an orientable surface of genus at most $g(G)$.*

It could be probably derivable from the results of [9] that $\tau_{\text{odd}}(G)$ is bounded by a function of $\nu_{\text{odd}}(G)$ and $g(G)$.

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