Nowhere-zero 3-flows in Cayley graphs of Abelian groups

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Abstract

We characterize Cayley graphs of abelian groups which admit a nowhere-zero 3-flow. In particular, we prove that every k-valent Cay-

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ley graph of an abelian group, where $k \geq 4$, admits a nowhere-zero 3-flow.

1 Introduction

Vertex-set, edge-set and the automorphism group of a graph X will be denoted by $V(X)$, $E(X)$ and $\mathrm{Aut}\, X$, respectively. The graph X is *vertex-transitive* if $\mathrm{Aut}\, X$ acts transitively on the set of vertices of X. A *dart* of a graph X is an ordered pair (u, v), where $\{u, v\}$ is an edge of X. The set of all darts of X will be denoted by $D(X)$. Let $(A, +)$ be an abelian group. A function $f\colon D(X) \to A \setminus \{0\}$ is a *nowhere-zero A-flow* if the following two conditions are satisfied:

(i) $f(u, v) = -f(v, u)$, for each dart $(u, v) \in D(X)$;

(ii) $\sum_{(u,v)\in D(X)} f(u, v) = 0$, for each vertex $u \in V(X)$.

If $A = \mathbb{Z}$ and the image of f is contained in the interval $[-k + 1, k - 1]$, then f is called a *nowhere-zero k-flow.* It is easy to see that every Eulerian graph admits a nowhere-zero 2-flow.

In [9, 10] Tutte initiated a study of nowhere-zero k-flows. He proved some important results on the existence of nowhere-zero k-flows for certain families of graphs, as well as posed some deep conjectures, most of them still being open. He proved that a graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow for some abelian group A of cardinality k. It is easy to see that if a graph admitting a nowhere-zero k-flow for some k has no bridges. On the other hand, Tutte conjectured that there is a positive integer k, such that every bridgeless graph admits a nowhere-zero k-flow. His conjecture was proved for $k = 8$ independently by Kilpatrick [4] and Jaeger [2, 3]. This result is now superseded by the 6-flow theorem of Seymour [8]. The question wheter $k = 5$ satisfies the condition in the conjecture remains to be unanswered and is know as the 5-flow conjecture. Since the Petersen graph is bridgeless and has no nowhere-zero 4-flows, the number $k = 5$ is the best possible. Avoiding the peculiarity of the Petersen graph Tutte posed the so-called the 4-flow conjecture saying that every bridgeless graph without a Petersen minor admits a nowhere-zero 4-flow. He also posed the so called the 3-flow conjecture, claiming that every bridgeless graph without 3-edge-cuts admits a nowhere-zero 3-flow.

Jaeger [2, 3] showed that every 4-edge-connected graph admits a nowhere-zero 4-flow. His result, together with the well known fact that every vertex-transitive graph of valence k is k-edge-connected, motivates the investigation of the interplay between the symmetry properties of graphs and the existence of nowhere-zero k-flows. It follows trivially from the above results that every vertex-transitive graph of valence at least 4 admits a nowhere-zero 4-flow. For the result on 4-flows in cubic vertex-transitive graphs we refer the reader to [1, 6, 7]. For further results on nowhere-zero flows we recommend [11].

In this paper we shall consider the Tutte's 3-flow conjecture restricted to a subclass of vertex-transitive graphs, namely the Cayley graphs of abelian groups. A *Cayley graph* $\mathit{Cay}(G, S)$ *of the group* G *and with symbol* S, where S is a subset of $G \backslash \{1\}$ such that $S^{-1} = S$, is the graph with vertex-set G and edge-set $\{\{g, gs\} \mid g \in G, s \in S\}$. The main result of this paper is Theorem 3.3, saying that every Cayley graph of an abelian group of degree at least 4 admits a nowhere-zero 3-flow. This immediately implies that a Cayley graph of an abelian group admits a nowhere-zero 3-flow if and only if it is of valence 2 (and thus isomorphic to a cycle or to a disjoint union of cycles), or of valence 3 and bipartite, or of valence greater than 3 (see Corollary 3.4). In the last section we consider the case of Cayley multigraphs, and show that the above result applies for them as well.

The path and the cycle on n vertices are denoted by P_n and C_n, respectively. The (multi)graph comprised of two vertices connected by k parallel edges is denoted by Θ_k. Denote by $C(n, k)$ the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \{-1, 1, -k, k\})$, that is the graph which can be constructed from the n-cycle $x_0 x_1 \cdots x_{n-1} x_0$ by connecting x_i with x_{i+k} (index modulo n) for each $i \in \{0, \dots, n-1\}$.

2 Nowhere-zero 3-flows of some product graphs

In this section we will construct a nowhere-zero 3-flow in some particular Cayley graphs. All of these graphs are cartesian product graphs, therefore we will use the following notion. The Cartesian product $X \square Y$ of two graphs X and Y is defined on the vertex-set $V(X) \times V(Y)$, where two vertices (x_1, y_1) and (x_2, y_2) of $X \square Y$ are adjacent if $x_1 = x_2$ and y_1 is adjacent to y_2 in Y, or if x_1 is adjacent to x_2 in X and $y_1 = y_2$. Edges of the form $\{(x_1, y_1), (x_1, y_2)\}$ are called Y-*edges*, and the others X-*edges*. Let $v = (v_1, v_2) \in X \square Y$. Then the subgraph induced in $X \square Y$ by the vertices of the form $\{(v_1, y) \mid y \in V(Y)\}$ is called the Y*-layer through* v. The X-layer through v is defined similarly. In what follows, we will use the following result of Tutte [9].

Theorem 2.1 *A cubic graph admits a nowhere zero 3-flow if and only if it is bipartite.*

Proposition 2.2 *Let $m, n \geq 3$ be integers. Then the graph $C_m \square C_n \square K_2$ admits a nowhere-zero 3-flow.*

Proof. Let $X = C_m \square C_n \square K_2$. If m is an even number, then remove the edges in X which correspond to the layers of C_n. As a result we obtain n vertex disjoints copies of $C_m \square K_2$. This graph is bipartite and cubic and by Theorem 2.1 admits a nowhere-zero 3-flow. Note that the removed edges induce the graph with m connected components isomorphic to C_n. This graph admits a nowhere-zero 3-flow. But then the superposition X of the two graphs admits a nowhere-zero 3-flow as well.

We may therefore assume that both m and n are odd numbers. The two copies of X isomorphic to $C_m \square C_n$ will be refered to as 0- and 1-layer. The vertices in each layer will be labelled by (x, y), where $x \in Z_m$, $y \in Z_n$, so that the vertices with the same label, but in different layers, are adjacent.

We shall now remove the edges of appropriately chosen cycles of X so that the remaining graph X' will be cubic and bipartite. By Theorem 2.1 this graph will then admit a nowhere-zero 3-flow. Since the removed cycles will be edge disjoint we will easily extend the 3-flow of X' to a nowhere-zero 3-flow of X.

First, in each of the two layers remove the edges of the 4-cycles induced by vertices $(0, j)$, $(1, j)$, $(1, j+1)$ and $(0, j+1)$, for each $j = 1, 3, \ldots, (m-2)$.

In each i-layer we let the path P_i contain the following edges

(i) $\{(j, 0), (j, 1)\}, \{(j, 1), (j, 2)\}, \ldots, \{(j, m-2), (j, m-1)\}$ for $j = 2, 3, \ldots,$ $n-1$;

(ii) $\{(j, m-1), (j+1, m-1)\}$ for $j = 2, 4 \ldots, n-3$;

(iii) $\{(j, 0), (j+1, 0)\}$ for $j = 1, 3, \ldots, n-2$;

(iv) $\{(0, 0), (1, 0)\}$.

Let C be the cycle comprised of the edges of P_1, P_2, the edge connecting both vertices labelled by $(0, 0)$ and the edge connecting both vertices labelled by $(m-1, n-1)$. Finally, remove the edges of C from X to obtain X'.

Let the subset A_i of the vertex-set of the i-layer (for each $i \in \mathbb{Z}_2$) contain the following vertices:

(i) $(0, j)$ and $(1, j)$ for $j = 0, 2, 4, \ldots, m-3$;

(ii) $(j, m-2)$ and $(j, m-1)$ for $j = 2, 4 \ldots, n-1$;

(iii) (j, l) for $2 \leq j \leq n-1$, $0 \leq l \leq m-3$, where $j + l$ is odd.

Further, for each $i \in \mathbb{Z}_2$ let B_i denote the complement of the set A_i relative to the vertex-set of the i-layer. It is easy to see that the sets $A_0 \cup B_1$ and $A_1 \cup B_0$ form the bipartition of the graph X'. ■

Note that in the following proposition the graph is not simple for the case $n = 1$, and for the case m even and $n = \frac{m}{2}$. We need these cases in the last section, where the nowhere-zero 3-flow is considered for Cayley multigraphs.

Proposition 2.3 *Let m, n be two integers such that $m \geq n \geq 1$ and $m \geq 3$. Then the graph $C(m,n) \,\square\, K_2$ admits a nowhere-zero 3-flow.*

Proof. In the graph $C(m, n)$, let $C = x_0 x_1 \cdots x_{m-1} x_0$ be the outer cycle, and let $\bar{C} = C(m, n) - C$. Edges from C and $\bar{C}$ are called *m-edges* and *n-edges*, respectively. Note that if $\gcd(m, n) \neq 1$ then $\bar{C}$ is comprised of more than one cycle.

Let $X = C(m, n) \,\square\, K_2$. If m is an even number, then we can decompose X into one 2-factor (induced by the n-edegs) and one bipartite 3-factor (induced by the remaining edges), and the result follows. So, we assume that m is odd. Since $C(m, n)$ is isomorphic to $C(m, m-n)$, we can assume that n is odd.

In both layers of $C(m, n)$ in X, split every x_i $(i \neq 0, n)$ into two vertices x_i', x_i'' so that the vertex x_i' is incident with the m-edges and the K_2-edge of x_i and the vertex x_i'' is incident with the n-edges of x_i. Finally, in each layer, split x_0 into x_0', x_0'' so that x_0' is adjacent to x_{m-1}' and it is incident with the K_2-edge of x_0. And, similarly split x_n into two vertices x_n', x_n'' so that x_n' is adjacent to x_{n+1}' and incident with the K_2-edge of x_n. Note that x_0'' and x_n'' are connected by an edge and by a path of length ≥ 2 whose intermediate vertices are of degree 2 and whose all edges are n-edges. Contract this path into an edge. Afterwards, x_0'' and x_n'' are connected by two edges. Denote by X' the new constructed graph. The component of X' which contains vertices $x_n', x_{n+1}', \ldots, x_{m-1}', x_0'$ (from both layers) is isomorphic to $P_{m-n+1} \,\square\, K_2$; we left the reader to check that this graph admits nowehere 3-flow. Since n is an odd integer, it follows that the component of X' which contains vertices x_0'', x_n'' (and $x_1', x_2', \ldots, x_{n-1}'$) is a bipartite cubic graph. Note that this component contains $n-1$ K_2-edges. In case that $\gcd(m,n) \neq 1$, the graph X' has also components which are cycles comprised of n-edges.

By Theorem 2.1, X' admits a nowhere-zero 3-flow. But than it easily follows that X also admits a nowhere-zero 3-flow. ■

Proposition 2.4 *Let $m \geq 3$ be an odd integer. Then graph $K_4 \,\square\, C_m$ admits a nowhere-zero 3-flow.*

Proof. Denote the vertices of K_4 by x, y, z and w, and the vertices of the cycle C_m by the elements of $\mathbb{Z}_m$, so that the edges of C_m are of the form $\{i, i+1\}$, $i \in \mathbb{Z}_m$. The vertices of the graph $X = K_4 \,\square\, C_m$ are $(x,i), (y,i), (z,i), (w,i)$, $i \in \mathbb{Z}_m$, which are briefly denoted by x_i, y_i, z_i, w_i, respectively. Now remove from X the edges of the following 4-cycles: $(x_0, y_0, y_{m-1}, x_{m-1})$, $(z_0, w_0, z_{m-1}, w_{m-1})$, and (x_i, y_i, w_i, z_i) for $i = 1, 2, \ldots, m-2$. Note that thus obtained graph X' is cubic and bipartite (where one of the two bipartition sets is comprised of the vertices x_{m-1}, y_{m-1}, and $x_i, y_i, z_{i+1}, w_{i+1}$, for $i = 0, 2, \ldots, m-3$). By Theorem 2.1 the graph X' admits a nowhere-zero 3-flow. But then so does X, as required. ■

3 Nowhere-zero 3-flows of Cayley graphs

In the proof of the main theorem we are going to use the notion of a quotient graph. Let X be a graph and $\mathcal{B} = \{V_1, V_2, \ldots, V_k\}$ a partition of the vertex-set $V(X)$. The *quotient graph* $X_{\mathcal{B}}$ of the graph X relative to the partition $\mathcal{B}$ is the graph with vertex-set $\mathcal{B}$ and an edge between V_i and V_j if and only if there exist vertices $v \in V_i$ and $u \in V_j$, such that $\{v, u\} \in E(X)$. If for each $V_i \in \mathcal{B}$ the valence of V_i in the graph $X_{\mathcal{B}}$ is the same as the valence of any vertex $v \in V_i$, then the graph $X_{\mathcal{B}}$ is refereed to as a *regular quotient graph* of X. The following lemma and proposition will be used in the proof of the main theorem.

Lemma 3.1 *Let $X = Cay(G; S)$ be a Cayley graph, and let H be a normal subgroup of G satisfying the following conditions:*

(i) $S \cap H = \emptyset$;

(ii) $H \cap \{s^{-1}t \mid s, t \in S, s \neq t\} = \emptyset$.

Then the Cayley graph $Cay(G/H, S/H)$*, where* $S/H = \{sH \mid s \in S\} \subset G/H$*, is a regular quotient graph of the graph* X*.*

Proof. The Cayley graph $\mathrm{Cay}\,(G/H, S/H)$ is clearly the quotient graph of X relative to the partition consisting of the orbits of the action of H by left multiplication. The conditions (i) and (ii) ensure the regularity of the quotient. ■

Proposition 3.2 *Let* $X_{\mathcal{B}}$ *be a regular quotient graph of a graph* X *and let* A *be an abelian group. If the graph* $X_{\mathcal{B}}$ *admits a nowhere-zero* A*-flow, then the graph* X *also admits a nowhere-zero* A*-flow.*

Proof. Let f be a nowhere-zero A-flow in the graph $X_{\mathcal{B}}$. Define $\tilde{f}\colon D(X) \to A$ by the rule $\tilde{f}(v, u) = f(V_i, V_j)$, where $v \in V_i$, $u \in V_j$. The function $\tilde{f}$ is clearly a nowhere-zero A-flow in X. ■

We are now ready to prove the main result of this paper.

Theorem 3.3 *Let* $X = Cay(G, S)$ *be a Cayley graph of an abelian group* G *and of degree at least* 4*. Then* X *admits a nowhere-zero 3-flow.*

Proof. Since every graph with vertices of even degrees admits a nowhere-zero 2-flow we may assume that X is an r-regular graph with $r \geq 5$ odd integer. Moreover, since the connected components of a disconnected Cayley graph of a group G are Cayley graphs of a subgroup of G, we can assume that X is conneted, and therefore that S is a generating set of G.

Suppose that F is the set of edges from X induced by two involutions or by a generator of order ≥ 3. Then, F is a 2-factor of X. If $X - F$ admits a nowhere-zero 3-flow, then so does X. It is therefore sufficient to prove the theorem for $r = 5$. As non-involutory elements of S comes in pairs, the number of involutions in S is odd.

Suppose first that S consists of 5 involutions, $S = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$. Then at least one of the involutions τ_3, τ_4, τ_5 (say τ_3) is not contained in the group $\langle \tau_1, \tau_2 \rangle$. But then the graph obtained from X by deleting the 4-cycles generated by τ_4- and τ_5-edges is cubic bipartite (since isomorphic to a disjoint union of cubes Q_3) and thus (by Theorem 2.1) admits a nowhere-zero 3-flow. As the graph induced by the deleted edges admits a nowhere-zero 3-flow as well, so does the graph X.

Suppose now that S contains 3 involutions τ_1, τ_2, τ_3 and elements c, c^{-1} of order $r \geq 3$. If the three involution generate the group isomorphic to $\mathbb{Z}_2^3$, then the existence of a nowhere-zero 3-flow in X follows similarly as in the previous paragraph (delete the c-edges to obtain a bipartite graph isomorphic to a disjoint union of cubes Q_3). We can thus assume that each of the three involutions is the product of the other two, and that the graph induced by the τ_1-, τ_2- and τ_3-edges is isomorphic to a disjoint of complete graphs K_4. We shall split the proof into two cases. Suppose first that the group $\langle c \rangle$ intersects the set $\{\tau_1, \tau_2, \tau_3\}$ trivially. In this case the graph X is isomorphic to the cartesian product $K_4 \,\square\, C_r$ and it admits a nowhere-zero 3-flow by Proposition 2.4. Suppose, on the contrary, that $\langle c \rangle$ contains at least one of the involutions τ_1, τ_2, τ_3, say τ_3. In this case r is even and the graph induced by τ_3- and c- edges is isomorphic to the prism $K_2 \,\square\, C_r$, which is bipartite and cubic, and therefore admits a nowhere-zero 3-flow. This implies that the graph X admits a nowhere-zero 3-flow as well.

We are now left with the case where $S = \{c, c^{-1}, d, d^{-1}, \tau\}$, and c, d are of respective orders $n_c, n_d \geq 3$, while τ is an involution. As the group G is an abelian group of even order, it contains a normal subgroup N of index 2 in G. We shall consider several cases with respect to the intersection $S \cap N$. Note that, for the sake of connectivity of X we have that $|S \cap N| \leq 4$. In most of the cases we will find a 2-factor F in the graph X, such that the cubic graph $X - F$, obtained from X by deleting the edges of F, will be bipartite. In this case both graphs F and $X - F$ admit a nowhere-zero 3-flow, implying that their superposition X admits a nowhere-zero 3-flow as well.

Case 1 If $|S \cap N| = 0$ then the graph is bipartite and admits a nowhere-zero 3-flow by Theorem 2.1.

Case 2 If $|S \cap N| = 1$, then $S \cap N = \{\tau\}$. Note that in this case the orders n_c and n_d of elements c and d in G are even. At every vertex x of X, consider the 4-cycle whose vertices are $x, cx, \tau cx$, and τx. All these 4-cycles comprise a 2-factor F with $X - F$ being bipartite and cubic.

Case 3 If $|S \cap N| = 2$ then we may assume that $S \cap N = \{c, c^{-1}\}$. Let F be the set of c-edges of X. The set F is a 2-factor of X, such that $X - F$ is bipartite.

Case 4 If $|S \cap N| = 3$ we may assume that $S \cap N = \{c, c^{-1}, \tau\}$. Note that n_c must be even. Let F be the set of c-edges of X. The graph $X - F$ is then isomorphic to the graph $\mathrm{Cay}\,(G; \{\tau, d, d^{-1}\})$. Since the subgroups $\langle \tau \rangle$

and $\langle d \rangle$ of G intersect trivially, the graph $X - F$ is isomorphic to a disjoint union of bipartite graphs isomorphic to $K_2 \square C_{2k}$.

Case 5 Finally, suppose that $|S \cap N| = 4$. Then, $S \cap N = \{c, c^{-1}, d, d^{-1}\}$ and the graph X is isomorphic to the graph $Y \square K_2$, where $Y = \mathrm{Cay}\,(N; S \cap N)$. Moreover, if n_c (or n_d) is an even number then let F be the 2-factor induced by d-edges (or c-edges, resp.). In this case, $X - F$ is a bipartite cubic graph. So, we may assume that both n_c and n_d are odd. Consider now the subgroup $K = \langle c \rangle \cap \langle d \rangle \leq N$. We shall consider three subcases.

Subcase a) Suppose that $K \cap S \neq \emptyset$. Then either $\langle c \rangle \subset \langle d \rangle$ or $\langle d \rangle \subset \langle c \rangle$, and X is isomorphic to the graph $C(m, n) \square K_2$ for some $m \in \{n_c, n_d\}$ and some $n \leq m$. By Proposition 2.3 the graph X admits a nowhere-zero 3-flow.

Subcase b) Suppose that $K \cap S = \emptyset$ and $K \cap \{s^{-1}t \mid s, t \in S, s \neq t\} \neq \emptyset$. Then either $c^2 \in \langle d \rangle$ or $d^2 \in \langle c \rangle$. But then (since $K \cap S = \emptyset$) the either n_c or n_d is even, contradicting the assumption on the parity of n_c and n_d.

Subcase c) Finally suppose that $K \cap S = \emptyset$ and $K \cap \{s^{-1}t \mid s, t \in S, s \neq t\} = \emptyset$. By Lemma 3.1 the Cayley graph $\mathrm{Cay}\,(G/K; S/K)$ is a regular quotient of X. Note that the intersection $\langle cK \rangle \cap \langle dK \rangle$ is trivial and that the group G/K is isomorphic to the direct product $\langle cK \rangle \times \langle dK \rangle \times \langle \tau K \rangle$. The graph $\mathrm{Cay}\,(G/K; S/K)$ is thus isomorphic to the graph $C_{n'_c} \square C_{n'_d} \square K_2$, where $n'_c = n_c/|K|$ and $n'_d = n_d/|K|$. By Proposition 2.2 it follows that the graph $\mathrm{Cay}\,(G/K; S/K)$ admits a nowhere-zero 3-flow. But then by Proposition 3.2 the graph X admits a nowhere-zero 3-flow as well. ■

From the above theorem, one can easily obtain the following characterization.

Corollary 3.4 *Let $X = Cay(G, S)$ be a Cayley graph of an abelian group G. Then X admits nowhere-zero 3-flow if and only if it is of valence 2, or of valence 3 and bipartite, or of valence at least 4.*

4 Nowhere-zero 3-flows of Cayley multigraphs

The concept of nowhere-zero flows is based on the class of multigraphs and not stricly on simple graphs. This is our motivation to extend Theorem 3.3 to Cayley multigraphs, that is Cayley graphs $\mathrm{Cay}\,(G; S)$ where the symbol S is a multiset containing each element s with the same multiplicity as its inverse s^{-1}. Moreover, we will allow the symbol S to contain the unit 1 of the group G. The Cayley multigraphs defined in this way may contain multiple edges as well as loops. Denote by C_n^2 the graph constructed from the cycle C_n by doubling every edge. Thus, C_n^2 is a 4-valence graph with each edge of multiplicity two. If $n \geq 4$ is an even number then let $C^*(n, \frac{n}{2})$ be the graph obtained from C_n^2 by connecting each two antipodal vertices.

First we show that $C^*(n, \frac{n}{2})$ admits a nowhere-zero 3-flow. We consider the following two cases. If $n = 4k+2$ for some k, then $C^*(n, \frac{n}{2})$ is a bipartite 5-valence graph. Then, we can decompose this graph into edge-disjoint one 2-factor and one 3-factor. Since the 3-factor is bipartite, we easily construct a nowehere zero 3-flow in $C^*(n, \frac{n}{2})$. Otherwsie, $n = 4k$ for some k. Then, we can decompose $C^*(n, \frac{n}{2})$ into $2k$ copies of Θ_2 and k copies of $\Theta_2 \,\square\, K_2$ which are pairwise edge-disjoint. Again, we easily construct a nowehere zero 3-flow in $C^*(n, \frac{n}{2})$.

Theorem 4.1 *Let $X = Cay(G, S)$ be a Cayley multigraph of an abelian group G. Then X admits nowhere-zero 3-flow if and only if after deleting the loops of X, the resulting graph is of valence 2, or of valence 3 and bipartite, or of valence grater than 3.*

Proof. Note that X admits a nowhere-zero 3-flow if and only if the graph obtained from X by removing all its loops admits a nowhere-zero 3-flow. So, we may assume that X is loopless. We may also assume that X is connected. It is easy to see that if X is of degree 1 or X is non-bipartite cubic graph, then X has no nowhere-zero 3-flows. Further if X is of even valence, then it admits such flow. So, assume that X is of odd valence. Similarly, as in the proof of Theorem 3.3, we may assume that X is of valence 5. This implies that the symbol S contains at least one involution, say τ.

Let μ be the maximum multiplicity of an edge in X (or generator from S). If $\mu = 1$ then X is a simple graph, and the proof follows by Theorem 3.3. Now, we consider the folowing cases:

Case 1 If $\mu = 2$ suppose first that τ is an involution of multiplicity two in S. If there exists $c \in S$ which is not an invoultion, then X can be decomposed into edge-disjoint cycles (induced by the edges of c) and copies of $\Theta_2 \,\square\, K_2$, and the result follows. Otherwise, all elements of S are involutions, which implies that X ismorphic to $C(4, 2)$ or $C^*(4, 2)$. Note that both graphs admit a nowhere-zero 3-flow. Suppose now that $c \in S$ is of multiplicity two and c is not an involution. Then X is isomorphic to $C^*(n, \frac{n}{2})$ or $C_n^2 \,\square\, K_2$ for some n. By the argument above and by Propositions 2.3 these graphs admit a nowhere-zero 3-flow.

CASE 2   If $\mu = 3$ the graph X can be decomposed into edge disjoints cycles (possibly of length 2) and copies of Θ_3. Since Θ_3 is bipartite it admits a nowhere-zero 3-flow and the result follows.

CASE 3   If $\mu = 4$ the graph X can be decomposed into edge-disjoint cycles and copies of Θ_3, similarly as in Case 2.

CASE 4   Finally if $\mu = 5$ then X is isomorphic to Θ_5, which admits a nowhere-zero 3-flow. ∎

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