Antisymmetric flows and strong oriented coloring of planar graphs

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Abstract

In [NR] Nešetřil and Raspaud defined antisymmetric flow, which is a variant of nowhere zero flow, and the dual notion to strong oriented coloring. We give an upper bound on the number of colors needed to a strong oriented coloring of a planar graph, and hereby we find a small antisymmetric flow for any planar graph. In the proof we use a result from combinatorial number theory—existence of large Sidon sets.

1 Introduction

Vertex set of a graph (or oriented graph) G is denoted by $V(G)$, its edge set by $E(G)$. We consider only graphs without 1-cycles (i.e. loops) and 2-cycles.

An important part of graph theory concerns in study of nowhere zero flows, which form dual notion to colorings of unoriented graphs. Nešetřil and Raspaud [NR] defined a similar notion, antisymmetric flow, which is dual to a variant of oriented coloring of oriented graphs. First we present the definitions of the two variants of oriented coloring (the first one is more common).

An *oriented coloring* of an oriented graph G is a mapping $c : V(G) \to \{1, \dots, n\}$ such that

1. for no edge (u, v) of G there is $c(u) = c(v)$; and

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2. for no edges (u,v), (x,y) of G there is $c(u)=c(y)$ and $c(v)=c(x)$.

The *oriented chromatic number* of G (denoted by $\vec{\chi}(G)$) is the minimal n in this definition, for which the oriented coloring exists. Note that this is a natural extension of the usual coloring for oriented graphs; using the language of homomorphisms, one can say that oriented coloring is a homomorphism into some tournament, whilst usual coloring is a homomorphism into some complete graph. (Tournament is any orientation of a complete graph. Homomorphism of graphs (or oriented graphs) is a mapping $f: V(G) \to V(H)$ such that two vertices connected by an edge are mapped to two vertices connected by an edge, and in case of the oriented graphs, the orientation is preserved as well.

The following is the principle notion of this paper. Let Γ be an abelian group. A mapping $c: V(G) \to \Gamma$ is a *strong oriented coloring* of an oriented graph G such that

1. for no edge (u,v) of G there is $c(u)=c(v)$; and

2. for no edges (u,v), (x,y) of G there is $c(v)-c(u)=c(x)-c(y)$.

The *strong oriented chromatic number* of G (denoted by $\vec{\chi}_s(G)$) is the minimal size of a group Γ, for which there is a strong oriented coloring with values in Γ. This notion was first defined in [NR] (it is called just strong coloring there). Again, there is an alternative definition, strong oriented coloring is a homomorphism into some oriented Cayley graph. (If Γ is any abelian group and A a subset of Γ such that A is disjoint from $-A$, the oriented Cayley graph $\mathrm{Cay}(\Gamma, A)$ has the vertex set Γ and (x,y) is its edge, iff $y-x \in A$. It is important to have A disjoint from $-A$, for otherwise the defined graph would have loops or 2-cycles.)

It is easy to see that each strong oriented coloring is also an oriented coloring and each oriented coloring is also a coloring. Hence

$$\chi(G) \le \vec{\chi}(G) \le \vec{\chi}_s(G)\,.$$

Let again Γ be an abelian group and G an oriented graph. We say that a mapping $\varphi: E(G) \to \Gamma$ is an *antisymmetric flow* (or *antiflow* for short), iff

1. for every vertex $v \in V(G)$

$$\sum_{(u,v)\in E(G)} \varphi\big((u,v)\big) = \sum_{(v,u)\in E(G)} \varphi\big((v,u)\big);$$

2. for no edge $e \in E(G)$ we have $\varphi(e) = 0$; and finally
3. for no edges $e, f \in E(G)$ we have $\varphi(e) = -\varphi(f)$.

(The first condition alone says that φ is a flow, the first two condition say that φ is a nowhere zero flow.) The *antiflow number* of G (denoted by $\mathrm{AF}(G)$) is the minimal size of an abelian group Γ such that G admits some antiflow with values in Γ. Some graphs G admit no antiflow, in which case we define $\mathrm{AF}(G) = \infty$. Obviously every bridge receives value 0 in every flow, so graph with bridge has no antiflow. Similarly if G has an oriented 2-cut (partition of $V(G)$ in two sets A and B, such that exactly two edges connect A and B and both of them are oriented from A to B) than the edges of the 2-cut receive opposite values, hence neither graphs with an oriented 2-cut have antiflow. In [NR] it is shown that these are the only obstacles, hence every graph without bridge and oriented 2-cut has antiflow in some group.

Now we can say, in which sense are antisymmetric flow and strong oriented coloring dual notions. If we have an oriented plane graph G, we can find its (oriented) dual $G^* = (V^*, E^*)$: V^* is the set of faces of G, and $E^* = \{e^* \mid e \in E(G)\}$, where e^* connects the face to the left of e to the face to the right of e. Let again Γ be an abelian group. For a mapping $c : V(G) \to \Gamma$ it is natural to define the mapping $\partial c : E(G) \to \Gamma$ by $\partial c((u,v)) = c(v) - c(u)$ and the mapping $(\partial c)^* : E(G^*) \to \Gamma$ by $(\partial c)^*(e^*) = \partial c(e)$. Then it holds (and is easy to prove) that c is a strong oriented coloring iff $(\partial c)^*$ is an antisymmetric flow. From this it follows that

$$\vec{\chi}_s(G) = \mathrm{AF}(G^*)\,.$$

(In particular G has a strong oriented coloring iff G^* has an antiflow. This follows more directly from the following observation: there are no loops or 2-cycles in G iff there are no bridges or 2-cycles in G^*.)

It is natural to ask how large $\vec{\chi}_s(G)$ and $\mathrm{AF}(G)$ can get, when G is a member of some class of graphs. In view of the previous paragraph, for planar graph G the answer is the same for $\vec{\chi}_s(G)$ and $\mathrm{AF}(G)$. The best known bound so far was

- for a planar graph G it holds $\vec{\chi}_s(G) \leq 7776$
- for a planar graph G without bridge and oriented 2-cut it holds $\mathrm{AF}(G) \leq 7776$

from [NR]. The main contribution of this article is the use of Sidon sets to improve this bound—see Theorem 4 and Corollary 1. For a general graph G, one obviously cannot bound $\vec{\chi}_s(G)$. As a (rather surprising) side remark let us mention that for any G, we have either $\mathrm{AF}(G) = \infty$ or $\mathrm{AF}(G) \le c$ for a (large) universal constant c (see [DJS] or [DNR] for a better constant).

The following definition from a combinatorial number theory has a crucial role in our proof. Let Γ be a group and M be its subset. We say that M is a *Sidon subset* of Γ, iff the differences $m-n$ (for $m, n \in M$ and $m \neq n$) are pairwise different. (Note that equivalent definition is to ask all of the sums $m+n$ to be different.) For example the set $\{0, 1, 4, 14, 16\}$ is a Sidon subset of $\mathbb{Z}_{21}$ (by $\mathbb{Z}_n$ we denote the additive group of integers modulo n). The following theorem claims the existence of large Sidon subsets of certain groups. It was first proved by Singer [S], the reader may find the proof in [HR].

Theorem 1 *(Singer, 1938) If m is a power of a prime, then there is a Sidon subset of $\mathbb{Z}_{m^2+m+1}$ with $m+1$ elements.*

Acyclic coloring of an unoriented graph is such assignment of colors to the vertices, for which each cycle of the graph contains vertices of at least three colors. Acyclic coloring of an oriented graph is an acyclic coloring of its underlying unoriented graph. Minimal number of colors needed for an acyclic coloring of a graph G is called the *acyclic chromatic number* of G and denoted by $a(G)$. We will use the celebrated result of Borodin [B].

Theorem 2 *Acyclic chromatic number of any planar graph is at most* 5.

2 The upper bound

Denote by s_k the size of the smallest group, which contains a Sidon subset with k elements. In the proof of Theorem 3 and 4 below we need an upper estimate on s_k, which is the subject of the following consequence of Theorem 1.

Lemma 1 *For every $k \ge 1$,*

$$k(k-1)+1 \le s_k \le 4k^2 .$$

If $k-1$ is a power of a prime, then the lower bound is tight.

Proof: The lower bound is rather obvious: if M is a k-element Sidon set in a group Γ, there are $k(k-1)$ ordered pairs (m, n), with $m \neq n \in M$. Each of this pairs corresponds to a different nonzero element of Γ (namely $m-n$), hence $|\Gamma| \geq k(k-1)+1$. The tightness of the lower bound for $k-1$ being a power of a prime follows from Theorem 1,

For $k=1$ the upper bound holds (since $s_1 = 1$, as $\{0\}$ is a Sidon set in a trivial group). For a larger k, Bertrand postulate claims that there is a prime p for which $k < p < 2k$. Now, according to Theorem 1,

$$s_k \leq s_{p+1} \leq p^2 + p + 1 \leq (p+1)^2 \leq 4k^2 \,.$$

□

Theorem 3 *Let G be an oriented graph with $\vec{\chi}(G) \leq k$. Then G has a strong oriented coloring in a group with s_k elements (this coloring uses just k elements of the group), hence*

$$\vec{\chi}_s(G) \leq s_k \leq 4k^2 \,.$$

Proof: Let M be a k-element Sidon subset of a group Γ of size s_k. Let c be an oriented coloring of G, using k colors; we may suppose that it uses the elements of M. We need to prove that c is strong oriented coloring. For no edge (u, v) of G there is $c(u) = c(v)$ (since c is a coloring). So suppose for contradiction that there are edges (u, v), (x, y) of G, for which

$$c(v) - c(u) = c(x) - c(y)$$

holds. As M is a Sidon set and $c(v) \neq c(u)$, this implies $c(u) = c(y)$ and $c(v) = c(x)$. As c is an oriented coloring, this is a contradiction. □

The previous theorem itself improves the bound on strong oriented chromatic number of planar graphs. If G is an oriented planar graph, it is known ([RS]) that $\vec{\chi}(G) \leq 80$, hence

$$\vec{\chi}_s(G) \leq s_{80} = s_{79+1} = 80 \cdot 79 + 1 = 6321,$$

which improves the previous known bound $\vec{\chi}_s(G) \leq 7776$. To improve this bound even further, we use the following theorem, which provides an estimate of strong oriented chromatic number of a graph in terms of its acyclic number. For an oriented planar graph G this gives the currently best known upper bound $\vec{\chi}_s(G) \leq 672$ (however note that no graphs are known to have $\vec{\chi}_s$ larger then 17).

Theorem 4 *Let G be an oriented graph with $a(G) \leq k$. Then G has a strong coloring in a group with $s_k \cdot 2^k$ elements, hence*

$$\vec{\chi}_s(G) \leq s_k \cdot 2^k \leq 4k^2 \cdot 2^k .$$

The mentioned coloring uses just $k \cdot 2^{k-1}$ elements of the group.

The bound given by this theorem is almost asymptotically tight—Kostochka, Sopena, and Zhu [KSZ] proved that there are graphs G for which

$$\vec{\chi}(G) = \Omega(2^{a(G)}) ,$$

so the main part of the upper bound (i.e. 2^k) cannot be removed. Let us also mention that in the proof we use the technique that Raspaud and Sopena [RS] used to prove that $\vec{\chi}(G) \leq 80$ for planar G.

Corollary 1 *If G is an oriented planar graph, then $\vec{\chi}_s(G) \leq 672$. If G has no bridge nor oriented 2-cut, then $\mathrm{AF}(G) \leq 672$.*

Proof: The corollary follows easily from Theorem 4, Theorem 2, Lemma 1, and the duality of antisymmetric flow and strong oriented coloring. To prove Theorem 4, let $c : V(G) \to \{0, 1, \dots, k-1\}$ be some acyclic coloring of G. Let further Γ be a group with s_k elements and $M = \{m_0, m_1, \dots, m_{k-1}\}$ its Sidon subset.

Now we construct a mapping $c_{\{i,j\}} : V(G) \to \{0, 1\}$ for every $0 \leq i < j < k$. To this end, denote by $T_{i,j}$ the subgraph of G induced by vertices with c-color equal to either i or j. Since c is acyclic, $T_{i,j}$ is a forest, and we can define $c_{\{i,j\}}$ so that $(c, c_{\{i,j\}})$ is a homomorphism of $T_{i,j}$ into an oriented 4-cycle $C_{i,j}$: $(i,0) \to (j,1) \to (i,1) \to (j,0) \to (i,0)$.

For every vertex v of G and for $0 \leq i < k$ we define

$$c_i(v) = \begin{cases} c_{\{i,c(v)\}}(v), & \text{iff } c(v) \neq i, \\ 0, & \text{iff } c(v) = i. \end{cases}$$

Finally we define mapping $f : V(G) \to \Gamma \times \mathbb{Z}_2^k$

$$f(v) = \left(m_{c(v)}, c_0(v), c_1(v), \dots, c_{k-1}(v)\right) .$$

Because of the first coordinate, f is a coloring of G. We prove that it is even a strong oriented coloring. Suppose for the contrary that there are edges $(u,v), (x,y) \in E(G)$, for which

$$f(v) - f(u) = f(x) - f(y) . \qquad (*)$$

From the first coordinate we know that

$$m_{c(v)} - m_{c(u)} = m_{c(x)} - m_{c(y)}\ .$$

Since c is a coloring and (u, v) an edge, there is $c(v) \neq c(u)$, hence $m_{c(v)} \neq m_{c(u)}$. Since M is a Sidon set, this implies that $c(v) = c(x)$ and $c(u) = c(y)$. Denote $i = c(v) = c(x)$ and $j = c(u) = c(y)$. By considering the i-th and the j-th coordinate of $(*)$ and using the definition of c_i and c_j we have

$$\begin{aligned} -c_i(u) &= c_i(v) - c_i(u) = c_i(x) - c_i(y) = -c_i(y), \\ c_j(v) &= c_j(v) - c_j(u) = c_j(x) - c_j(y) = c_j(x). \end{aligned}$$

The first equality gives us $c_{\{i,j\}}(u) = c_{\{i,j\}}(y)$, the second one $c_{\{i,j\}}(v) = c_{\{i,j\}}(x)$. Thus the homomorphism $(c, c_{\{i,j\}})$ identifies the vertex u with y and also v with x, hence it maps the edges (u, v) and (x, y) to an oriented 2-cycle. However, there is no oriented 2-cycle in the 4-cycle $C_{i,j}$, which is the desired contradiction. □

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