On Pattern Coloring of Cycle Systems*

Zdeněk Dvořák Jan Kára Daniel Král'
Ondřej Pangrác
Department of Applied Mathematics and
Institute for Theoretical Computer Science†
Charles University
Malostranské nám. 25, 118 00 Prague, Czech Republic
E-mail: {`rakdver,kara,kral,pangrac`}`@kam.ms.mff.cuni.cz`

Abstract

A k–cycle system is a system of cyclically ordered k–tuples of a finite set. A pattern is a sequence of letters. A coloring of a k–cycle system with respect to a set of patterns of length k is proper iff each cycle is colored consistently with one of the patterns, i.e. the same/distinct letters correspond to (the) same/distinct color(s). The feasible set of a cycle system is the set of all l's such that there exists a proper coloring of it using exactly l colors.

For all combinations of a pattern set $\mathcal{P}$ and a number l, we either find a polynomial algorithm or prove NP–completeness of the problem whether a given cycle system with a set of patterns $\mathcal{P}$ can be colored by at most l colors. We further construct a cycle system with a prescribed feasible set for almost every set of patterns containing only two different letters.

1 Introduction

Coloring problems for different combinatorial objects are ones of intensively studied mathematical problems. Original problems dealing with colorings of graphs have been generalized to hypergraphs where one allows edges of

---

*The research was done as a part of DIMACS/DIMATIA REU 2001 project.

†Supported by Ministry of Education of Czech Republic as project LN00A056.

arbitrary sizes (see [1]) and one demands that the vertices in one edge are colored consistently with certain prescribed rules. The original notion of proper colorings of hypergraphs, demanding that no edge of a hypergraph is monochromatic, has been generalized to lots of other structures: Steiner triple and quadruple systems ([2, 4, 11, 12, 13]), mixed hypergraphs ([14, 15]), mixed hypertrees ([7, 8]), mixed multigraphs ([9, 15]), block–pattern cycle systems and designs ([10]).

A *k–cycle system* is a pair $\mathcal{C} = (V_\mathcal{C}, C_\mathcal{C})$ where $V_\mathcal{C}$ is a finite set and $C_\mathcal{C}$ is set of cyclically ordered k–tuples of $V_\mathcal{C}$. The members of $V_\mathcal{C}$ are called *vertices* and the members of $C_\mathcal{C}$ are called *cycles*. A k–cycle system is a *k–cycle design* if for each pair of vertices u and v there is exactly one cycle containing uv or vu. Some of the properties of k–cycle systems, especially of k–cycle designs, can be found in [6], e.g. the 4–cycle designs on n vertices exist precisely for $n \bmod 8 = 1$.

A *pattern* of length k is a sequence of letters of length k. The *coloring* c of vertices of a k–cycle system $\mathcal{C}$ with the set of patterns $\mathcal{P}$ of length k is *proper* iff for each cycle C of $\mathcal{C}$ there is a pattern $p \in \mathcal{P}$ which satisfies the following: There is a rotation of C such that the vertices on the positions with the same letters of p are colored by the same color and the vertices on the positions with mutually different letters of p are colored by mutually different colors. This notion of pattern coloring was introduced in [10] and is actually generalization of several previously introduced notions. Coloring of 2–cycle systems with the pattern AB is just graph coloring: The pattern forces the vertices in the same cycle to be colored by different colors and thus the cycles correspond to the edges of a graph and vice versa. Coloring of 3–cycle systems with the pattern AAB is just bicoloring of triple systems considered in [2, 12, 13]. The pattern forces that each triple is colored by exactly two colors. Coloring of k–cycle systems with suitable set of patterns also correspond to coloring of uniform mixed bihypergraphs as considered in [7, 8, 14, 15].

The feasible set $\mathcal{F}(\mathcal{C})$ of a cycle system $\mathcal{C}$ is the set of all l's for which there exists a proper coloring of $\mathcal{C}$ using exactly l colors. It turned out that feasible sets of certain types of combinatorial objects need not to be intervals, e.g. any finite set omitting 1 is a feasible set of a mixed hypergraph (and even a mixed bihypergraph), see [5]. Although uniform mixed bihypergraphs are cycle systems with a certain special pattern set, the result of [5] does not extend to cycle systems with arbitrary pattern sets.

We address two problems related to coloring of cycle systems with prescribed pattern sets in this paper. The first one is: "For which sets I of integers and for which sets $\mathcal{P}$ of patterns does there exist a cycle system with the pattern set $\mathcal{P}$ and the feasible set I?" The analogous problem for mixed hypergraphs was considered in [15] and solved in [5] as described in the previous paragraph. We prove that there exists a cycle system with the pattern set $\mathcal{P}$ and the feasible set I for any set $\mathcal{P}$ of patterns consisting of two letters which omits the monochromatic pattern and which does not contain only the alternating pattern and for any set of integers I. If the pattern set $\mathcal{P}$ contains only the alternating pattern, then the feasible set of any cycle system with the pattern set $\mathcal{P}$ is an interval. We refer the reader for the definition of the monochromatic and alternating pattern to Section 2.

The second problem which we address is the following: "What is the complexity of decision whether a given cycle system $\mathcal{C}$ with the pattern set $\mathcal{P}$ can be colored by at most l colors ($\mathcal{P}$ and l are fixed)?" We describe all the pattern sets $\mathcal{P}$ and the numbers l for which this problem can be solved in polynomial time (unless $P = NP$) and we prove that the problem is NP–complete for all the remaining combinations of the pattern set $\mathcal{P}$ and the number l.

We introduce several definitions and prove some basic properties related to coloring of cycle systems in Section 2. We actually prove that certain patterns can be "simulated" by almost any set of patterns in Lemma 1. We later prove a similar lemma for larger sets of patterns with an additional restriction on the number of used colors in Section 4, Lemma 3.

The existence of cycle systems with prescribed feasible sets and pattern sets is addressed in Section 3. We restrict our attention only to pattern sets containing patterns with at most two letters which omits the monochromatic pattern. If the pattern set contains only the alternating pattern, then the feasible set of any cycle system with this pattern set is an interval due to Observation 2. But we construct a cycle system for any other pattern set omitting the monochromatic pattern and for any finite set I of integers omitting one such that its feasible set is equal to I in Theorem 1.

The complexity of coloring of cycle system is studied in Section 4, Section 5 and Section 6. We study the decision problem whether a given cycle system with a fixed pattern set $\mathcal{P}$ can be colored by at most fixed number l of colors for $l \geq 3$ in Section 4. The problem is trivial if $l = 1$ (since the answer is yes iff the given cycle system is empty or the pattern set contains the monochromatic pattern). The problem for $l = 2$ is studied in Section 5. The problem is, not very surprisingly, NP–complete for any $l \geq 3$ and any set $\mathcal{P}$ of patterns omitting the monochromatic pattern — see Theorem 2.

We deal with the problem for $l = 2$ in Section 5. It is obvious, since the coloring may use at most two colors, that the pattern set $\mathcal{P}$ contains w.l.o.g. only the patterns consisting of at most two letters. It is not hard to see that if we get an affine subspace over $\mathbb{GF}(2)$ through replacing the letters in the patterns of $\mathcal{P}$ by ones and zeroes and considering all the rotations of the patterns of $\mathcal{P}$, then the problem of finding a coloring of the given cycle system can be reduced to the problem of finding a solution of a system of linear equations. Consult the beginning of the Section 5 and Lemma 5 for definitions and more details. Thus the problem can be solved in polynomial time in all such cases. Lemma 8 states that all the remaining cases (except for the trivial cases that $\mathcal{P}$ contains the monochromatic pattern) are NP–complete. The results of the whole section are summarized in Theorem 3.

The characterization in Theorem 3 is general and thus it does not provide any examples of patterns for which the problem can be solved in polynomial time. Thus we address and answer the following question in Section 6: "For which patterns p consisting of l distinct letters, the decision problem whether a given cycle system with the pattern set $\{p\}$ can be colored by at most l colors is solvable in polynomial time?" The results of Section 4 and 5 imply that this is possible only for the pattern consisting of one or two letters. In the former case, the problem is actually trivial. In the latter case, these are exactly the patterns whose rotations induce affine subspaces over $\mathbb{GF}(2)$. We list all such patterns in Theorem 4; these patterns are exactly the following ones: A^k, $(AB)^k$, $(AABB)^k$ and $(AAAB)^k$ for all k's (we write X^k for the concatenation of k copies of X). We would like to point the attention of the reader to quite interesting linear algebra Lemma 9 in Section 6; note that the condition that λ is a power of two is essential, since for any other λ the statement of the lemma is false.

2 Definitions and Basic Properties

The notion of cycle systems and their colorings with prescribed patterns was introduced in Section 1. A pattern is called an *l–pattern* if it consists of at most l different letters. We assume throughout the paper to simplify the statements that the letters contained in a l–pattern are the first l letters of the Roman alphabet. The *monochromatic pattern* of length k is A^k (we write throughout the paper for shortness A^k instead of the sequence consisting of k A's), the *alternating pattern* of length k is $(AB)^{k/2}$ (for k even) and the *multichromatic pattern* of length k is a pattern consisting of k mutually different letters, e.g. the multichromatic pattern of length 3 is ABC. We say that the pattern p is *periodic* if it is a concatenation of two or more copies of another pattern. We say that the pattern is *aperiodic* if it is not periodic.

We say that the pattern p of length k' *can be simulated* by a set of patterns $\mathcal{P}$ of length k if there exists a k–cycle system $\mathcal{C}$ with k' special vertices $v_1, \ldots, v_{k'}$ with the following properties:

- Let l be the number of different letters in p. Then any proper coloring of $\mathcal{C}$ uses exactly l colors.
- The cycle consisting of $v_1, \ldots, v_{k'}$ is in any proper coloring of $\mathcal{C}$ colored consistenly with the pattern p.
- Any precoloring of the vertices $v_1, \ldots, v_{k'}$ consistent with the pattern p can be extended to a proper coloring of $\mathcal{C}$.

If the conditions above are satisfied, we say that $\mathcal{C}$ *simulates* the pattern p.

The just established notion of simulating allows us to state the following observation which demonstrates several just defined terms and which will be usefull in Section 3:

Observation 1 *Let $\mathcal{C}_p$ be a k'–cycle system with a pattern p of length k' and assume that p can be simulated by a set of patterns $\mathcal{P}$ of length k. Then there exists a k–cycle system $\mathcal{C}_{\mathcal{P}}$ with the pattern set $\mathcal{P}$ such that $\mathcal{F}(\mathcal{C}_p) = \mathcal{F}(\mathcal{C}_{\mathcal{P}})$.*

Proof: Let $\mathcal{C}$ be the k–cycle system with the pattern set $\mathcal{P}$ which simulates p and let $v_1, \ldots, v_{k'}$ be its special vertices which are described in the definition of simulating above. We create the k–cycle system $\mathcal{C}_{\mathcal{P}}$ as follows: We replace each cycle $w_1, \ldots, w_{k'}$ of $\mathcal{C}_p$ with a copy of the k–cycle system $\mathcal{C}$ identifying vertices v_i and w_i for $1 \le i \le k'$. Since no copy of $\mathcal{C}$ contains in any proper coloring a vertex colored by a color different to all its special vertices, it is clear that $\mathcal{F}(\mathcal{C}_p) = \mathcal{F}(\mathcal{C}_{\mathcal{P}})$. $\Box$

In the paper, we will need to assure that certain vertices get the same color or that certain vertices get mutually different colors. The following technical lemma allows us this for most sets of patterns:

Lemma 1 *Let $\mathcal{P}$ be any set of l–patterns of length k which contains neither the monochromatic pattern nor the multichromatic pattern of length k. Let*

p be the multichromatic pattern of length l. Then p^κ can be simulated by $\mathcal{P}$ for any $\kappa \geq 1$.

Proof: Since p^{κ_1} can be trivially simulated by p^{κ_2} for $\kappa_1 \leq \kappa_2$, we may assume w.l.o.g. that $\kappa > kl$. We create the k–cycle system $\mathcal{C}_{p^\kappa}$ on κl vertices v_i^j, $1 \leq i \leq l$, and $1 \leq j \leq \kappa$ as follows: We add to $\mathcal{C}_{p^\kappa}$ all the k–cycles such that the coloring $c_0(v_i^j) = i$ colors them properly with respect to the set of patterns $\mathcal{P}$. We claim that the coloring c_0 is the only proper coloring (upto renaming the colors) of $\mathcal{C}_{p^\kappa}$. Then setting $v_1^1, \ldots, v_l^1, v_1^2, \ldots, v_l^2, \ldots, v_1^\kappa, \ldots, v_l^\kappa$ to be special vertices of C_{p^κ} would finish the proof of the lemma.

We prove the claim in rest of the proof of this lemma. Let c be any proper coloring of $\mathcal{C}$ and let $V_i = \{v_i^j | 1 \leq j \leq \kappa\}$. We first prove that each V_i contains at least k vertices colored by c with the same color. Let us assume the opposite for the contradiction. We may w.l.o.g. due to symmetry assume that each of the sets $V_1, \ldots, V_{i_0}$ contains at least k vertices colored by c with the same color and none of the remaining sets does. Let U_i be a set of k vertices of V_i colored by c with the same color, for $1 \leq i \leq i_0$ and let γ_i be the common color of the vertices of U_i. The colors $\gamma_1, \ldots, \gamma_{i_0}$ are mutually distinct: If $\gamma_i = \gamma_{i'}$, let p_1 be one of the possibly more patterns of $\mathcal{P}$ which contain the most number of occurencies of the same letter and let λ_1 be this number. Let C be a cycle containing λ_1 vertices of U_i and a vertex of $U_{i'}$ — there is certainly a cycle colored properly by the coloring c_0 from the first paragraph with respect to the pattern p_1 (which is surely not monochromatic) which contains a λ_1–tuple of vertices of V_i and a vertex of $V_{i'}$ for each prescribed λ_1–tuple of vertices of V_i and each prescribed vertex of $V_{i'}$. But the cycle C contains at least $\lambda_1 + 1$ vertices colored by c with the same color and this contradicts the choice of p_1 and λ_1.

If none of $V_{i_0+1}, \ldots, V_l$ contains k vertices colored by c with the same color, then each of them has to contain at least $l + 1$ vertices colored by mutually distinct colors due to pigeonhole principle. Let p_2 be a pattern of $\mathcal{P}$ containing maximum number of different letters and let λ_2 be this number. Since p_2 is not multichromatic, it contains at least one letter at least twice. Let w_i be any vertex of U_i for $1 \leq i \leq i_0$, let $w_i \in V_i$ for $i_0 < i \leq l$ be a vertex of V_i colored by c with the color distinct to the color of any vertex $w_{i'}$ for $i' < i$ and let $w_{l+1} \in V_l$ be a vertex of V_l colored by c with the color distinct to the color of any vertex w_i for $1 \leq i \leq l$. These vertices exist since each of V_i for $i > i_0$ contains at least $l + 1$ vertices colored by c with mutually different colors. The colors of $w_1, \ldots, w_{l+1}$ are mutually different due to the choice of the vertices. Let C be a cycle containing the vertices

$w_{l-\lambda_2+1},\dots,w_{l+1}$ — there is certainly such a cycle, since for any choice of λ_2-1 vertices from different V_i's and additional choice of two other vertices from another V_i, there is a cycle containing these chosen vertices colored properly by the coloring c_0 of the first paragraph with respect to p_2. But C contains λ_2+1 vertices colored by c with mutually distinct colors and thus C cannot be colored properly due to choice of p_2 and λ_2. This implies that $i_0=l$. Moreover, the common colors γ_i's are different for different V_i's.

We prove that the common color γ_i of the vertices of U_i share actually all the vertices of V_i. Let us assume the opposite. We may assume that V_1 contains two vertices colored by different colors. If the color of one of them is different from all γ_i's, we proceed as in the previous paragraph: We choose p_2 to be a pattern of $\mathcal{P}$ containing the maximum number of different letters and we set λ_2 to be this number. We construct a cycle containing vertices colored by at least λ_2+1 distinct colors and we obtain a contradiction. Thus w.l.o.g. the color of one of the two vertices of V_1 is γ_1 and the color of the other one is γ_2. Let p_1 be a pattern of $\mathcal{P}$ containing the most number of occurencies of the same letter and let λ_1 be this number, consult the end of the second paragraph of this proof. Let C be a cycle containing the vertex of V_1 colored by γ_2 and λ_1 vertices of V_2 colored by γ_2; its existence follows from the arguments used in the second paragraph. But the cycle C cannot be properly colored by c. This implies that all the vertices of each V_i have to share the same color γ_i and thus the proper coloring c_0 of $\mathcal{C}$ is its only proper coloring upto renaming the colors. $\Box$

3 Feasible Sets of Cycle Systems with 2–Patterns

We first prove that the feasible sets of cycle systems with the single alternating pattern are only intervals. Next we prove, that the feasible sets of cycle systems with any set of 2–patterns omitting the monochromatic pattern can be any finite set of integers omitting one except for the case that the set consists only of the alternating pattern.

Observation 2 *Let $\mathcal{P}$ be the set of 2–patterns containing only the alternating pattern. Then the feasible set of any cycle system with the pattern set $\mathcal{P}$ is an interval.*

Proof: Let $\mathcal{C}$ be a cycle system with the pattern set $\mathcal{P}$ which consists only of the alternating pattern. The pattern forces that any two vertices which are in an even distance in some of the cycles of $\mathcal{C}$ have to get the same color and any two vertices in an odd distance have to get different colors. Thus if we first identify all the vertices in even distances and then we join the vertices in odd distances by edges, we get a graph whose proper colorings one–to–one correspond to the proper coloring of the original cycle system. If the obtained graph contains a loop, the original cycle system cannot clearly be colored properly. $\Box$

We prove that for any other set of 2–patterns omitting the monochromatic pattern, there exists a cycle system which simulates the pattern $AABB$. Note that the pattern $AABB$ is one of the patterns for which 2–colorability of 4–cycle system with this pattern can be decided in polynomial time (Theorem 4) and thus the following lemma has no corollaries regarding complexity of coloring cycle systems with at most 2 colors.

Lemma 2 *Let $\mathcal{P}$ be any set of 2–patterns of length $k \geq 3$ which does not contain the monochromatic pattern of length k and which contains a pattern different from the alternating pattern of length k. Then the pattern $AABB$ can be simulated by $\mathcal{P}$.*

Proof: Let $\mathcal{C}^1_{AB}$ and $\mathcal{C}^2_{AB}$ be two vertex–disjoint copies of a k–cycle system with the pattern set $\mathcal{P}$ which simulates the pattern $(AB)^k$; its existence follows from Lemma 1. Let V_1 and V_3 be the sets of special vertices of $\mathcal{C}^1_{AB}$ such that the vertices of V_1 have to share the same color, the vertices of V_3 have to share the same color and the colors of vertices of V_1 and V_3 are different; let V_2 and V_4 be the corresponding sets of special vertices of $\mathcal{C}^2_{AB}$.

Let p be any pattern of $\mathcal{P}$ different from the monochromatic and the alternating pattern of length k. The pattern p consists w.l.o.g. of the letters A and B and it has to either contain a subsequence $BA^\lambda B$ or be equal to $A^\lambda B$ for some $\lambda \geq 2$. Let p' be the pattern p rotated by one letter to the right and let p_i (p'_i) be the i–th letter of the pattern p (p'). There are certainly indices i_{AA}, i_{AB} and i_{BA} such that the following holds (since p contains a subsequence $BA^\lambda B$ or is equal to $A^\lambda B$ for some $\lambda \geq 2$):

- $p_{i_{AA}} = A$ and $p'_{i_{AA}} = A$
- $p_{i_{AB}} = A$ and $p'_{i_{AB}} = B$
- $p_{i_{BA}} = B$ and $p'_{i_{BA}} = A$

We create a k–cycle $v_1, \dots, v_k$ such that:

- $v_i \in V_1$ iff $p_i = p'_i = A$
- $v_i \in V_2$ iff $p_i = A$ and $p'_i = B$
- $v_i \in V_3$ iff $p_i = p'_i = B$
- $v_i \in V_4$ iff $p_i = B$ and $p'_i = A$

The choice of vertices $v_1, \ldots, v_k$ except for the above conditions can be arbitrary. This cycle forces that either the color of the vertices of V_2 or the color of the vertices of V_4 is equal to the color of the vertices of V_1. It does not necessarilly force the same for the vertices of V_3, since the cycle may contain no vertex of V_3. On the other hand, if the above condition is satisfied, then the cycle is colored properly (it is consistent with the pattern p). Next, we create similary a k–cycle $v_1, \ldots, v_k$ such that:

- $v_i \in V_1$ iff $p_i = p'_i = B$
- $v_i \in V_2$ iff $p_i = A$ and $p'_i = B$
- $v_i \in V_3$ iff $p_i = p'_i = A$
- $v_i \in V_4$ iff $p_i = B$ and $p'_i = A$

The choice of vertices $v_1, \ldots, v_k$ can be again arbitrary (when obeying the above conditions). This cycle forces that the color of the vertices of V_3 is equal to either the color of the vertices of V_2 or the color of the vertices of V_4. On the other hand, if this condition is satisfied, then the cycle is colored properly (it is consistent with the pattern p). The just added two cycles force that any proper coloring of the vertices of $\mathcal{C}^1_{AB}$ and $\mathcal{C}^2_{AB}$ uses at most two colors and thus choosing any four vertices $v_i \in V_i$ for $i = 1, 2, 3, 4$ to be the special ones finish creating the k–cycle system with the pattern set $\mathcal{P}$ simulating $AABB$: The only allowed pattern for coloring these vertices is due to the above mentioned reasons $AABB$. On the other hand, if we have precolored the special vertices v_1, v_2, v_3, v_4 consistently with the pattern $AABB$, then assigning all the vertices of V_i the color of v_i yields a proper coloring. $\Box$

We are now ready to prove the main theorem of this section:

Theorem 1 *Let $\mathcal{I}$ be any finite set of integers omitting* 1*. Let $\mathcal{P}$ be any set of* 2*–patterns of length $k \geq 3$ which does not contain the monochromatic* *pattern of length k and which contains a pattern different from the alternating pattern of length k. Then there exists a k–cycle system $\mathcal{C}$ with the pattern set $\mathcal{P}$ such that $\mathcal{F}(\mathcal{C}) = \mathcal{I}$.*

Proof: The theorem is sufficient to prove for $\mathcal{P} = \{AABB\}$ due to Observation 1 and Lemma 2.

The proof proceeds by induction on $\max(\mathcal{I})$ (setting $\max(\emptyset) = 0$). If $\mathcal{I}$ is empty then the 4–cycle system with the pattern $AABB$ on four vertices containing all the 6 possible 4–cycles is clearly uncolorable. Otherwise, $\mathcal{I}$ is non–empty. If $\max(\mathcal{I}) = 2$ then we set $\mathcal{C}$ to the 4–cycle system on four vertices containing a single 4–cycle; it is clear that $\mathcal{F}(\mathcal{C}) = \mathcal{I} = \{2\}$.

We deal with the remaining cases in the rest of the proof. Let $\mathcal{I}' = \{i | i \geq 2 \ \wedge \ i+1 \in \mathcal{I}\}$. Since $\max(\mathcal{I}) > 2$, $\mathcal{I}' \neq \emptyset$. Let $\mathcal{C}'$ be the 4–cycle system such that $\mathcal{F}(\mathcal{C}') = \mathcal{I}'$ (it exists due to the induction). Let $v_1, \ldots, v_n$ be the vertices of $\mathcal{C}'$. We distinguish two cases:

- If $2 \notin \mathcal{I}$ (i.e. $\mathcal{I} = \{i+1 | i \in \mathcal{I}'\}$), then we add to the 4–cycle system $\mathcal{C}'$ $n+2$ new vertices x, y and v'_i for $1 \leq i \leq n$ and n new 4–cycles x, y, v_i, v'_i for $1 \leq i \leq n$. We claim that the feasible set $\mathcal{F}(\mathcal{C})$ of the just obtained 4–cycle system $\mathcal{C}$ with the pattern $AABB$ is equal to $\mathcal{I}$.

  Let c' be any proper coloring of $\mathcal{C}'$. Then setting $c(v_i) = c(v'_i) = c'(v_i)$ and $c(x) = c(y)$ to a completely new color yields a proper coloring c of $\mathcal{C}$ using exactly one more color than c' does. Thus $\mathcal{I} \subseteq \mathcal{F}(\mathcal{C})$.

  On the other hand, let c be a proper coloring of $\mathcal{C}$. It is clear that $c(x) \neq c(v_i)$ for any $1 \leq i \leq n$ due to the presence of the cycles x, y, v_i, v'_i. Thus setting $c'(v_i) = c(v_i)$ yields a proper coloring c' of $\mathcal{C}'$ which uses exactly one less color than c does (note that $c(y)$ or $c(v'_i)$ cannot be different both from $c(x)$ and $c(v_i)$). Thus $\mathcal{I} = \mathcal{F}(\mathcal{C})$.

- If $2 \in \mathcal{I}$ (i.e. $\mathcal{I} = \{2\} \cup \{i+1 | i \in \mathcal{I}'\}$), we create a 4–cycle system $\mathcal{C}$ on $2n+2$ vertices x, y, w_i and w'_i for $1 \leq i \leq n$. We add a 4–cycle $w_{i_1}, w_{i_2}, w'_{i_3}, w'_{i_4}$ to $\mathcal{C}$ for each 4–cycle $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ of $\mathcal{C}'$. Moreover, we add a 4–cycle x, y, w_i, w'_i to $\mathcal{C}$ for each $1 \leq i \leq n$. Coloring all the vertices x and w'_i with a common color and coloring all the vertices y and w_i with another common color yields a proper 2–coloring of $\mathcal{C}$. On the other hand, this is the only proper coloring of $\mathcal{C}$ which colors the vertices x and y with different colors.

  We focus our attention to the colorings of $\mathcal{C}$ which assign x and y the same color in this paragraph. Any such coloring assigns the pair of vertices w_i and w_i' the same color. Thus the proper colorings c of $\mathcal{C}$ and c' of $\mathcal{C}'$ one–to–one correspond each other through setting $c(w_i) = c(w_i') = c'(v_i)$. This immediately implies that $\mathcal{F}(\mathcal{C}) = \mathcal{I}$.

□

4 NP–completeness of Coloring of Cycle–Systems

We prove that for any pattern set omitting the monochromatic pattern the problem whether a given cycle system can be colored by at most fixed number of colors (for three or more) is NP–complete. In order to do this, we first prove a modification of Lemma 1:

Lemma 3 *Let $\mathcal{P}$ be any fixed set of l–patterns of length $k \geq 2$ omitting the monochromatic pattern of length k and let $m \geq \max\{3, l\}$ be a fixed integer. Then there exists a k–cycle system $\mathcal{C}$ with m special vertices $v_1, \ldots, v_m$ such that any proper coloring of it with at most m colors with respect to the pattern set $\mathcal{P}$ assigns the vertices $v_1, \ldots, v_m$ distinct colors.*

Proof: We adopt the proof of Lemma 1. We create the k–cycle system $\mathcal{C}$ on km^2 vertices v_i^j for $1 \leq i \leq m$ and $1 \leq j \leq km$. We add to $\mathcal{C}$ all the k–cycles such that the coloring $c_0(v_i^j) = i$ colors them properly with respect to the set of patterns $\mathcal{P}$. Let $V_i = \{v_i^j | 1 \leq j \leq km\}$.

Let c be any proper coloring of $\mathcal{C}$ using at most m colors. Each V_i contains at least k vertices colored by c with the same color due to pigeonhole principle; let $U_i \subseteq V_i$ be a set of such vertices and let γ_i be their common color. We claim that there are no vertices other than those of V_i colored by the color γ_i. Assume that a vertex $v \in V_{i'}$ is colored by the color γ_i ($i \neq i'$). Let p_1 be a pattern of P which contains the most number of occurencies of the same letter and let λ_1 be this number. Let C be a cycle containing λ_1 vertices of U_i and the vertex v of $V_{i'}$ — there is certainly a cycle colored properly by the coloring c_0 from the first paragraph with respect to the pattern p_1 (which is surely not monochromatic) which contains a λ_1–tuple of vertices of V_i and a vertex of $V_{i'}$ for each prescribed λ_1–tuple of vertices of V_i and each prescribed vertex of $V_{i'}$ (consult the proof of Lemma 1). But the cycle C contains at least $\lambda_1 + 1$ vertices colored by c with the same color γ_i and this contradicts that c is proper due to the choice of p_1 and λ_1. Thus the vertices colored by c with the color γ_i are only in V_i. This implies that the colors $\gamma_1, \ldots, \gamma_m$ are mutually distinct. Since c uses at most m colors, all the vertices of V_i have to be colored by c with the color γ_i and thus c_0 is upto renaming the colors the only proper coloring of $\mathcal{C}$. Choosing vertices $v_1^1, \ldots, v_m^1$ to be the special vertices of $\mathcal{C}$ completes the proof. $\square$

It is quite easy to prove the NP–completeness result of this section using Lemma 3. Note that the condition that $l' \geq l$ in the statement of the next theorem is necessary, since if the pattern set does not contain a pattern with less than l different letters, then the cycle system cannot be properly colored by at most l' colors for $l' < l$.

Theorem 2 *Let $\mathcal{P}$ be any fixed set of l–patterns of length $k \geq 2$ omitting the monochromatic pattern of length k and let $l' \geq \max\{3, l\}$ be a fixed integer. Then the decision problem whether a given k–cycle system with the pattern set $\mathcal{P}$ can be colored by at most l' colors is NP–complete.*

On the other hand, if $\mathcal{P}$ contains the monochromatic pattern, any cycle system with $\mathcal{P}$ can be colored by one color and the problem is trivial.

Proof: We may w.l.o.g. assume that $k = l = l' \geq 3$ and $\mathcal{P}$ contains only the multichromatic pattern of length k due to Observation 1 and due to Lemma 3. We present the reduction from the well–known NP–complete problem (see [3]) which is to decide whether a given graph can be colored by at most k colors (for any fixed $k \geq 3$). Let G be any instance of this problem; let $v_1, \ldots, v_n$ be the vertices of G and $e_1, \ldots, e_m$ the edges of G. We create a k–cycle system $\mathcal{C}$ with $m(k-2)+n$ vertices and m cycles: The vertices of $\mathcal{C}$ are $v_1, \ldots, v_n$ and w_i^j for $1 \leq i \leq m$ and $1 \leq j \leq k-2$. We add a cycle $u, v, w_i^1, \ldots, w_i^{k-2}$ for each edge $e_i = uv$, $1 \leq i \leq m$. The proper coloring of G using at most k colors can be extended to a proper coloring of $\mathcal{C}$ which uses at most k colors by setting $w_i^1, \ldots, w_i^{k-2}$ to mutually distinct colors distinct to the color of u and v for all $e_i = uv$, $1 \leq i \leq m$. On the other hand, any proper coloring of $\mathcal{C}$, restricted only to the vertices $v_1, \ldots, v_n$ yield a proper coloring of G. Thus G can be properly colored by at most $k = l = l'$ colors iff $\mathcal{C}$ can. This completes the proof of the theorem. $\square$

5 Complexity of Two–Coloring of Cycle Systems

We restrict our attention to the algorithmic issues of two–colorings of cycle systems. We first state an easy lemma which allows us to force two vertices to have different colors or the same color:

Lemma 4 *Let $\mathcal{P}$ be any set of 2–patterns of length $k \geq 2$ which does not contain the monochromatic pattern of length k. Then the pattern AB can be simulated by $\mathcal{P}$.*

Proof: If $k = 2$, then $\mathcal{P} = \{AB\}$ and the statement is trivial. If $k > 2$, $\mathcal{P}$ cannot contain a multichromatic pattern (since it contains only 2–patterns) and thus the lemma follows from Lemma 1. □

We develop in this section connection between colorings of cycle systems using at most two colors and linear algebra. The calculation will be done over the field $\mathbb{GF}(2)$ and the elements of $\mathbb{GF}(2)$ will represent the colors. Let $\mathcal{P}$ be the set of 2–patterns of length k. Let $\mathcal{A}(\mathcal{P})$ be the set of all the vectors over $\mathbb{GF}(2)$ of length k such that they are consistent with the set of patterns $\mathcal{P}$, i.e. there exists a cyclically rotation p of a pattern of $\mathcal{P}$ such that the vector contains zeroes exactly in those positions where p contains A's and ones in those with B's (or vice versa). We say that $\mathcal{P}$ *can be described by a system of linear equations* iff $\mathcal{A}(\mathcal{P})$ forms an affine subspace of $\mathbb{GF}(2)^k$, i.e. there exists a matrix A of size $k' \times k$ and a vector b of size k' for some k' such that $\mathcal{A}(\mathcal{P}) = \{x | Ax = b\}$. We say in this case that the matrix A and the vector b describe the set of patterns $\mathcal{P}$. Recall that the calculations are done over $\mathbb{GF}(2)$.

We first state and prove a lemma which guarantees that under certain conditions the problem whether a given cycle system can be colored by at most two colors is solvable in polynomial time for a certain set of patterns. We later prove that unless $P = NP$ these are the only polynomial–time solvable cases.

Lemma 5 *If the set $\mathcal{P}$ of 2–patterns of length $k \geq 2$ can be described by a system of linear equations, then the decision problem whether a given k–cycle system with the pattern set $\mathcal{P}$ can be colored by at most 2 colors, can be solved in polynomial time.*

Proof: Let A be the matrix of size $k' \times k$ and let b the vector of size k' which describe $\mathcal{P}$. $\mathcal{A}(\mathcal{P})$ forms an affine subspace of $\mathbb{GF}(2)^k$ in this case. Let $\mathcal{C}$ be a given k–cycle system with the pattern set $\mathcal{P}$, let $v_1, \dots, v_n$ be the vertices of $\mathcal{C}$ and let m be the number of cycles of $\mathcal{C}$. We form a system of mk' equations with n variables $x_1, \dots, x_n$. We add for each cycle $v_{i_1}, \dots, v_{i_k}$ of $\mathcal{C}$ the following k' equations:

$$A \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_k} \end{pmatrix} = b$$

The solutions $x_1, \dots, x_n$ of this system of equations clearly one–to–one correspond to proper 2–colorings c of $\mathcal{C}$ through a simple equalities $c(v_i) = x_i$ for all $1 \leq i \leq n$. Since the size of the system of equations is polynomial in the size of the cycle system $\mathcal{C}$, the decision problem whether $\mathcal{C}$ can be properly colored by at most 2 colors can be solved in polynomial time. $\square$

We prove that the decision problem whether a given 3–cycle system with the pattern AAB can be colored by at most 2 colors is NP–complete. Later, we reduce all the remaining cases to this one with the aim of Lemma 7 which immediately follows the next lemma.

Lemma 6 *The decision problem whether a given* 3*–cycle system with the pattern AAB can be colored by at most* 2 *colors, is NP–complete.*

Proof: We present an almost trivial reduction from the well–known NP–complete problem of not–all–equal satisfiability (NAE-SAT), see [3] for details. The problem is to decide whether for a given formula there exists a truth assignment such that each clause contains both a positive and a negative literal. This problem remains NP–complete even if restricted to the formulas such that the sizes of all the clauses are exactly three, i.e. each clause contains exactly three literals.

Let Φ be a given formula whose all the clauses have size exactly three. Let $x_1, \dots, x_n$ be the varibles of Φ. Let $\mathcal{C}$ be a 3–cycle system with the pattern AAB which simulates the pattern AB; its existence follows from Lemma 4. We take n copies of $\mathcal{C}$ and we write for their special vertices $v_1, \dots, v_n$ and $v'_1, \dots, v'_n$ (the vertices v_i and v'_i belong to the same copy of $\mathcal{C}$). We add for each clause of the formula Φ a 3–cycle which contains v_i iff the clause contains x_i and which contains v'_i iff the clause contains the negation of x_i. We claim that the constructed 3–cycle system can be 2–colored iff the formula Φ can be NAE–satisfied. Let c be a proper coloring of the constructed 3–cycle system and let us assume w.l.o.g. that c colors the vertices with the color 0 and 1. We set x_i to the false value iff $c(v_i) = 0$ and we set x_i to the truth otherwise ($c(v_i) = 1$). The obtained truth assignment clearly NAE–satisfies the formula Φ: The coloring of a vertex v_i (v_i') by the color 0/1 represents that the value of (the negation of) x_i is false/truth. The same correspondence works for obtaining a proper coloring of the 3–cycle system from a NAE–satisfying truth assignment. $\Box$

Lemma 7 *Let $\mathcal{P}$ be the set of 2–patterns of length $k \geq 2$ which omits the monochromatic pattern. If there exists a k–cycle system $\mathcal{C}$ with three special vertices u, v and w such that there are only three (up to renaming the colors) ways of coloring u, v and w by proper colorings of $\mathcal{C}$ using at most 2 colors, then the decision problem whether a given k–cycle system with the pattern set $\mathcal{P}$ can be colored by at most 2 colors is NP–complete.*

Proof: The possible ways of coloring u, v and w by proper colorings are w.l.o.g. (after permutation of the three special vertices if necessary) either $\{AAB, ABA, BAA, ABB, BAB, BBA\}$ or $\{AAA, BBB, AAB, BBA, ABA, BAB\}$. In the first case, $\mathcal{C}$ simulates the pattern AAB with the additional condition that the whole coloring can use at most two colors and thus the problem is NP–complete due to Lemma 6.

There exists a k–cycle system $\mathcal{C}_{AB}$ with the pattern set $\mathcal{P}$ with two special vertices u' and u'' which simulates the pattern AB due to Lemma 4. We create a cycle system $\mathcal{C}'$ from a copy of $\mathcal{C}$ and a copy of $\mathcal{C}_{AB}$ by identifying the vertices u'' and u. The possible ways of coloring its vertices u', v and w are $\{BAA, ABB, BAB, ABA, BBA, AAB\}$ and thus $\mathcal{C}'$ simulates the pattern AAB with the additional condition that the whole coloring can use at most two colors and the problem is NP–complete due to Lemma 6. $\Box$

We prove that Lemma 5 actually described all the polynomial cases:

Lemma 8 *If the set $\mathcal{P}$ of 2–patterns of length $k \geq 2$ which omits the monochromatic pattern cannot be described by a system of linear equations, then the decision problem whether a given k–cycle system with the pattern set $\mathcal{P}$ can be colored by at most 2 colors, is NP–complete.*

On the other hand, if $\mathcal{P}$ can be described by a system of linear equations, and the dimension of $\mathcal{A}(\mathcal{P})$ is k', then there exists $\beta_1, \ldots, \beta_{k'+1} \in \mathbb{GF}(2)$ such that the following system of k equations describes $\mathcal{P}$:

$$\sum_{i=1}^{k'+1} \beta_i x_{1+(j+i-2) \bmod k} = 1 \quad \textit{for } 1 \leq j \leq k$$

Proof: We find a system of equations describing $\mathcal{P}$ or we prove that the decision problem from the statement of the lemma is NP–complete for the given set $\mathcal{P}$.

Let $\mathcal{A}(\mathcal{P})$ be the set of zero–one vectors defined in the beginning of this section. We understand the vectors of $\mathcal{A}(\mathcal{P})$ as sequences of length k. We say that $\alpha_1, \ldots, \alpha_\kappa$ *uniquely determines* the $(\kappa+1)$-th coordinate iff the $(\kappa+1)$-th coordinate of each vector of $\mathcal{A}(\mathcal{P})$, whose first κ coordinates are equal to $\alpha_1, \ldots, \alpha_\kappa$, is uniquely determined. If both $\alpha_1, \ldots, \alpha_\kappa$ and $\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_\kappa$ uniquely determines the $(\kappa+1)$-th coordinate, we say that α_i is *essential* $(1 \leq i \leq \kappa)$ iff the $(\kappa+1)$-th coordinate differs for $\alpha_1, \ldots, \alpha_\kappa$ and $\alpha_1, \ldots, \alpha_{i-1}, \alpha_i+1, \alpha_{i+1}, \ldots, \alpha_\kappa$. We say that α_i is *non–essential* otherwise.

Let κ be the smallest number such that there exists a sequence $\alpha_1, \ldots, \alpha_\kappa$ such that it uniquely determines the $(\kappa+1)$-th coordinate. For each sequence $\alpha'_1, \ldots, \alpha'_\kappa$ there exists a vector of $\mathcal{A}(\mathcal{P})$ whose first κ coordinates are equal to $\alpha'_1, \ldots, \alpha'_\kappa$: Otherwise, let $\alpha'_1, \ldots, \alpha'_\kappa$ be the sequence such that there is not a vector of $\mathcal{A}(\mathcal{P})$ whose first κ coordinates are equal to $\alpha'_1, \ldots, \alpha'_\kappa$ and among all such sequences let this be the sequence with the longest initial subsequence common with $\alpha_1, \ldots, \alpha_\kappa$. Let κ' be the smallest number such that $\alpha_{\kappa'+1} \neq \alpha'_{\kappa'+1}$. Then due to choice of $\alpha'_1, \ldots, \alpha'_\kappa$ and κ', the sequence $\alpha'_1, \ldots, \alpha'_{\kappa'}$ determines the $(\kappa'+1)$-th coordinate (it forces it to be $\alpha_{\kappa'+1} = \alpha'_{\kappa'+1} + 1$) and it contradicts the choice of $\alpha_1, \ldots, \alpha_\kappa$ and κ.

We prove that actually each sequence $\alpha'_1, \ldots, \alpha'_\kappa$ uniquely determines the $(\kappa+1)$-th coordinate. Let us assume the opposite and let $\alpha'_1, \ldots, \alpha'_\kappa$ be the sequence which does not uniquely determine the $(\kappa+1)$-th coordinate and which among all such sequences has the longest initial subsequence common with $\alpha_1, \ldots, \alpha_\kappa$. Let κ' be the smallest number such that $\alpha_{\kappa'} \neq \alpha'_{\kappa'}$; note that $\kappa' \geq 2$ since $\mathcal{A}(\mathcal{P})$ contains together with each vector also its negation. Due to Lemma 4 there exists a k–cycle system $\mathcal{C}_{AB}$ with the pattern set $\mathcal{P}$ which simulates the pattern AB. Thus we can force two vertices to have different colors and with additional condition that the coloring does not use more than two colors we can force also two vertices to have the same color by using the k–cycle system $\mathcal{C}_{AB}$ twice. We create a k–cycle system $\mathcal{C}$ with vertices $v_1, \ldots, v_k$ as follows: We add for each $1 < i \leq \kappa, i \neq \kappa'$ either one copy of $\mathcal{C}_{AB}$ or two copies of $\mathcal{C}_{AB}$ obeying the following rules:

- If $\alpha_i = \alpha'_i = \alpha_1 = \alpha'_1$, then we force the vertex v_i to have the same color as v_1.
- If $\alpha_i = \alpha'_i \neq \alpha_1 = \alpha'_1$, then we force the vertex v_i to have a different color from v_1.

- If $\alpha_i \neq \alpha'_i$ and $\alpha_i = \alpha_{\kappa'}$, $\alpha'_i = \alpha'_{\kappa'}$, then we force the vertex v_i to have the same color as $v_{\kappa'}$.
- If $\alpha_i \neq \alpha'_i$ and $\alpha_i = \alpha'_{\kappa'}$, $\alpha'_i = \alpha_{\kappa'}$, then we force the vertex v_i to have a different color from $v_{\kappa'}$.

Next, we add a cycle $v_1, \ldots, v_k$. Note that the vertices $v_1, \ldots, v_\kappa$ are colored consistently with either $\alpha_1, \ldots, \alpha_\kappa$ or $\alpha'_1, \ldots, \alpha'_\kappa$ by any proper coloring of $\mathcal{C}$ using at most two colors. In the first case the colors of v_1 and $v_{\kappa'}$ uniquely determine the color of $v_{\kappa+1}$, in the second case the color of $v_{\kappa+1}$ can be arbitrary. $\mathcal{C}$ is a k–cycle system with the pattern set $\mathcal{P}$ such that the vertices v_1, $v_{\kappa'}$ and $v_{\kappa+1}$ can be colored by colorings using at most two colors only in three different ways upto renaming the colors. Thus the decision problem of 2–colorability of k–cycle systems with the pattern $\mathcal{P}$ is NP–complete due to Lemma 7 in this case.

The results of the previous two paragraphs can be summarized as follows: **Unless the problem is NP–complete, there exists κ such that each sequence $\alpha_1, \ldots, \alpha_\kappa$ uniquely determines the $(\kappa+1)$-th coordinate.**

The preceding immediately implies, since $\mathcal{A}(\mathcal{P})$ is closed under rotations, that $\mathcal{A}(\mathcal{P})$ contains exactly 2^κ vectors — the first κ coordinates uniquely determines the rest of the vector. We prove that α_i is either essential for the $(\kappa+1)$-th coordinate for all the choices of $\alpha_1, \ldots, \alpha_\kappa$ or it is non–essential for all the choices. Let us assume the opposite: Let $\alpha_1, \ldots, \alpha_\kappa$ be the sequence where α_i is essential for $\alpha'_{\kappa+1}$ and $\alpha'_1, \ldots, \alpha'_\kappa$ be the sequence where α'_i is non–essential for $\alpha_{\kappa+1}$. We may assume w.l.o.g. that $\alpha_1 = \alpha'_1$, since $\mathcal{A}(\mathcal{P})$ contains together with each vector also its negation. We further may assume w.l.o.g. that the $(\kappa+1)$-th coordinate determined by $\alpha_1, \ldots, \alpha_\kappa$ and by $\alpha'_1, \ldots, \alpha'_\kappa$ are the same (since otherwise we could change α_i in the first sequence to $\alpha_i + 1$) and let $\alpha_{\kappa+1}$ be this value. Let further κ' be the smallest number different from i such that $\alpha_{\kappa'} \neq \alpha'_{\kappa'}$. We create a k–cycle system $\mathcal{C}$ with vertices $v_1, \ldots, v_k$ similarly to the previous paragraph. We add for each $1 < j \leq \kappa+1, j \notin \{\kappa', i\}$ either one copy of $\mathcal{C}_{AB}$ or two copies of $\mathcal{C}_{AB}$ obeying the following rules:

- If $\alpha_j = \alpha'_j = \alpha_1 = \alpha'_1$, then we force the vertex v_j to have the same color as v_1.
- If $\alpha_j = \alpha'_j \neq \alpha_1 = \alpha'_1$, then we force the vertex v_j to have a different color from v_1.

- If $\alpha_j \neq \alpha'_j$ and $\alpha_j = \alpha_{\kappa'}$, $\alpha'_j = \alpha'_{\kappa'}$, then we force the vertex v_j to have the same color as $v_{\kappa'}$.
- If $\alpha_j \neq \alpha'_j$ and $\alpha_j = \alpha'_{\kappa'}$, $\alpha'_j = \alpha_{\kappa'}$, then we force the vertex v_j to have a different color from $v_{\kappa'}$.

Next, we add a cycle $v_1, \dots, v_k$. Each proper coloring of $\mathcal{C}$ with at most two colors colors the vertices $v_1, \dots, v_\kappa, v_{\kappa+1}$ consistently with either $\alpha_1, \dots, \alpha_\kappa, \alpha_{\kappa+1}$ or with $\alpha'_1, \dots, \alpha'_{i-1}, 0, \alpha'_{i+1}, \dots, \alpha'_\kappa, \alpha'_{\kappa+1}$ or with $\alpha'_1, \dots, \alpha'_{i-1}, 1, \alpha'_{i+1}, \dots, \alpha'_\kappa, \alpha'_{\kappa+1}$. Thus $\mathcal{C}$ is a k–cycle system with the pattern set $\mathcal{P}$ such that the vertices v_1, v_i and $v_{\kappa'}$ can be colored by colorings using at most two colors only in three different ways upto renaming the colors. Thus the decision problem of 2–colorability of k–cycle systems with the pattern $\mathcal{P}$ is NP–complete due to Lemma 7.

The previous paragraphs can be summarized as follows: **Unless the problem is NP–complete, there exists κ such that each sequence $\alpha_1, \dots, \alpha_\kappa$ uniquely determines the $(\kappa+1)$-th coordinates and α_i is either essential or non–essential for the $(\kappa+1)$-th coordinate regardless the choice of $\alpha_1, \dots, \alpha_\kappa$.**

Let I be the set of all i's for which α_i is essential for the $(\kappa+1)$-th coordinate and let γ be the value of the $(\kappa+1)$-th coordinate determined by the sequence of κ zeroes. Let $\alpha_1, \dots, \alpha_\kappa$ be any sequence of length κ, then this sequence forces the $(\kappa+1)$-th coordinate to be $\gamma + \sum_{i \in I} \alpha_i$, since each change of α_i for $i \in I$ from 0 to 1 changes the value of the $(\kappa+1)$-th coordinate and the change of α_i for $i \notin I$ from 0 to 1 does not affect the $(\kappa+1)$-th coordinate. Let $I' = I \cup \{\kappa+1\}$. The previous can be restated (due to the fact that $\mathcal{A}(\mathcal{P})$ is closed under rotation) as follows:

$$\sum_{i \in I'} \alpha_{1+(i+j-2) \bmod k} = \gamma \quad \text{for all } 1 \leq j \leq k$$

The first $k-\kappa$ equations above, for $1 \leq j \leq k-\kappa$, are linearly independent and since $\mathcal{A}(\mathcal{P})$ contains 2^κ vectors these equations describe $\mathcal{A}(\mathcal{P})$. Since the all–zero vector does not belong to $\mathcal{A}(\mathcal{P})$, the constant γ has to be one (otherwise the all–zero vector would belong to $\mathcal{A}(\mathcal{P})$). The above system of equations can be clearly rewritten to the form from the statement of the lemma by setting $k' = \kappa$, $\beta_i = 1$ for $i \in I'$ and $\beta_i = 0$ for $i \notin I'$. □

We summarize the results of this section and we state its main theorem:

Theorem 3 *Let $\mathcal{P}$ be a set of 2–patterns of length $k \geq 2$. The decision problem whether a given k–cycle system with the pattern set $\mathcal{P}$ can be colored* *by at most 2 colors is solvable in polynomial time iff at least one of the following two conditions holds (unless P=NP):*

- *$\mathcal{P}$ contains the monochromatic pattern of length k.*
- *$\mathcal{P}$ can be described by a system of linear equations.*

Otherwise, the problem is NP–complete.

Proof: If $\mathcal{P}$ contains the monochromatic pattern, then assigning all the vertices the same color yields a proper coloring and thus the system can always be properly colored by at most 2 colors. If $\mathcal{P}$ can be described by a system of linear equations, the 2–coloring problem is solvable in polynomial time due to Lemma 5. If neither of the preceding two conditions is satisfied, then the 2–coloring problem is NP–complete due to Lemma 8. $\Box$

6 Single Polynomial Patterns

We answer the question for which 2–patterns there is a polynomial–time algorithm for deciding whether a given cycle system can be colored by at most 2 colors consistently with this pattern. We first state and prove quite surprising linear algebra lemma about existence of a solution of a system of equations of a certain special type.

Lemma 9 *Let λ be a power of two and let $\alpha_1, \ldots, \alpha_\lambda$ be any sequence of 0's and 1's which contains at least one 1. Then the following system of equations has a solution over $\mathbb{GF}(2)$:*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ddots & \alpha_{\lambda-1} & \alpha_\lambda \\ \alpha_\lambda & \alpha_1 & \ddots & \alpha_{\lambda-2} & \alpha_{\lambda-1} \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \alpha_3 & \alpha_4 & \ddots & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \ddots & \alpha_\lambda & \alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\lambda-1} \\ x_\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

Proof: The proof proceeds by induction on λ. The statement is trivial for $\lambda = 1$, since it has to be $\alpha_1 = 1$ in this case and $x_1 = 1$ is the solution.

Otherwise, let A be the matrix consisting of α_i's from the statement of the lemma. Let A_1 and A_2 be the $\lambda/2 \times \lambda/2$ matrices forming the matrix A:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$$

We distinguish two cases:

- $A_1 + A_2$ is a non–zero matrix ($A_1 \neq A_2$).
  Then the matrix $A_1 + A_2$ is the matrix for the sequence $\alpha_1 + \alpha_{\lambda/2+1}, \ldots, \alpha_{\lambda/2} + \alpha_\lambda$. Let $x_1, \ldots, x_{\lambda/2}$ be the solution for the sequence $\alpha_1 + \alpha_{\lambda/2+1}, \ldots, \alpha_{\lambda/2} + \alpha_\lambda$. Then setting $x_{\lambda/2+1} = x_1, \ldots, x_\lambda = x_{\lambda/2}$ yields a solution of the original system of linear equations.

- $A_1 + A_2$ is a zero matrix ($A_1 = A_2$).
  Then $\alpha_i = \alpha_{i+\lambda/2}$ for all $1 \leq i \leq \lambda/2$. The matrix $A_1 = A_2$ is the matrix obtained for the sequence $\alpha_1, \ldots, \alpha_{\lambda/2}$. Let $x_1, \ldots, x_{\lambda/2}$ be the solution for the sequence $\alpha_1, \ldots, \alpha_{\lambda/2}$. Then setting $x_{\lambda/2+1} = \ldots = x_\lambda = 0$ yields a solution of the original system of linear equations.

□

We prove that aperiodic patterns for which the above stated problem can be solved in polynomial time have only lengths equal to powers of two:

Lemma 10 *If $\mathcal{P}$ is the set containing a single aperiodic 2–pattern of length k and if $\mathcal{P}$ can be described by a system of linear equations, then k is either $2^{k'-1}$ or $2^{k'}$ where k' is the dimension of the affine subspace $\mathcal{A}(\mathcal{P})$ considered over $\mathbb{GF}(2)$.*

Proof: The affine subspace $\mathcal{A}(\mathcal{P})$ contains all the rotations and negations of the only pattern contained in $\mathcal{P}$. If it can be described by a system of linear equations, then the size of $\mathcal{A}(\mathcal{P})$ is $2^{k'}$ where where k' is the dimension of the affine subspace $\mathcal{A}(\mathcal{P})$. If all the vectors of $\mathcal{A}(\mathcal{P})$ can be obtained just by the rotations of the pattern, then $k = 2^{k'}$. Otherwise, each rotation of the pattern adds to $\mathcal{A}(\mathcal{P})$ itself and its negation and thus the size of $\mathcal{A}(\mathcal{P})$ is $2k$ and $k = 2^{k'-1}$. □

We characterize all the patterns for which the above stated problem can be solved in polynomial time in the next theorem:

Theorem 4 *The only patterns p for which the decision problem whether a given k–cycle system with the pattern set $\{p\}$ can be colored by at most l colors where l is the number of distinct letters of p can be solved in polynomial time (unless P=NP) are the following:*

- $p = A^k$
- $p = (AB)^{k/2}$ *for k even*
- $p = (AABB)^{k/4}$ *for k divisible by four*
- $p = (AAAB)^{k/4}$ *for k divisible by four*

The problem is NP–complete for all the remaining ones.

Proof: If the pattern p contains three or more different letters the problem is NP–complete due to Theorem 2. On the other hand, if p is the monochromatic pattern, then the problem is trivial and can be solved in polynomial time. Thus we assume in the rest of the proof that p contains exactly two letters and l is equal to 2. We set $\mathcal{P} = \{p\}$.

Let us assume that there is a polynomial algorithm for 2–coloring of k–cycle systems with the set of patterns $\mathcal{P} = \{p\}$. In this case, $\mathcal{P}$ can be described by a system of linear equations due to Theorem 3. If the only pattern p of $\mathcal{P}$ is periodic, i.e. of the form p'^{κ} where p' is an aperiodic 2–pattern and $\kappa \geq 2$ is an integer, then also $\{p'\}$ can be described by a system of linear equations: Let $\beta_1, \ldots, \beta_{k'+1}$ be the coefficients of the linear equations from Lemma 8 for $\mathcal{P}$. The following system of linear equations clearly describes $\{p'\}$:

$$\sum_{i=1}^{k'+1} \beta_i x_{1+(i+j-2) \bmod (k/\kappa)} = 1 \quad \text{for } 1 \leq j \leq k/\kappa$$

On the other hand, if $\{p'\}$ can be described by a system of linear equations, then clearly also each pattern p'^{κ} for $\kappa \geq 2$ can be described by a system of linear equations. Thus it is enough to prove in order to establish the statement of the theorem that the only aperiodic patterns which can be described by a system of linear equations are AB, $AABB$ and $AAAB$. These patterns can be indeed described by a system of linear equations (and thus also the patterns obtained by concatenation of their several copies too) through the following choice of β_i's:

$$\begin{array}{ll} \text{AB}: & \beta_1 = 1, \beta_2 = 1 \\ \text{AABB}: & \beta_1 = 1, \beta_2 = 0, \beta_3 = 1 \\ \text{AAAB}: & \beta_1 = 1, \beta_2 = 1, \beta_3 = 1, \beta_4 = 1 \end{array}$$

We prove that the three above mentioned aperiodic patterns are the only ones which can be described by a system of linear equations. We further assume w.l.o.g. that $p = p'$. Let k' be the dimension of $\mathcal{A}(\mathcal{P})$. The length of p is either $2^{k'-1}$ or $2^{k'}$ due to Lemma 10. We prove that there exists a pattern q of length $K = 2^{\lceil \log_2(k'+1) \rceil}$ such that the vector corresponding to $q^{k/K}$ is a solution of the system of linear equations for $\mathcal{A}(\mathcal{P})$. Thus such an aperiodic pattern of length k cannot exist for $K < k$. Since $2^{k'-1} \leq k$ and $K = 2^{\lceil \log_2(k'+1) \rceil}$, there are no aperiodic patterns for $k' \geq 5$. It's routine to check that the only single aperiodic patterns which can be described by a system of linear equations for $k' = 1, 2, 3, 4$ are AB, $AABB$ and $AAAB$.

Note that $K \geq k' + 1$ due to choice of K. Let us set $\alpha_i = \beta_i$ for $1 \leq i \leq k' + 1$ and $\alpha_i = 0$ for $k' + 2 \leq i \leq K$. Let $x_1, \ldots, x_K$ be the solution of the following system of equalities (its existence follows from Lemma 9):

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ddots & \alpha_{K-1} & \alpha_K \\ \alpha_K & \alpha_1 & \ddots & \alpha_{K-2} & \alpha_{K-1} \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \alpha_3 & \alpha_4 & \ddots & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \ddots & \alpha_K & \alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{K-1} \\ x_K \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

Let q be the pattern corresponding to x through changing zeroes to A's and ones to B's. Then the periodic pattern $q^{k/K}$ has to belong to $\mathcal{P}$ contradicting the fact that $\mathcal{P}$ contains only a single aperiodic pattern. This completes the proof of the theorem. $\square$

7 Conclusion

We deal only with sets of two–patterns in Section 3. Thus the problem which remains open is the following:

- For which sets of l–patterns $\mathcal{P}$ does there exist a cycle system with the pattern set $\mathcal{P}$ and the feasible set $\mathcal{F}$ for any choice of $\mathcal{F}$ as a set of integers omitting the integers $1, \ldots, l-1$ for $l \geq 3$?

This is true for $l = 2$ for all $\mathcal{P}$ omitting the monochromatic pattern and containing a pattern different from the alternating one due to Theorem 1 and this is not true for $l = 2$ for $\mathcal{P}$ consisting only of the alternating pattern.

The answers to complexity questions provided in Section 4, Section 5 and Section 6 are exhausting in the sense that they fully describe (unless $P = NP$) the choices of the pattern set $\mathcal{P}$ and the integer l such that the decision problem whether a given cycle system with the pattern set $\mathcal{P}$ can be colored by at most l colors is solvable in polynomial time. On the other hand, it would be interesting to describe such sets of patterns containing more than just one pattern in a manner similar to Theorem 4.

Acknowledgement

The research was done as a part of DIMACS/DIMATIA Research Experience for Undergraduates programme. This is a joint programme of DIMACS at Rutgers University and DIMATIA at Charles University. Our REU supervisor was Jeff Kahn from Rutgers University.

References

[1] C. Berge: Graphs and Hypergraphs, North Holland, 1973.

[2] C. Colbourn, J. Dinitz, A. Rosa: Bicoloring Triple Systems, Electronic J. Combin. 6# 1, paper 25, 16 pages.

[3] M. R. Garey, D. S. Johnson: Computers and Intractability, A Guide to the Theory of NP–completeness, Freeman, San Franscisco, Cal., 1979.

[4] L. Haddad: On the chromatic numbers of Steiner triple systems, J. Combinat. Designs 7, 1999, 1–10.

[5] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. B. West: Chromatic spectrum is broken, 6th Twente Workshop on Graphs and Combinatorial Optimization, 26–28, May, 1999, H. J. Broersma, U. Faigle and J. L. Hurink (eds.), University of Twente, May, 1999, 231–234.

[6] C. C. Lindner, C. A. Roger: Decomposition into cycles II: Cycle Systems, Contemporary Design Theory: A Collection of Surveys, J. H. Dinitz and D. R. Stinson (eds.), John Wiley and Son, New York, 1992, 325–369.

[7] D. Král': On Complexity of Colouring Mixed Hypertrees, to appear in Proceedings 13th International Symposium on Fundamentals of Computing Theory, 1st International Workshop on Efficient Algorithms, LNCS, 2001.

[8] D. Král', J. Kratochvíl, A. Proskurowski, H.-J. Voss: Coloring mixed hypertrees, Proceedings 26th Workshop on Graph-Theoretic Concepts in Computer Science, LNCS vol. 1928, 2000, p. 279–289.

[9] D. Král', J. Kratochvíl, H.-J. Voss: Complexity Note on Coloring Mixed Hypergraphs, to appear in Proceedings 26th International Symposium on Mathematical Foundation of Computer Science, LNCS, 2001.

[10] G. Quattrocchi: Colouring 4-cycle systems with specified block colour patterns: the case of embedding P_3–designs, Electronic J. Combinatorics 8, 2001, #R24.

[11] L. Milazzo: On upper chromatic number for SQS(10) and SQS(16), Le Matematiche L (Catania, 1995), 179–193.

[12] L. Milazzo and Zs. Tuza: Upper chromatic number of Steiner triple and quadruple systems, Discrete Math. 174 (1997), 247–259.

[13] L. Milazzo and Zs. Tuza: Strict colorings for classes of Steiner triple systems, Discrete Math. 182 (1998), 233–243.

[14] V. Voloshin: The mixed hypergraphs, Computer Science Journal of Moldova 1, 1993, 45–52.

[15] V. Voloshin: On the upper chromatic number of a hypergraph, Australasian Journal of Combinatorics 11, 1995, 25–45.