A Counter–Example to Voloshin's Hypergraph Co–perfectness Conjecture

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Abstract

The upper chromatic number $\overline{\chi}(H)$ of a hypergraph H is the maximum number of colors in a coloring avoiding a polychromatic edge. The stability number $\alpha(H)$ of a hypergraph H is the cardinality of the largest set of vertices of H which does not contain an edge. A hypergraph is k–uniform if the sizes of all its edges are k. A hypergraph H is co–perfect if $\overline{\chi}(H') = \alpha(H')$ for each induced subhypergraph H' of H.

Voloshin conjectured that an r–uniform hypergraph H ($r \geq 3$) is co–perfect iff it contains neither of two special r–uniform hypergraphs (a so–called monostar and a complete circular r–uniform hypergraph on $2r-1$ vertices) as an induced subhypergraph. We disprove this conjecture for all r's.

1 Introduction

A *hypergraph* H is a pair (V, E) where V is its vertex set and $E \subseteq 2^V$ is its edge set; we do not restrict the sizes of the edges to two as in case of graphs. Throughout the paper we write $V(H)$ for a vertex set of a hypergraph H

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and $E(H)$ for its edge set. Recently, the topic of coloring of vertices of hypergraphs avoiding a *polychromatic* edge (i.e., the edge whose vertices have mutually different colors) has drawn an attention of different researchers, cf. [4, 6, 10, 15, 16], and related extremal (anti–Ramsey) questions were studied in [1, 3, 7, 17]. In this case, we want to color a hypergraph with a maximum possible number of colors (coloring all the vertices with the same color is clearly proper and thus minimizing the number of colors is not interesting); the maximum possible number $\overline{\chi}(H)$ of colors such that the vertices of the hypergraph H can be colored avoiding a polychromatic edge is called the *upper chromatic number* of H. Besides studying this type of coloring, the researchers also study a so–called *mixed hypergraphs* where the coloring has to prevent some of the edges to be monochromatic and some of them to be polychromatic, cf. [5, 9, 11, 12, 13].

We study in this paper coloring hypergraphs which avoids a polychromatic edge as described in the previous paragraph. The *stability number* $\alpha(H)$ of a hypergraph H is the cardinality of the largest set A such that no edge of H is fully contained in A; such set A is called *stable.* If c is a coloring of the vertices of H, then a *color class* with respect to c is a set of the vertices of H colored with the same color. It is clear that $\alpha(H) \geq \overline{\chi}(H)$, since we can create a stable set by taking one vertex from each color class of a coloring using $\overline{\chi}(H)$ colors (and this is actually a stable set, since the coloring avoids a polychromatic edge). The natural question is: "For which hypergraphs H does it hold that $\alpha(H) = \overline{\chi}(H)$?" A conjecture on a possible answer to this question was stated in [18].

A hypergraph H is r*–uniform* if the sizes of all its edges are r; a hypergraph H is r*–regular* if each of its vertices is contained in precisely r edges of H. A *subhypergraph* H' of a hypergraph H is a hypergraph whose both vertex and edge sets are subsets of a vertex set and an edge set of H; a subhypergraph H' is *induced* if $E(H') = E(H) \cap 2^{V(H')}$, i.e., all the edges of H whose all the vertices are in $V(H')$ are also the edges of H'. A famous strong perfect graph conjecture asserts that $\alpha(G') = \chi(G')$ for each induced subgraph G' of G (such graphs are called *perfect*). iff G nor its complement contains an odd cycle or a complete graph as an induced subgraph. Voloshin inspired by this famous conjecture made a similar conjecture in [18]:

Conjecture 1 *For each $r \geq 3$, an r–uniform hypergraph H is co–perfect if and only if it contains neither a monostar nor a C^r_{2r-1} as an induced subhypergraph.*

We postpone the missing definitions to the next paragraph. The co–perfectness of hypergraphs has been introduced in [18]; Conjecture 1 can be found as Conjecture 1 in [18]; the other conjecture stated in [18], Conjecture 2 of [18], has been recently answered in affirmative by the author in [9]. Other problems posed in [18] has been considered in [2, 8, 14] (Problem 8 of [18]), in [9] (Problem 10 and Problem 11 of [18]), in [12] (Problem 13 of [18]) and in [11, 13] (Problem 14 of [18]).

A hypergraph H is *co–perfect* if for each its induced subhypergraph H' it holds that $\alpha(H') = \overline{\chi}(H')$. A *monostar* is a hypergraph H such that the cardinality of the intersection of all the edges of H is exactly one, i.e., there exists a vertex v which is contained in all the edges and v is a unique vertex with this property; we call such a vertex the *center vertex* of a monostar. It is clear that $\alpha(H) = n - 1$ for a monostar H on n vertices and $\overline{\chi}(H) < n-1$; hence monostars are certainly not co–perfect. A hypergraph H is *circular* if there exists a cycle (in the usual graph theory sense) on the vertices of H such that the edges of H form its paths; we write C_n^r for an r–uniform hypergraph whose edges are precisely all the paths consisting of r vertices of the n–vertex cycle, i.e., C_n^r is the complete r–uniform circular hypergraph on n vertices. The hypergraph C_n^r for $n \geq 2r$ contains a monostar as an induced subhypergraph and thus it is not certainly co–perfect; but also C_{2r-1}^r is not co–perfect, since $\alpha(C_{2r-1}^r) = 2r - 3$ and $\overline{\chi}(C_{2r-1}^r) < 2r - 3$ (cf. [18]). These two examples of non–co–perfect hypergraphs lead to Conjecture 1 which is similar to the strong perfect graph conjecture, but besides this similarity there is no other connection between these two conjectures. Conjecture 1 has attracted attention of researchers, e.g., Tuza discussed Conjecture 1 during his invited talk at the Workshop Cycles and Colorings 2001 in Stara Lesna, Slovakia. We provide a counter–example to this conjecture for any $r \geq 3$.

Conjecture 1 is clearly equivalent to the following conjecture:

Conjecture 2 *If an r–uniform hypergraph H ($r \geq 3$) contains neither a monostar nor C_{2r-1}^r as an induced subhypergraph, then $\alpha(H) = \overline{\chi}(H)$.*

Due to Conjecture 2, it is enough to find an r–uniform hypergraph H (for each $r \geq 3$) which contains neither a monostar nor C_{2r-1}^r as an induced subhypergraph and for which $\overline{\chi}(H) < \alpha(H)$. We prove the existence of such hypergraphs in Theorem 1 in Section 2. The paper is structured as follows: We give additional definitions in Subsection 1.1. We define the counter–example r–uniform hypergraph H^r in Definition 1 of Section 2; we study properties of the hypergraph H^r in the following lemmas in Section 2 and we finish Section 2 proving Theorem 1. We conclude in Section 3.

1.1 Definitions and Used Notation

Let H be a hypergraph. We write $H \setminus V_0$ where $V_0 \subseteq V(H)$ for the induced subhypergraph of H on the vertex set $V(H) \setminus V_0$. Let c be a coloring of the vertices of H. If H contains no polychromatic edge, we say that the coloring c is *proper.* A color of a vertex v is *unique* if v is the only vertex colored with this color. An *isomorphism* between two hypergraphs H_1 and H_2 is a one–to–one mapping $\varphi : V(H_1) \to V(H_2)$ such that the images of the edges of H_1 are precisely the edges of H_2. An isomorphism is an *automorphism* if $H_1 = H_2$; an automorphism is *non–trivial* if it is not an identity. A hypergraph H is *vertex–transitive* if for any two vertices v and w of H there is an automorphism φ of H such that $\varphi(v) = w$.

The *incidence matrix* of a hypergraph H with $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$ is $n \times m$ matrix $I(H)$ such that $I(H)_{ij} = 1$ if $v_i \in e_j$ and $I(H)_{ij} = 0$ otherwise. Note that if H is r–uniform, then each column sum is precisely r; if H is k–regular, then each row sum is precisely k. We deal with different uniform hypergraphs in the paper: We try to use the notation such that the superscript is equal to the common sizes of edges, e.g., C^r_n (defined earlier) is an r–uniform hypergraph.

2 The Counter–Example

We first define the counter–example (to Conjecture 2) r–uniform hypergraph H^r:

Definition 1 *Let $r \geq 3$ be a fixed integer. Let H^r be the r–uniform hypergraph with $2r$ vertices and $2r+2$ edges whose incidence matrix $2r \times (2r+2)$ is the following (the incidence matrices for $r = 3$ and $r = 4$ can be found* *below):*

$$I(H^r) = \left(\begin{array}{cccccccccccc|cc}
1 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 1 & 0 & 1
\end{array}\right)$$

We write $v_1, \ldots, v_{2r}$ for the vertices of H^r; the vertex v_i corresponds to the i–th row of the incidence matrix. We write $e_1, \ldots, e_{2r}$ for the edges corresponding to the first $2r$ columns of the incidence matrix; the edge e_i corresponds to the i–th column of the incidence matrix:

$$e_i = \{v_i, v_{i+1}, v_{i+3}, \ldots, v_{i+2r-3}\}$$

where the subscripts of the vertices are taken modulo $2r$. We write e_o and e_e (odd and even in correspondence to the parity of the indices of the subscripts of the vertices contained in e_o and e_e) for the edges corresponding to the last but one and the last column of the incidence matrix.

In order to illustrate the definition, we include the incidence matrices for H^3 and H^4:

$$I(H^3) = \left(\begin{array}{cccccc|cc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$I(H^4) = \left(\begin{array}{cccccccc|cc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

Lemma 1 *The hypergraph H^r is a vertex–transitive r–uniform $(r+1)$–regular hypergraph.*

Proof: The proof of the uniformity and the regularity of H^r follows immediately from Definition 1. In order to prove the vertex–transitivity of H^r, note that the function $\varphi : V(H^r) \to V(H^r)$ defined as follows is an automorphism of H^r:

$$\varphi(v_i) = \begin{cases} v_1 & \text{if } i = 2r, \\ v_{i+1} & \text{otherwise.} \end{cases}$$

■

We next find the stability number of H^r:

Lemma 2 *The stability number of H^r is $2r-3$.*

Proof: Let $r \geq 3$ be a fixed integer through the proof. The set of vertices of H^r $\{v_1, \ldots, v_{2r-3}\}$ is stable; thus $\alpha(H^r) \geq 2r-3$. If $\alpha(H^r) > 2r-3$, let $A \subseteq V(H^r)$ be the stable set of size $2r-2$. We can assume that $v_1 \notin A$ since H^r is vertex–transitive. Let v_i be the only vertex different from v_1 not contained in A. If i is odd, then $e_e \subseteq A$. Hence i has to be even. If $i = 2$, then $e_3 \subseteq A$; but if $i \geq 4$, then $e_2 \subseteq A$.

■

We next prove a lemma which allows us to use a proper coloring of H^r to get a proper coloring of H^{r-1}:

Lemma 3 *Let c be a coloring of the hypergraph $H^r, r \geq 4$, using k colors such that the colors of v_{2r-1} and v_{2r} are unique. Then $\overline{\chi}(H^{r-1}) \geq k-2$.*

Proof: We write e'_i for the i–th edge of H^{r-1} and e_i for the i–th edge of H^r in the proof; we consider the vertices $v_i, 1 \leq i \leq 2r-2$ of H^{r-1} and H^r to be the same ones; we write V_0 for these vertices, i.e., $V_0 = \{v_1, v_2, \ldots, v_{2r-2}\}$. Let c' be the coloring of H^{r-1} obtained by restriction of c to the first $2r-2$ vertices, i.e., $c'(v_i) = c(v_i), 1 \leq i \leq 2r-2$. We claim that there is no polychromatic edge in H^{r-1}. The edge e'_e, resp. e'_o, is not polychromatic, since $e'_e = e_e \cap V_0$, resp. $e'_o = e_o \cap V_0$ (recall that the colors of the vertices v_{2r-1} and v_{2r} are different and not used to color any of the vertices of V_0). The edges $e'_i, 2 \leq i \leq 2r-2$, are not polychromatic, since $e'_i = e_i \cap V_0$. The remaining edge e'_1 is not polychromatic, since $e'_1 \subset e_{2r-1} \cap V_0$.

■

We next prove that the upper chromatic number of H^r is smaller than its stability number:

Lemma 4 *The upper chromatic number of H^r is $2r-4$.*

Proof: Let c be the following coloring of H^r:

$$c(v_i) = \begin{cases} i & \text{for } 1 \leq i \leq 2r-4, \\ 2r-4 & \text{for } 2r-5 \leq i \leq i. \end{cases}$$

The coloring c is a proper coloring of H^r and thus $\overline{\chi}(H^r) \geq 2r-4$. In the rest, we prove that $\overline{\chi}(H^r) \leq 2r-4$; the proof proceeds by induction on r (although it might not seem so in the beginning of it):

- $r = 3$
  Let c be a coloring of H^3 using 3 colors. The coloring c colors some consecutive vertices by different colors and hence we may assume that $c(v_1) \neq c(v_2)$ due to the vertex–transitivity of H^3. We distinguish three cases according to the possibilities which vertex is assigned the third color (the one different from $c(v_1)$ and $c(v_2)$):
  - The third color is assigned to v_3. (This case is symmetric to the case that the third color is assigned to v_6).

The color $c(v_5)$ is $c(v_3)$ due to the edges $e_2 = \{v_2, v_3, v_5\}$ and $e_o = \{v_1, v_3, v_5\}$. The color $c(v_6)$ is $c(v_3)$ due to the edges $e_5 = \{v_5, v_6, v_2\}$ and $e_6 = \{v_6, v_1, v_3\}$. Then, $c(v_4)$ has to be $c(v_2)$ due to the edges $e_1 = \{v_1, v_2, v_4\}$ and $e_e = \{v_2, v_4, v_6\}$. But then the edge $e_4 = \{v_4, v_5, v_1\}$ is polychromatic — contradiction.

- The third color is assigned to v_4.
  This is impossible, since $e_1 = \{v_1, v_2, v_4\}$.
- The third color is assigned to v_5.
  Then $c(v_3) \in \{c(v_2), c(v_5)\}$ due to the edge $e_2 = \{v_2, v_3, v_5\}$ and $c(v_3) \in \{c(v_1), c(v_5)\}$ due to the edge $e_o = \{v_1, v_3, v_5\}$. Hence $c(v_3) = c(v_5)$, the third color is assigned to v_3 and this case has been dealt in the first subcase.

- $r = 4$
  Let c be a coloring of H^4 using 5 colors; we distinguish several cases according to the sizes of color classes of c:
  - $4:1:1:1:1$
    If there were two consecutive (with respect to their subscripts) vertices with unique colors, Lemma 3 would imply that $\overline{\chi}(H^3) \geq 3$. Since there are four vertices with unique colors, we can assume w.l.o.g. that they are v_1, v_3, v_5, v_7 (no two of them can be consecutive and H^4 is vertex–transitive). But then edge e_o is polychromatic — contradiction.
  - $3:2:1:1:1$
    Let C_1 be the color class of size 3 and C_2 be the color class of size 2. Since it cannot hold both $|C_1 \cap e_o| \geq 2$ and $|C_1 \cap e_e| \geq 2$, we may assume that $C_2 \subseteq e_e$. Then $|C_1 \cap e_o| \geq 2$. We assume that $v_1 \in C_1$ (recall that H^4 is vertex–transitive). We prove that $C_1 \subseteq \{v_1, v_2, v_3, v_5, v_7\}$. Consider the edge $e_2 = \{v_2, v_3, v_5, v_7\}$. Since $|e_2 \cap C_2| \leq 1$, it has to be $|e_2 \cap C_1| = 2$. Hence $C_1 \subseteq \{v_1, v_2, v_3, v_5, v_7\}$. Let v_i be one of the two vertices of C_2; note that $i \in \{2, 4, 6, 8\}$. We assume that $i \neq 8$ (otherwise take the other vertex of C_2). Consider the edge e_{i+1}: $|e_{i+1} \cap C_2| \leq 1$ and thus it has to be $|e_{i+1} \cap C_1| \geq 2$ and $C_1 = \{v_1, v_2, v_{i+1}\}$. Then the edge e_{i+2} is polychromatic ($e_{i+2} \cap C_1 \subseteq \{v_1\}$ and $e_{i+2} \cap C_2 \subseteq \{v_{i+2}\}$) — contradiction.

- $2:2:2:1:1$
  Let C_1, C_2 and C_3 be the three color classes of sizes 2. We assume w.l.o.g. that $C_2 \subset e_o$ and $C_3 \subset e_e$ (otherwise, the edge e_o or the edge e_e is polychromatic). Let v_i be a vertex of either C_2 or C_3. Hence $|e_{i+1} \cap C_2| \leq 1$ and $|e_{i+1} \cap C_3| \leq 1$; this gives that $C_1 \subseteq e_{i+1}$ (the subscript is considered modulo 8). But then C_1 has to be in the intersection of four different edges $e_{k_1}, e_{k_2}, e_{l_1}, e_{l_2}$ where k_1 and k_2 are odd and l_1 and l_2 are even; but any such intersection is empty, since $e_{k_1} \cap e_{k_2} \subseteq \{v_2, v_4, v_6, v_8\}$ and $e_{l_1} \cap e_{l_2} \subseteq \{v_1, v_3, v_5, v_7\}$ — contradiction.

- $r = 5$
  Let c be a coloring of H^5 using 7 colors; we distinguish several cases according to the sizes of color classes of c:

  - $4:1:1:1:1:1:1$
    In this case, there are certainly two consecutive (with respect to their subscripts) vertices with unique colors, w.l.o.g. v_9 and v_{10}. But then Lemma 3 would imply that $\overline{\chi}(H^4) \geq 5$ — contradiction.
  - $3:2:1:1:1:1:1$
    There cannot be two consecutive vertices with unique colors as dealt in the previous subcase. But then w.l.o.g. the colors of v_1, v_3, v_5, v_7, v_9 have to be unique (there are five vertices with unique colors and H^5 is vertex–transitive) and the edge e_o is polychromatic — contradiction.
  - $2:2:2:1:1:1:1$
    Let C_1, C_2 and C_3 be the three color classes of sizes 2. We assume w.l.o.g. that $C_2 \subset e_o$ and $C_3 \subset e_e$ (otherwise, these edges would be polychromatic). Let v_i be a vertex of either C_2 or C_3. Hence $|e_{i+1} \cap C_2| \leq 1$ and $|e_{i+1} \cap C_3| \leq 1$; this implies that $C_1 \subseteq e_{i+1}$ (the subscript is considered modulo 10). But then C_1 has to be in the intersection of four different edges $e_{k_1}, e_{k_2}, e_{l_1}, e_{l_2}$ where k_1 and k_2 are odd and l_1 and l_2 are even; but any such intersection is empty, since $e_{k_1} \cap e_{k_2} \subseteq \{v_2, v_4, v_6, v_8, v_{10}\}$ and $e_{l_1} \cap e_{l_2} \subseteq \{v_1, v_3, v_5, v_7, v_9\}$ — contradiction.

- $r = 6$
  Let c be a coloring of H^6 using 9 colors; H^6 contains at least 6 vertices

with unique colors. If there were two consecutive (with respect to their subscripts) vertices with unique colors, we may assume that these are v_{11} and v_{12} since H^6 is vertex–transitive, Lemma 3 would imply that $\overline{\chi}(H^5) \geq 7$. If there were more than 6 vertices with unique colors, then (due to pigeonhole principle) there would be two consecutive ones. Thus there are exactly 6 vertices colored with unique colors and these are w.l.o.g. $v_1, v_3, v_5, v_7, v_9, v_{11}$ (no two of them can be consecutive and H^6 is vertex–transitive). But then edge e_o is polychromatic — contradiction.

- $r \geq 7$
  Let c be a coloring of H^r using $2r - 3$ colors. H^r contains at least $2r - 6 > r$ vertices with unique colors. Due to pigeonhole principle, there are two consecutive (with respect to their subscripts) vertices colored with unique colors. We may assume that these vertices are v_{2r-1} and v_{2r}, since H^r is vertex–transitive, but then Lemma 3 implies that $2r - 5 \leq \overline{\chi}(H^{r-1})$ — contradiction.

■

It remains to check that H^r contains neither a monostar nor C^r_{2r-1} as an induced subhypergraph:

Lemma 5 *The hypergraph H^r does not contain a monostar as an induced subhypergraph.*

Proof: Let $r \geq 3$ be a fixed integer through the proof. We assume that H^r contains a monostar with the center vertex equal to v_1. Let $V_0 \subseteq V(H^r)$ be the vertices which induce the monostar and let $E_0 = E(H^r) \cap 2^{V_0}$. Note that the following hold due to the definition of a monostar and an induced subhypergraph:

$$V_0 = \bigcup_{e \in E_0} e$$

$$\forall e' \in E(H) : e' \subseteq V_0 \Rightarrow e' \in E_0$$

$$\{v_1\} = \bigcap_{e \in E_0} e = \bigcap_{e \subseteq V_0, e \in E(H')} e$$

We distinguish several cases in the proof:

- $e_1 \in E_0$ and $e_o \in E_0$
  It has to be that $V_0 \supseteq e_o \cup e_1 = V(H^r) \setminus \{v_{2r}\}$. But then $e_2 \in E_0$ — contradiction.

- $e_1 \in E_0$ and $e_o \notin E_0$
  The edge e_1 cannot be the only edge of E_0. Because the intersection of the edges of E_0 is $\{v_1\}$, E_0 can contain besides e_1 only the edges e_i for even $i, 4 \leq i \leq 2r$. If $e_{2r} \in E_0$, then $e_1 \cup e_{2r} = V(H^r) \setminus \{v_{2r-1}\} \supseteq V_0$ and $e_e \in E_0$ which is impossible. Let e_{i_1} be an edge of E_0 different from e_1; i_1 has to be an even integer between 4 and $2r-2$. Since $e_1 \cap e_{i_1} = \{v_1, v_{i_1}\}$, the edge set E_0 has to contain an edge e_{i_2} different from e_1 and e_{i_1}. But then $e_1 \cup e_{i_1} \cup e_{i_2} = V(H^r) \setminus \{v_{2r}\} \supseteq V_0$. Hence $e_2 \in E_0$ — contradiction.

- $e_1 \notin E_0$ and $e_o \in E_0$
  The only two edges of E_0 which contain v_1 and do not contain v_i for odd $3 \leq i \leq 2r-1$ are e_1 and e_{i+1}. Since $e_1 \notin E_0$, the intersection of the edges of E_0 consists of the single vertex v_1 and $v_i \in e_o$ for all odd i's, $3 \leq i \leq 2r-1$, it has to be $e_{i+1} \in E_0$. But then, $V_0 \supseteq V(H^r) \setminus \{v_2\}$ and e_3 has to be contained in E_0 — contradiction.

- $e_1 \notin E_0$ and $e_o \notin E_0$
  In this case it has to be $E_0 \subseteq \{e_4, e_6, \ldots, e_{2r}\}$. The only edge of e_4, e_6, …, e_{2r} which does not contain v_i for odd $i, 3 \leq i \leq 2r-1$ is e_{i+1}. Hence, it has to be $E_0 = \{e_4, \ldots, e_{2r}\}$. But then, $V_0 \supseteq V(H^r) \setminus \{v_2\}$ and thus e_3 has to be contained in E_0 — contradiction.

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Lemma 6 *The hypergraph H^r does not contain the complete circular hypergraph C^r_{2r-1} on $2r-1$ vertices as an induced subhypergraph.*

Proof: Let $r \geq 3$ be a fixed integer through the proof. If H^r contains C^r_{2r-1} as an induced subhypergraph, then $H^r \setminus v_1$ is isomorphic to C^r_{2r-1} (recall that H^r is vertex–transitive). But $H^r \setminus v_1$ consists of only $2r+2-(r+1) = r+1$ edges and C^r_{2r-1} consists of $2r-1$ edges.

■

We can conclude the section:

Theorem 1 *The r–uniform hypergraph H^r contains neither a monostar nor the complete circular hypergraph C^r_{2r-1} on $2r-1$ vertices, but $\overline{\chi}(H^r) < \alpha(H^r)$ for any $r \geq 3$.*

Proof: The proof immediately follows from Lemma 2, Lemma 4, Lemma 5 and Lemma 6.

■

3 Conclusion

The co–perfectness conjecture of Voloshin is wrong, but the following interesting problem remains open:

Problem 1 *For which (r–uniform) hypergraphs H does the equality $\alpha(H) = \overline{\chi}(H)$ hold?*

The answer suggested by Conjecture 2 for r–uniform hypergraphs is false. We do not think there is some hope to find a finite number of cases of minimal non–perfect hypergraphs different from monostars, but even though the answer to Problem 1 could be nice and provide new insights to the structure of colorings avoiding polychromatic edges.

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