# On Maximum Face–Constrained Coloring of Plane Graphs of Girth at least 5

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**Abstract**

We prove that each plane graph of girth at least five on $n \geq 4$ vertices can be colored by at least $\lceil n/2 \rceil + 1$ colors in such a way that it does not contain a multicolored face, i.e. the face whose all the vertices have mutually distinct colors.

## 1 Introduction

Coloring vertices of plane graphs under restrictions on the coloring with respect to its faces has become recently a very intensively studied topic. The most natural problems in this area include: Coloring the vertices with the minimum possible number of colors in such a manner that a plane graph does not contain a monochromatic face and coloring the vertices with the maximum possible number of colors in such a manner that a plane graph does not contain a multicolored face. A face is *monochromatic* if all its vertices are colored with the same color; a face is *multicolored* if all its vertices are colored with mutually distinct colors.

Hypergraphs $H$ for which there is a plane graph $G$ such that for each edge of $H$ there is a face of $G$ whose vertices are exactly the vertices of

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the edge are called *planar hypergraphs*; this definition was introduced by Zykov in [16] and it is equivalent to the statement that the bipartite incidence graph of $H$ formed by the vertices and the edges of $H$ is planar as discussed in [2, 12]. Coloring vertices of a plane graph under the restriction that there is no monochromatic face was studied in [12]; under this condition, coloring faces of a plane graph which correspond to a planar hypergraph is just usual hypergraph coloring. Coloring vertices of a plane graph under the restriction that there is no multicolored face was studied in [13]; we remark that coloring vertices of graphs avoiding a certain multicolored pattern has been intensively studied in extremal combinatorics (cf. [1, 3, 6, 7, 8, 14]). Coloring vertices of a plane graph under both restrictions (some of them possibly imposed only to some of its faces) was studied in [2, 9, 11] — this model is also called *planar mixed hypergraphs.*

We study in this paper colorings of vertices of plane graphs under the constraint that there is no multicolored face. In such case, we say that a coloring of vertices of a plane graph $G$ is *valid.* We write $\chi_f(G)$ for the maximum possible number of colors for which there is a valid coloring (using all the $\chi_f(G)$ colors). The following inequality relating $\chi_f(G)$ of a plane graph $G$ to its independence number $\alpha(G)$ and its chromatic number $\chi(G)$ was proved in [13]:

$$\chi_f(G) \geq \alpha(G) + 1 \geq \left\lceil \frac{n}{\chi(G)} \right\rceil + 1$$

This implies that $\chi_f(G) \geq \lceil \frac{n}{4} \rceil + 1$ for all plane graphs $G$ from the Four Color Theorem and $\chi_f(G) \geq \lceil \frac{n}{3} \rceil + 1$ for plane graphs with girth at least four from Grötzsch's theorem (cf. [4, 13, 15]); the girth of a graph $G$ is the length of the shortest cycle in $G$. This lower bound on $\chi_f(G)$ is tight for plane graphs $G$ on $n$ vertices with chromatic number equal to 2 or 3 for all $n$'s and is within one from the optimal one for other plane graphs. The following conjecture has been stated in [13]:

**Conjecture 1** *If $G$ is a plane graph on $n$ vertices of girth at least four, then $\chi_f(G) \geq \lceil n/2 \rceil + 1$.*

One can find a proof of this conjecture for plane graphs of girth at least six in [13]. We prove this conjecture for plane graphs of girth at least five in this paper.

We introduce notations and recall some definitions in Subsection 1.1. We prove that a graph with minimum degree can be covered by stars of small degree in Section 2. We relate the number of edges in certain subgraphs in the dual graph of a plane graph $G$ to $\chi_f(G)$ in Theorem 2 in Section 3. Then, we use the results of Section 2 to prove Theorem 3 which contains the main result of this paper. We conclude in Section 4.

## 1.1 Definitions and Notation

We mean by a *plane graph* a graph together with a fixed embedding of it to the plane; if we consider just a graph which can be embedded to the plane, we refer to it as to a *planar graph*. If $G$ is a plane graph, we write $V(G)$, resp. $E(G)$, $F(G)$, for the set of its vertices, resp. its edges, its faces. A *loop* is an edge whose both end–vertices are the same; two edges are *parallel* if they join the same pair of vertices. If a graph can contain parallel edges, we refer to it as to a *multigraph*. We do not consider graphs containing loops in this paper. If a graph cannot contain parallel edges, we refer to it as to a *simple graph*; unless stated otherwise, we mean by a graph, for shortness, a simple graph. The *girth* of a graph $G$ is the length of the shortest cycle in $G$; if $G$ contains parallel edges, then its girth is equal to two.

We write $\deg_G(v)$ for the degree of a vertex $v$ in $G$, i.e. the number of edges incident to $v$ in $G$; if $G$ is a directed graph, then the *indegree*, resp. *outdegree*, of a vertex $v$ is the number of edges leading to, resp. from, $v$. We call a directed edge also an *arc*. A graph $G$ is *connected* if it contains a path between any two of its vertices. A graph $G$ is *$k$–edge connected* if it remains connected after removal of any $k-1$ edges of it. An edge whose removal increases the number of connected components of a graph is called a *bridge*; if a graph does not contain a bridge, we say that it is *bridgeless*. The *dual (multi)graph $G^\star$ of a plane graph $G$* is a graph whose vertices correspond to the faces of $G$ and the edges correspond to the edges of $G$ in such a manner that the edge $e$ joins the vertices corresponding to the two faces sharing the edge $e$ in $G$. If $G$ is bridgeless, $G^\star$ does not contain loops.

We use a notion of a covering of a graph to prove our results. A *covering* of a graph $G$ is a subgraph $G'$ of $G$ which contains all the vertices of $G$ such that the degree of any vertex in $G'$ is at least one. A *partial covering* is a subgraph of $G$ which contains all the vertices of $G$ but it may contain vertices of degree zero. A vertex is *covered* by a partial covering if its degree in it is at least one. If $\mathcal{G}$ is a set of graphs, we say that $G$ can be *covered* by $\mathcal{G}$ if there is a covering such that each its connected component is isomorphic to a graph in $\mathcal{G}$. A *star* is a complete bipartite graph such that one of its parts is a single vertex; we refer to this vertex as to a *center vertex* and the *order* of a star is the degree of its center vertex.

## 2 Covering graphs by stars of small orders

The following lemma is a "folklore" application of Hall's theorem [5], but we include its proof for completeness:

**Lemma 1** *Let $G$ be a connected multigraph. Then the edges of $G$ can be oriented in such a way that the in–degree of a vertex $v$ of $G$ is at most* $\lceil \deg_G(v)/2 \rceil$.

**Proof:** Consider the following bipartite graph $G' = (V_1, V_2, E)$:

- The vertex set $V_1$ contains $\lceil \deg_G(v)/2 \rceil$ different vertices which correspond to a vertex $v$ of $G$ for each vertex $v$ of $G$.
- The vertex set $V_2$ consists of the edges of $E$.
- A vertex $u$ of $V_1$ which corresponds to a vertex $v$ of $G$ is joined by an edge to a vertex $e$ of $V_2$ (which is an edge of $G$) if and only if $v$ is an end–vertex of $e$.

The bipartite graph $G'$ contains a matching of size $|V_2|$ due to Hall's theorem. If an edge $e$ is matched to a vertex which corresponds to a vertex $v$ of $G$, then we orient the edge $e$ to lead to $v$. There are clearly at most $\lceil \deg_G(v)/2 \rceil$ edges leading to a vertex $v$ of $G$.

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**Theorem 1** *Let $G$ be a connected multigraph of minimum degree* 3*. Then $G$ can be covered by stars such that the order of a star with a center vertex $v$ is at most* $\lceil \deg_G(v)/2 \rceil$.

**Proof:** Fix an orientation of $G$ such that the in–degree of a vertex $v$ of $G$ is at most $\lceil \deg_G(v)/2 \rceil$ (it exists due to Lemma 1). We construct a covering of $G$ from an empty partial covering through the following procedure: Take a vertex $v$ which is not covered. Let $w$ be any vertex such that there is an arc leading from $v$ to $w$. Add an edge $vw$ to the partial covering. If there is an arc leading from $w$ included in the partial covering perform additionally the following steps: Let $ww'$ be this arc (there can be only one such arc). If the degree of $w'$ in the covering is at least two or the degree of $w$ in the covering is more than $\lceil \deg_G(w)/2 \rceil$, then remove the edge $ww'$ from the partial covering.

First, note that a vertex $w$ always exists: There are always at least $\deg_G(v) - \lceil \deg_G(v)/2 \rceil \geq 1$ arcs leading from the vertex $v$ (the inequality is due to the assumption on the minimum degree). Further, note that for each vertex $v$, there is at most one edge leading from $v$ included to the partial covering at any time. We prove that the algorithm described in the first paragraph ends and thus it finds the desired partial covering (the obtained covering clearly contains only stars of the claimed degree). Let $d_u$ be the number of arcs leading to $u$ included to the partial covering. We claim that at each step **either** the number of covered vertices is increased **or** the number of covered vertices is preserved and the sum $\sum_{u \in V(G)} d_u^2$ is increased. This claim implies that the above algorithm ends. Let us look at the algorithm when covering the vertex $v$: If we do not remove the edge $ww'$, then the number of covered vertices is increased. If we remove the edge $ww'$ and the degree of $w'$ in the partial covering is at least two, then the number of covered vertices is increased also in this case. If we remove the edge $ww'$ and the degree of $w'$ in the covering is one, then after performing the whole step of the algorithm $d_{w'}$ is decreased from 1 to 0, but $d_w$ is increased by one from a non–zero value, otherwise the degree of $w$ in the covering cannot exceed $\lceil \deg_G(w)/2 \rceil \geq 2$; note that we use that there is at most one edge leading from $w$ included in the covering and the minimum degree of $G$ is at least three.

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We remark that the preceding theorem is a modification of the following similar (but much easier) statement which can be found in [10]:

**Lemma 2** *Let $G$ be a connected graph of maximum degree $\Delta$. Then $G$ can be covered by stars of order at most $\Delta$.*

If we have no additional assumptions on a graph $G$, Lemma 2 is sharp, i.e. the bound on the orders of stars cannot be improved. Note also that the condition that the minimum degree of a graph is at least three cannot be weakened in Theorem 1 (as witnessed by an odd cycle).

## 3 The main result

**Theorem 2** *Let $G$ be a plane graph. If the dual multigraph of $G$ contains a covering subgraph with $m$ edges, then $\chi_f(G) \geq n-m$ where $n$ is the number of vertices of $G$.*

**Proof:** Let $E$ be the edges of $G$ corresponding to the $m$ edges of a subgraph covering the dual graph of $G$; since the subgraph covers the dual graph, each face of $G$ is incident to at least one edge of $E$. Let $G'$ be a graph with a vertex set equal to $V(G)$ and the edge set equal to $E$; $G'$ consists of at least $n-m$ connected components. Color the vertices of each of its components with the same color and the vertices of different components with different colors. This coloring is a valid coloring of $G$.

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**Lemma 3** *If there exists a plane graph $G$ on $n \geq 4$ vertices with girth at least five, such that $\chi_f(G) < \lceil n/2 \rceil + 1$, then there exists a plane graph $G'$ on $n' \geq 4$ vertices with girth at least five which is 2–edge–connected and $\chi_f(G') < \lceil n/2 \rceil + 1$.*

**Proof:** Let $n \geq 4$ be the smallest integer for which there exists a plane graph $G$ on $n$ vertices with girth at least five, such that $\chi_f(G) < \lceil n/2 \rceil + 1$. We claim that $G$ is 2–edge–connected. We assume that $G$ is connected, otherwise we may add some edges to make it connected in such a manner that we do not change structure of its faces. Let us assume that $G$ contains a bridge. Let $G_1$ and $G_2$ be the parts of $G$ separated by the bridge and let $n_1$ and $n_2$ be the numbers of vertices of $G_1$ and $G_2$. If both $n_1 \geq 4$ and $n_2 \geq 4$, then we can color $G_1$ by at least $\lceil n_1/2 \rceil + 1$ colors and $G_2$ by at least $\lceil n_2/2 \rceil + 1$ colors due to the choice of $G$. These colorings together give a valid coloring using at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + 2 \leq \lceil n/2 \rceil + 1$ colors — contradiction. If $n_1 \leq 4$ and $n_2 \geq 4$ (or vice versa), then $G_1$ has to be a tree due to the assumption on the girth of $G$; $G_2$ can be colored by at least $\lceil n_2/2 \rceil + 1$ colors due to choice of $G$ and extending this coloring from $G_2$ to $G$ by coloring the vertices of $G_1$ with $n_1$ completely new colors gives a valid coloring of $G$ using at least $n_1 + \lceil n_2/2 \rceil + 1 \leq \lceil n/2 \rceil + 1$ colors — contradiction. In the last case, $n_1 \leq 4$ and $n_2 \leq 4$; but both $G_1$ and $G_2$ are trees due to the assumption on the girth of $G$ and thus $G$ has a valid coloring using $n - 1 \geq \lceil n/2 \rceil + 1$ colors since the whole $G$ is a tree in this case — contradiction.

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**Theorem 3** *Let $G$ be a plane graph on $n$ vertices of girth at least five. Then $\chi_f(G) \geq \lceil n/2 \rceil + 1$.*

**Proof:** We may assume that $G$ is connected and bridgeless due to Lemma 3. Let $f$ be the number of faces of $G$ and let $G^\star$ be the dual multigraph of $G$. Since $G$ is bridgeless, $G^\star$ contains no loops. The minimum degree of $G^\star$ is at least three, otherwise $G$ contains a bigon and its girth is two. Let $C$ be a covering of $G^\star$ from Theorem 1, i.e. the degree of a vertex $v$ in $C$ is at most $\lceil \deg_{G^\star}(v)/2 \rceil$. We prove that $C$ contains at most $\lfloor n/2 \rfloor - 1$ edges. This implies together with Theorem 2 and Lemma 3 the statement of the theorem.

We use discharging technique to prove the bound on the number of edges in the covering $C$. Let $l_f$ be the number of edges on the boundary of a face $f$ of $G$. We give to a face $f$ in the beginning $\frac{l_f-2}{4}$ units. Note that the boundary of any face consists of at least 5 edges due to the assumption on the girth of $G$ and thus each face gets at least $3/4$ units. Each face gives all its units to a unique star of the covering $C$ in which it is contained. We prove that each star of the covering $C$ gets at least as many units as it has edges and the number of units given to the faces of $G$ in the beginning is at most $n/2-1$. This implies the desired bound on the number of edges of $C$; realize that $C$ contains an integer number of edges.

The number of edges of $G$ is equal to $\sum_{f\in F(G)} l_f/2$ (each edge is contained in two faces). We get using the Euler formula, $n+|F(G)| = |E(G)|+2$, the following:

$$\sum_{f\in F(G)} \frac{l_f-2}{4} = \frac{1}{2}\sum_{f\in F(G)}\left(\frac{l_f}{2}-1\right) = \frac{1}{2}\left(|E(G)|-|F(G)|\right) = \frac{n}{2}-1$$

Thus we give at most $\frac{n}{2}-1$ units to all the faces together in the beginning. Let $S$ be a star of order $o$ of a covering $C$ of $G^\star$. If $o \le 3$, then $S$ gets at least $\frac{3}{4}(o+1) \ge o$ units. If $o \ge 4$, we proceed as follows: Let $f$ be the face corresponding to the center vertex of $S$; the star $S$ gets at least $\frac{3}{4}o + \frac{l_f-2}{4}$ units. On the other hand, the degree of the center vertex which is equal to the order $o$ of the star is at most $\lceil l_f/2 \rceil$; thus $l_f \ge 2o-1$. This altogether gives the following:

$$\frac{3}{4}o + \frac{l_f-2}{4} \ge \frac{3}{4}o + \frac{2o-3}{4} = o + \frac{o}{4} - \frac{3}{4} \ge o$$

Thus each star of the covering gets at least as many units as it has edges. ∎

Note that in the proof of Theorem 3, we just need that the boundary of each face consists of at least five edges (under the assumption of 2–edge–

connectivity):

**Corollary 1** *Let $G$ be a connected bridgeless plane graph whose each face is incident to at least five edges. Than $\chi_f(G) \geq \lceil n/2 \rceil + 1$.*

# 4 Conclusion

We have proven in Theorem 3 in Section 3 that a plane graph $G$ of girth at least five has a valid coloring such that at least half of the vertices have mutually distinct colors. The original conjecture from [13] (Conjecture 1) remains open. Is it true that each plane graph with girth at least four has a valid coloring using at least as many colors as it is a half of its vertices? The proof of the original conjecture of Ramamurthi and West might require an approach different from that used in this paper: The same technique as used when proving Theorem 3 gives in case of plane graphs of girth at least four only the bound $\lceil n/3 \rceil + 1$ which meets the bound derived in [13] from Grötzsch's theorem.


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