On Free Minor Closed Classes of Graphs Outside Planarity

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Abstract

It is shown that the class of graphs on projective plane is not $M-$class [defined in [3, 4, 5]] but the class of graphs on torus is $M-$class, i. e. the forbidden minors for free minor closed class of graphs on the torus can be found using simplified formula [without split part] in a similar way as in case of planarity, but in the case of the projective plane can't.

Most of definitions of the topological graph theory are the same as in [1].

Graph is defined as a pair of sets (V, E), where V is the set of vertices and E – the set of edges. For graph G V(G) is its vertex set and $E(G)$ is its edge set. We denote by $G-e$ graph obtained by deleting edge $e \in E(G)$ from G. Similarly, $G-v$ is graph obtained by deleting vertex $v \in V(G)$ from G.

G/e is graph obtained by *contracting* edge $e \in E(G)$ in G. Reverse operation to edge adding and its contraction is the vertex *split* operation $G \odot u$ that is not unique. Thus, if in G by adding and contracting $e \notin E(G)$ appears a new vertex $u \in V(G')$ then there exists such vertex split $G' \odot u$ that we receive back previous graph G.

H is *subgraph* of G (denoting it $H \subset G$) if there is such a graph H' isomorphic to H that $V(H') \subset V(G)$ and $E(H') \subset E(G)$.

H is a *minor* of G (denoting it $H \prec G$) if H can be obtained performing some edge contractions in a subgraph of G. It is easy to see that if $H \prec G$ then H can be obtained from G by vertex deletions, edge deletions and edge contractions.

A class of graphs A is called *minor closed* if for each graph H belonging to A and arbitrary graph G from $G \prec H$ follows that G is in A.

For a minor closed class A, $\mathbf{Forb}(A)$ is the minimal set of forbidden minors, i.e.

$$\mathbf{Forb}(A) = \lfloor \{G \mid G \notin A\} \rfloor.$$

Here we use a notion $\lfloor B \rfloor$ denoting set which contains only minimal minors of B:

$$\lfloor B \rfloor \triangleq \{G \mid H \in B \wedge H \prec G \Rightarrow H \cong G\}.$$

Analogously, $\lceil B \rceil$ contains only maximal graphs of B:

$$\lceil B \rceil \triangleq \{G \mid H \in B \wedge G \prec H \Rightarrow H \cong G\}.$$

Proposition 1. *For a minor closed class A if G doesn't belong to A there exists such $H \in \mathbf{Forb}(A)$ that $H \prec G$ and conversely.*

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*This research is supported by the grant 00.0041 of the Latvian Council of Science.

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Theorem 2. *(Robertson, Seymour): **Forb**(A) is finite for any minor closed A.*

$N_\circ(B)$ denotes a minor closed class with B as its set of forbidden minors, i.e.

$$N_\circ(B) \triangleq \{G \mid \forall H \in B : H \not\prec G\},$$

i. e. we may say, that $N_\circ(B)$ is a minor closed class *generated* by its forbidden minors in B and $\mathbf{Forb}(N_\circ(B)) = B$. For example, $N_\circ(K_5, K_{3,3})$ is the class of planar graphs, as it is asserted by Kuratowski theorem.

A planar graph is called *free-planar*, if after adding an arbitrary edge to it it remains planar. In [4] it is proved, that the class of free-planar graphs is equal to $N_\circ(K_5^-, K_{3,3}^-)$, and its characterization in terms of the permitted 3-connected components is given.

In [3] a generalization of the notion of free-planar graphs is suggested. We denote by $Free(A)$ the class of graphs that consists of all graphs which should belong to A after adding an arbitrary edge to them. It is easy to see, that if A is minor closed $Free(A)$ is minor closed too [3]. Because of this we use to say, that $Free(A)$ is *free-minor-closed-class* for a minor closed class A.

In [3] Kratochvíl proved the following theorem:

Theorem 3.

$$\boldsymbol{Forb}(Free(A)) = \lfloor \boldsymbol{Forb}(A)^- \cup \boldsymbol{Forb}(A)^{\odot} \rfloor,$$

where

$$B^- \triangleq \{G - e \mid G \in B, e \in E(G)\}$$

and

$$B^{\odot} \triangleq \{H \mid H \cong G \odot v, G \in B, v \in V(G)\}.$$

Further, we denote by $Free^k(A)$ repeatedly applied $Free$ k times, i.e.

$Free^0(A) = A$;

$Free^k(A) = Free(Free^{k-1}(A))$.

Let for a minor closed class A $Free^m(A)$ do not consist of only empty graphs but $Free^{m+1}(A)$ do, then we say that A is *of depth* m.

In the graph G a vertex split $G \odot v$ for $v \in V(G)$ is called *proper* if both new vertices arising in the result of the split are of the degree at least two. Otherwise the vertex split is called *non-proper*.

Theorem 6 in [5] can be simplified.

Theorem 4. *Let a class A be of depth m. If there holds*

$$\boldsymbol{Forb}(Free(A)) = \lfloor \boldsymbol{Forb}(A)^- \rfloor$$

then there holds also

$$\boldsymbol{Forb}(Free^k(A)) = \lfloor \boldsymbol{Forb}(Free^{k-1}(A))^- \rfloor$$

for $k = 1, ..., m$.

Proof. We must proof that $\mathbf{Forb}(Free(A)) = \lfloor \mathbf{Forb}(A)^- \rfloor \Rightarrow \mathbf{Forb}(Free(Free(A)) = \lfloor \mathbf{Forb}(Free(A))^- \rfloor = \lfloor \mathbf{Forb}(A)^= \rfloor$. Let $H \in \mathbf{Forb}(A)$. Then members of $\mathbf{Forb}(Free(A))$ are minors of $\{H\}^-$ and $\{H\}^{\odot}$ and for the latter part being the proper minors thus excluding them from the set $\mathbf{Forb}(Free(A))$ completely. Then always $H \odot u$ has some minor from $\{H\}^-$ and [because arbitrary $H - e \odot u$ is a minor of $H \odot u$ in any case] $H - e \odot u$ has a minor from $\{H\}^=$. □

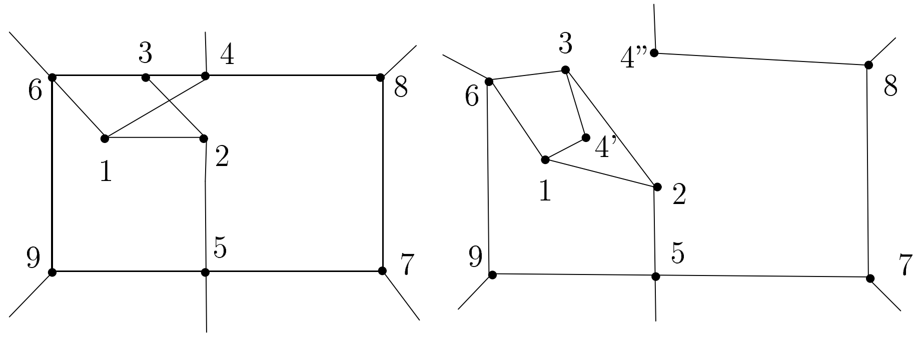

Figure 1: Forbidden minor for A_P F_1, p.340. [2]. Splitting vertex 4 we get planar graph [see fig. 2]. The only edge that can drive it out projective plane is $4'4$”.

The class A is called *M-class* [defined similarly as in [5]] if $\mathbf{Forb}(Free(A) = \lfloor \mathbf{Forb}(A)^- \rfloor$. Theorem 4 says that if A of depth m is $M-$class then classes $Free^k(A)$ are $M-$classes for all $k, 1 \leq k \leq m$. It is easy to see that the class of planar graphs is $M-$class. In [5] an example of $M-$class artificially is constructed.

The question is: if class of planar graphs is $M-$class, are graphs on other surfaces behaving similarly? Here we give answer to this question in the cases of the projective plane and the torus.

Theorem 5. *Let A_P be class of graphs that are embeddable on the projective plane. The class A_P is not M-class.*

Proof. Let us see graph $H = F_1 \in \mathbf{Forb}(A_P)$ [2, page 340][on fig. 1 left]. Splitting vertex 4 of H as in fig. 1 right we get graph H' which is planar [see fig. 2 without edge $4'8$]. The only possible edge added to H' that would drive it outside A_P would be $4'4$”. Truly, the only nontrivial case is depicted in fig. 2 with edge $4'8$ added to H'. Thus H' would give contribution to $\mathbf{Forb}(Free(A_P))$ and A_P is not $M-$class. Besides, H' belongs to $\mathbf{Forb}(Free(A_P))$. □

There are more such graphs from $\mathbf{Forb}(A_P)$ which split in such a way that it becomes planar giving contribution to $\mathbf{Forb}(Free(A_P))$. See e. g. graph B_7 from [2] [p.341] on fig. 3 [B_7 split in the way the thick line shows gives planar graph depicted right where only edge $1'1$” gives graph that does not belong to A_P]. Similarly behaves graph E_{22} see fig. 4. But not all forbidden minors behave in that way, for example, D_{17} can be made planar only splitting off an edge, see fig. 5. In all four examples we get samples of $\mathbf{Forb}(Free(A_P))$ [in last case D_{17} with edge off]. It is interesting that just the split contribution gives us planar forbidden minors of $Free(A_P)$, [except case D_{17}, where $D_{17} - 15$ is forbidden minor for $Free(A_P)$] .

Let us further discuss the class of graphs on the torus. It is easy to see that for every $H \in A_T$ arbitrary H^- is not planar. Is it right for vertex splits too? Does there exist $H \in A_T$ that for some vertex [v] split $H \odot v$ is planar and in the same time H belongs to forbidden minors of A_T? We are going to show that it is not true and that A_T is $M-$class.

Let A_T be class of graphs on the torus. Following three propositions are equivalent.

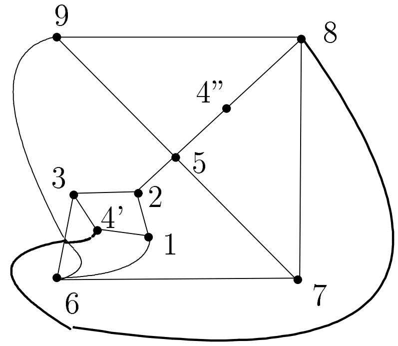

Figure 2: Illustration of the assertion in theorem 5 and fig. 1: F_1' is planar and $F_1' + 4'8$ is on A_P

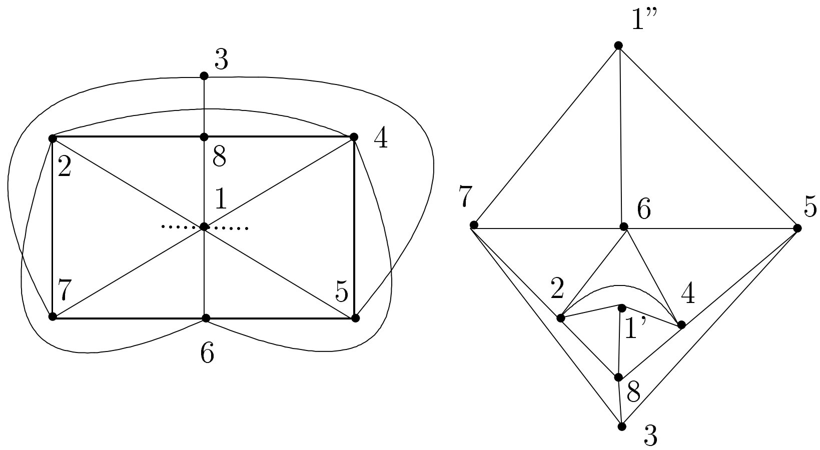

Figure 3: Forbidden minor for A_P B_7, p.341. [2]. Splitting vertex 1 we get planar graph. The only edge that can drive it out projective plane is $1'1$”.

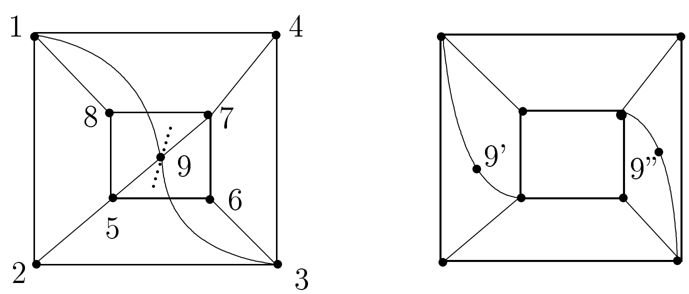

Figure 4: Forbidden minor for A_P E_{22}, p.339. [2]. Splitting vertex 9 we get planar graph. The only edge that can drive it out projective plane is $9'9$”.

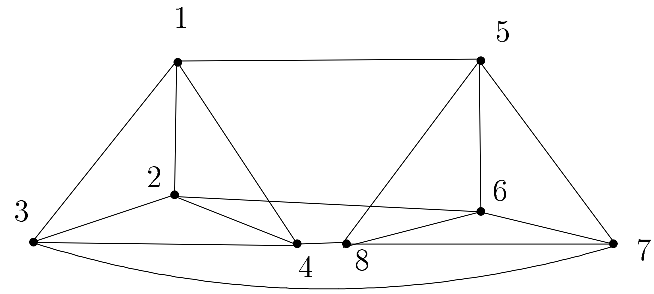

Figure 5: Forbidden minor for A_P D_{17}, p.344. [2]. The only split to make graph planar is to split off an end of an edge joining K_4, i.e. 15, 26, 37 or 48.

Proposition 6. *A_T is $M-$class.*

Proposition 7. *Let $G \notin A_T$ and for arbitrary split of $v \in V(G)$ G result in $G \odot v = G'$ with two new vertices v_1 and v_2 and $G' + v_1v_2 = G''$. Then either there is an edge $e \in E(G')$ that $G'' - e \notin A_T$ or G' can be augmented with an edge $e \notin E(G')$ differing from v_1v_2 such that $G' + e \notin A_T$.*

Proposition 8. *Let $G \notin A_T$ but for every edge $e \in E(G)$ $G - e \in A_T$ and edge f be such that $G/f \notin A_T$. Then there is at least one more edge h differing from f that $G/h \notin A_T$ or the edge f is an isthmus.*

It is easy to see that 7 follows from 6 immediately.

Assertion 8 is not trivial and the feature expressed by it may be used to build a counterexample to the assertion 6 or in case we prove that such counterexample does not exist assertion 6 is proved to be true.

Let us try to build counterexample to assertion 8 or come to the proof that it does not exist. Thus, we behave as if building the counterexample. Let us take a graph such as in 8 and split some chosen vertex into two new vertices s and t and name this graph the *counterexample (CE)* graph which is to disprove the assertion 8.

Let us say that the triple (G, s, t) defines a CE graph if:
1) $G \in A_T$ and $s, t \in V(G)$ and $st \notin E(G)$;
2) $G + st \notin A_T$ and $G + st/st \notin A_T$;
3) $\nexists e \notin E(G)$ distinct from st that $G + e \notin A_T$;
4) $\nexists e \in E(G)$ that $G + st/st - e \notin A_T$ and $G + st/st/e \notin A_T$.

If conditions 1) - 3) are true for some graph then we call such graph *weak* CE graph for CE graph [satisfying all conditions] saying that it is *strong* CE graph.

It is easy to see that if strong CE graph exists then it gives us immediately counterexample to assertion 6. Indeed, if $G \in \mathbf{Forb}(A_T)$ and for $v \in V(G)$ some split $G \odot v [= G']$ gives two new vertices v_1 and v_2 then (G, v_1, v_2) defines CE graph if and only if G' gives contribution in $\mathbf{Forb}(Free(A_T))$, i. e. there doesn't exist $H \in \mathbf{Forb}(A_T)$ that $H^- \prec G'$ for arbitrary H^-.

In place of proposition 8 we are going to prove what follows

Proposition 9. *Let e be edge of G that is not isthmus and $E(e)$ set of edges that are adjacent to e. Let $G \notin A_T$ but for every edge $f \in E(e)$ $G/f \in A_T$. Then $G/e \in A_T$ or G is a weak CE graph.*

We prove this assertion and the fact that strong CE graph does not exist in two theorems 10 and 15.

Theorem 10. *Let (G, s, t) define a [possibly weak] CE graph. Then either it does not have subgraph a subdivision of $K_{3,3}$ or it is a weak CE graph.*

Proof. Let us suppose that there exists in G some subgraph K that is a subdivision of $K_{3,3}$ and let the main vertices of K be numbered from 1 to 6. Let us suppose that K is chosen in the way that [as long as possible] no local non-trivial bridges are with respect to K.

1) First let us consider the case when there is a bridge Br with respect to K that has a subgraph that is subdivision of some Kuratowski graph.

a) Suppose Br has two attachment vertices to K, say 1 and 4, as depicted in figures 6 and 9.

(i) Let us first assume that at least on of them, say 4, is such that Br has only one leg, say $4b$, attached to it.

Let us number the main vertices of K [from 1 to 6] and one inner vertex of Br [b] as in figures fig. 6.

Let us consider different possible embeddings of $K + Br$ on torus. Flipping in three ways is possible due to changes in rotations of vertices in K. Firstly, around 14 up down we call *trivial flipping*. Secondly, flipping chains 2..5 and 3..6 from in to out with respect to the cycle 1..2..3..4..5..6..1 we call *non-trivial*. Thirdly, flipping chains 2..5 and 1..4 [together with the bridge Br] from in to out with respect to the cycle 1..2..3..4..5..6..1 we also call non-trivial. Let us divide all possible embedding of $K + Br$ into four subsets each of which have only these embedding where non-trivial flipping are not performed but from subset to subset we go over by non-trivial flipping in $K + Br$. Let us call these subsets *states*. Thus we have four states which change one into another by non-trivial flipping. Let us keep in the mind that the trivial flipping give us additional subdivision of each state into two substates.

Let A, B, C and D be four facial walks in $K + Br$ that contain only basic pieces from K [as in figures 6 and 7]. If a facial walk is not closed then we add in the denotation two points, e.g. for non closed facial walk A we write $A:$. Now comes the main observation of this proof. It is easy to see that the flipping in four possible states turn facial walks in facial walks, only changing closed facial walk in non closed and vice versa [see figure 6]. More over, this same is true for substates too except that facial walk may be affected more [see how in fig. 6] than simply changing its direction.

Let S be subset of $V(G)$ and $F[S]$ set of facial walks on S. Let K' be K together with the edge $1b$ and the chains $b..b$ and $1..1$ [enclosing the bridge Br]. Let F be $F[V(K)]$ and F' be $F[V(K')]$.

Let us define the visibility graph $W[U]$ where $U = S \cup F[S]$ and $S \subset V(G)$ and whose edges are these pairs of elements which see each other in some fixed state, where vertex see facial walk if it goes into it and two facial walks see each other if there is some other facial walk that comprise both. The visibility graph defined in this way is non-determinated for a fixed state [in different embeddings of the same state it may be different] but nevertheless it is well defined.

Lemma 11. *Let U be $V(K) \cup F[K]$ and $U' = V(K') \cup F[K']$. Then graph $W[U]$ does not change by flipping between states. The same is true for $W[U']$ if flipping is from states $\{1, 2\}$ to states $\{3, 4\}$.*

Proof. It is easy to see that the first assertion is true. Truly, changes between states by non-trivial flipping does not change facial walks what concerns their content as sets of vertices. But then visibility between vertices and vertices and facial walks and facial walks between themselves is kept also.

In the second case [with K'] the visibility graph changes only from state 1 to 2 and 3 to 4 where edges with 1..1 and $b..b$ are changing [see figures 6 and 7]. □

Now, constructing CE graph we should decide where to put vertices s and t. We prove a lemma.

Lemma 12. *Let chain p be non-local bridge with respect to K'. Then $W[K' + p]$ does not change by flipping [if possible] between states.*

Proof. The ends of p see each other in all states of flipping in $W[K' + p]$. Adding p to K_{Br} some fixed face is divided into two new ones in all states in the same way, because by flippings [if not obstructed by the new chain] facial walks change only their orientation, not their order otherwise, thus adding p to K' the flipping between states would not change the visibility graph otherwise as in lemma 11. □

The following technical lemma which follows immediately would be useful.

Lemma 13. *Vertex on 1..1 sees either A or D in different states but does not see B and C in any state. Vertex on $b..b$ sees either B or C in different states but does not see A and D in any state.*

Let us continue with considerations building CE graph.

Let f be some facial walk in K'. Then we say that a vertex $v \in V(G)$ but $v \notin V(K')$ is *f-facial* vertex if there is a bridge B_v [with respect to K'] with $v \in V(B_v)$ but not its attach vertex that B_v may be embedded in the face with f in its border. From lemma 12 follows that the visibility graph does not change when we add bridge B_v to K'.

We say that the vertex v *belongs* to $U = V(K) \cup F[K]$ if it either belongs to $V(K)$ or it is a $f-$facial vertex for some facial walk $f \in F[K]$. We say that v is *border* vertex if it is on chains 1..1 or $b..b$. We say that v is *outside* vertex if it is in Br otherwise.

Further we consider cases when both s and t are not in local bridges with respect to $K + Br$.

Let both s and t belong to $K \cup F[K]$. Then at least two of their neighbors from $V(G)$ should be on the border [in order to be seen and unseen in different states]. Say, s belongs to A, then t should belong to B, C or D. But then one neighbor of s, say s_1, should be on 1..1 and one, say s_2, on $b..b$ but then, if t be on B or C then it would not see s_1, i. e. the neighbor of s on 1..1, and if t be on D then it would no see s_2, i. e. the neighbor of s on $b..b$, and for the neighbors of t on the border similarly. Possible flipping in the bridge Br can not repair the contradiction.

Let s belong to $K \cup F[K]$ and t to border, say, $s \in A$ or $s \in D$ and $t \in b..b$. Then at least two neighbors of s should be on the border [because flip is needed], but two neighbors of t should be on the border too [because they can not be in $K \cup F[K]$ otherwise s would see them or not see at all without flipping]. But then t would either see all neighbors of s

or not see them at all [from lemma 13]. Possible flipping in the bridge Br can not repair the contradiction.

Let s belong to $K \cup F[K]$ and t be outside. But then neighbors of both must be on the border and previous situation repeats.

Let neither of s or t belong to $K \cup F[K]$. This case we address to other cases taking the subdivision of Kuratowski graph of Br in place of K.

Let us consider the case when trivial local bridges with respect to K are possible.

If both s and t belong to $K \cup F[K]$ then weak CE graph is possible [fig. 8]. But then some edge in $G' = G + st/st$ is contractible that it remains non-toroidal, because that subdivision of K_5 appears in G' where one edge of the rim through six vertices of the wheel W_4 around the new vertex is contractible.

If a weak CE graph would arise in the same way in some case with K with some more bridges then similarly some edge which could be contractible without loosing the non-toroidal feature. The following lemma says this.

Lemma 14. *Let CE graph (H, s, t) arise. Then it is only a weak CE graph.*

Proof. Let chains p_s and p_t contain correspondingly s and t and their ends are on $K + Br$. Then there exists in $K + Br$ a cycle C such that chains p_s and p_t are bridges against C and C with these chains comprise subdivision of K_4 with chains being its non-adjacent edges [because s and t do not see each other C is of length more than four and ends of chains all are distinct]. But then there is some edge h on C that $G + st/h$ remains non-toroidal. Thus, CE graph (H, s, t) is not strong. □

(ii) Let us suppose that the block Br both its legs are main vertices of Kuratowski graph in it. Then we have the situation which is showed in figure 9. But here an additional flipping is possible after which each basic piece in K' sees each other what excludes possibility to build CE graph. [Besides, in this case each facial walk on the left is not any more facial walk on the right and visibility graph can not be defined.] But if with some bridge configurations this flipping is obstructed then we return to the previous case which did not give us a strong CE graph. Contradiction.

b) Let Br has more than two attach vertices. Then either flipping as before is possible and the same considerations as before are right or some flipping is prohibited and in visibility graph edges may only be lost not augmented thus all considerations are the same as before. Contradiction.

2) Let there is not a single bridge with respect to K with Kuratowski graph in it. Previous considerations similar as in case 1) b) are right in this case too.

3) Let Br has only of vertex of attachment with respect to K. This case is by author considered, but here omitted. The case when K and Br are not connected is trivial.

All cases are considered. Thus CE graph does not have $K_{3,3}$. Contradiction. □

Theorem 15. *Let (G, s, t) define a CE graph. Then it does not have subgraph subdivision of K_5.*

Proof. All considerations as in the previous theorem are the same there.

Case corresponding to the case 1) a) (ii) is demonstrated in fig. 10. Contradiction. □

Maybe all the prove of the theorem 9 may be made much shorter all the impact putting on the lemma 14 without considering many cases as it is here done.

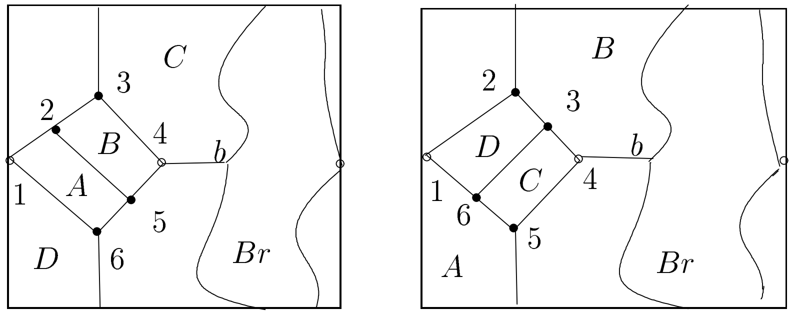

Figure 6: State 1 [with edges $(D, 1..1)$ and $(C, b..b)$] and state 2 [with edges $(A, 1..1)$ and $(B, b..b)$].

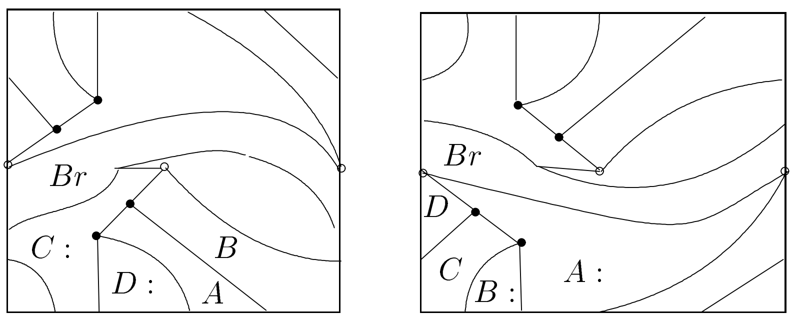

Figure 7: State 3 [with the same visibility graph as the state 1] and state 4 [with the same visibility graph as the state 2].

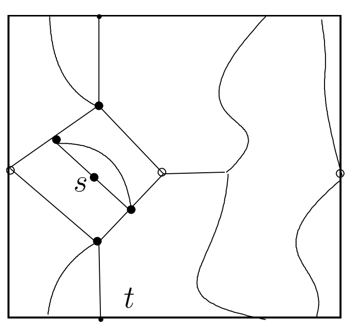

Figure 8: A weak CE graph

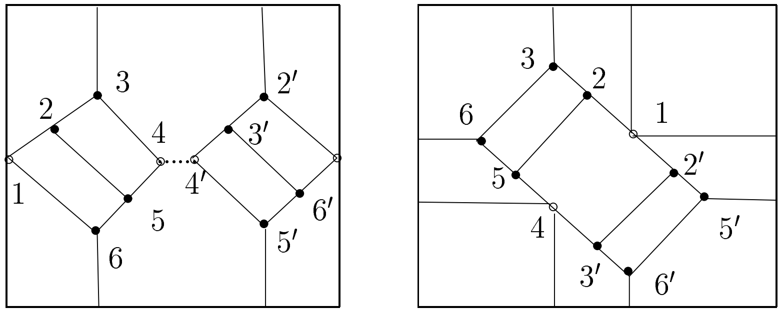

Figure 9: Only when $4 - 4'$ is contracted the flipping to the state right is possible

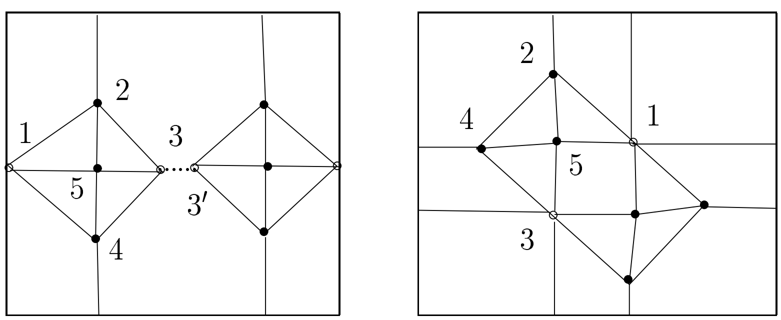

Figure 10: Only when $3 - 3'$ is contracted the flipping to the state right is possible

Acknowledgment

I would like to thank Jan Kratochvil and Paul Kikusts for invaluable help to prepare this note and Jaroslav Neshetril for giving me possibility in March of 2001 to visit Prague where part of this work was done.

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