LOCAL DIRECTEDNESS AND SOME CATEGORIES OF DOMAINS

ALEŠ PULTR AND ANNA TOZZI

Abstract. Information systems and approximable maps introduced in D. Scott's pioneering article [12] are well known to constitute a category equivalent to that of the ABC-domains (algebraic domains with joins of bounded subsets). We show that a slight generalization of information systems allows for similar representations of other categories of theoretical computer science (continuous lattices, continuous domains, algebraic lattices, and others). It turns out that the crucial role is being played by transitive relations $\lhd$ in which $a_i \lhd b$, $i = 1, 2$, can be simultaneously interpolated to $a_i \lhd a \lhd b$.

In [12], D.Scott introduced the notions of information systems and approximable maps. This was used as an approach to one of the categories of domains (algebraic domains with suprema of bounded sets). Usually, due to the usefulness of the domains, the interest is focused on this aspect of the theory. It should not be forgotten, however, that the information systems and approximable maps constitute an interesting structure in its own right. There are the logical connotations (we have here a fragment of conjunctive logic; for this, and some extensions, see [1],[3],[4],[8],[9]). Another aspect is that it is a model of localizing the solution of a task by approximation. This can be illustrated by the following example. In a space (think, e.g., of the plane) approximate, or localize, the points by open sets (the smaller the set is, the better the approximation); a finite system $U_1, \dots, U_n$ of such approximations is consistent if $\bigcap U_i \neq \emptyset$ (that is, if all the U_i have a common refinement, or, in other words, if they can approximate the same point), and the entailment $\{U_1, \dots, U_n\} \vdash U$ is given by $\bigcap U_i \subseteq U$ ("if we have approximated a point by the U_i we know that it is approximated by U").

The first author would like to express his thanks for the support by the Italian C.N.R. and by the project LN 00A056 of the Ministry of Education of the Czech Republic.

Then the relations f that are called approximable maps are indeed approximations of maps: because of the condition $\mathcal{U} f \mathcal{V}_i \Rightarrow \mathcal{U} f(\mathcal{V}_1 \cup \mathcal{V}_2)$, distinct values in one argument have to have a common refinement, that is, have to be able to approximate the same value).

In this paper we analyze the translation between information systems and the domains of [12] (also see [2]), and show that a natural generalization yields representations of various other important categories of theoretical computer science (continuous domains, continuous lattices, algebraic lattices, etc.) in similar vein. The central point is the "local directedness" of the generalized entailment, that is, the fact that if $a \vdash b_i$ for $i = 1, 2$ there is a b such that $a \vdash b$ and $b \vdash b_i$ for both i; it turns out that the desired representations are obtained by analyzing the structure of ideals in sets with transitive relations satisfying this condition.

The paper is divided into four sections. The first section contains preliminaries and in the second one a natural extension of the information systems is discussed. Section 3 is devoted to the locally directed sets, and in the last Section 4, representations of various categories of domains and lattices are presented.

1. Preliminaries

1.1. A (binary) relation R is said to be *interpolative* (resp. *strongly interpolative*) if for any a, b such that aRb there is a c such that $aRcRb$ (resp. $aRcRcRb$). Thus, each reflexive relation is strongly interpolative.

1.2. If a directed D in a partially ordered $(P, \leq)$ has a supremum, we speak of a *directed supremum* and write

$$\bigvee^{\uparrow} D \quad \text{(or, } \bigcup^{\uparrow} D \text{ if it is a union of sets).}$$

Thus, using the symbol $\bigvee^{\uparrow} D$ (or $\bigcup^{\uparrow} D$) we are stating (or making the assumption) that the set D is directed.

The term *directed* will be also used in connection with transitive relations that are not necessarily orders. Thus, they will be non-void D's such that

$$a, b \in D \quad \Rightarrow \quad \exists c \in D,\ a, b \lhd c.$$

In particular, in a directed $D \subseteq (X, \lhd)$,

for every $a \in D$ there is a $b \in D$ such that $a \lhd b$.

Note. The importance of the interpolativity and of the directedness in the context of not necessarily reflexive transitive relations has been recognized in the work of M. Erné. For instance see his representation of special topologies in [6].

1.3. Let $(P, \leq)$ be a partially ordered set. An element a is *way below* b in P, written $a \ll b$, if for any directed $D \subseteq P$, $b \leq \bigvee^{\uparrow} D$ implies that there is a $d \in D$ such that $a \leq d$ (see, e.g., [7]).

Obviously,

- $a \leq a' \ll b' \leq b \quad \Rightarrow \quad a \ll b$, and
- $a_1, a_2 \ll b \quad \Rightarrow \quad a_1 \vee a_2$, *whenever* $a_1 \vee a_2$ *exists.*

An element $a \in P$ is said to be *compact* if $a \ll a$. The set of all compact elements of P will be denoted by $\mathsf{K}(P)$.

1.4. A poset X is said to be *continuous* (resp. *algebraic*) if

$$\forall a \in X, \quad a = \bigvee^{\uparrow}\{b \mid b \ll a\} \quad (\text{resp.} \quad a = \bigvee^{\uparrow}\{b \mid b \ll b \leq a\}).$$

A *continuous* resp. *algebraic domain* is a continuous resp. algebraic poset in which every directed subset has a supremum. Moreover, throughout the paper,

we will assume it to possess a least element 0.

It is a standard fact that

- *in a continuous domain, if* $a_1, a_2 \ll b$ *then there is an* a *such that* $a_1, a_2 \ll a \ll b$

so that, in particular,

- *in a continuous domain,* $\ll$ *is interpolative.*

The category of continuous domains with the maps that preserve the suprema of directed sets will be denoted by

CDom.

1.5. Let X be a set. The set of all subsets (resp. all finite subsets) of X will be denoted by $\mathfrak{P}(X)$ (resp. $\mathfrak{P}_{\text{fin}}(X)$). The set $\mathfrak{P}(X)$, ordered by inclusion, is obviously an algebraic lattice, and $\mathsf{K}(\mathfrak{P}(X)) = \mathfrak{P}_{\text{fin}}(X)$.

1.6. From category theory only standard notions (morphism, functor, full embedding, equivalence of categories) are assumed. The reader may consult [10].

2. Generalized Scott information systems

2.1. Recall that a *Scott information system* (see [12]; briefly, *S.i.system*) $S=(X_S, C_S, \dashv_S)$ consists of a set X_S, a subset $C_S \subseteq \mathfrak{P}_{\text{fin}}(X_S)$ $(=\mathsf{K}(\mathfrak{P}(X_S)))$ and a relation $\dashv_S \subseteq C_S \times C_S$ such that

- (S1) $a \subseteq b \dashv c \subseteq d \;\Rightarrow\; a \dashv d$,
- (S2) $\emptyset \dashv a$ for all a, and $a_1, a_2 \dashv b \;\Rightarrow\; (a_1 \cup a_2) \dashv b$,
- (S3) $\dashv$ is transitive,
- (S4) $\dashv$ is reflexive, and
- (S5) for all $x \in X$, $\{x\} \in C$.

(We use the convention that $a \dashv b$ automatically indicates that $a, b \in C$; thus, (S1) includes the assumption that $a \subseteq b \dashv c$ implies that $a \in C$, and (S2) also states that if $a_i \dashv b$ then $a_1 \cup a_2 \in C$.)

An *approximable map* $f : S \to T$ between S.i.systems is a relation $f \subseteq C_T \times C_S$ such that

- (A1) $\emptyset f \emptyset$,
- (A2) if $a_i f b$, $i = 1, 2$, then $(a_1 \cup a_2) f b$, and
- (A3) if $a \dashv a' f b' \dashv b$ then afb.

They are composed as relations and the resulting category is denoted by

SInf.

Note that the relations $\dashv_S$ are the units $\dashv_S : S \to S$ in this category.

Remark. The notation we use is reversed as compared with [12] to have $\subseteq$ (not $\supseteq$) as the background order. Thus, "a entails b" is written "$b \dashv a$" instead of "$a \vdash b$".

2.2. It is well known (see e.g. [12],[2]) that the category **SInf** is equivalent with the category of Scott domains (algebraic domains such that each bounded couple has a supremum). To represent other important categories of theoretical computer science in a similar manner (for instance, cf. the representation of continuous lattices in [5] by "inductive systems") let us introduce the following notion.

A *generalized information system* (briefly, *g.i.system*) $S = (P_S, C_S, \dashv_S)$ consists of an algebraic lattice P_S, a subset $C_S \subseteq \mathsf{K}(P_S)$ and a relation $\dashv_S \subseteq C_S \times C_S$ such that

- (G1) $a \leq b \dashv c \leq d \;\Rightarrow\; a \dashv d$,
- (G2) $0 \dashv a$ for all a, and $a_1, a_2 \dashv b \;\Rightarrow\; (a_1 \vee a_2) \dashv b$, and
- (G3) $\dashv$ is transitive and interpolative.

An *approximable map* $f : S \to T$ between g.i.systems is a relation $f \subseteq C_T \times C_S$ such that

(A'1) $0f0$,
(A'2) if $a_i fb$, $i = 1, 2$, then there is an a such that $a_i \dashv a$ and afb,
(A'3) if $a \dashv a'fb$ or $afb' \dashv b$ then afb; if afb then there are a', b' such that $a \dashv a'fb' \dashv b$.

Approximable maps will be composed as relations and the resulting category will be denoted by

$$\mathbf{GInf}.$$

2.3. Observation. *Let S, T be Scott information systems. Then the approximable maps in the sense of 2.2 coincide with those of 2.1. Consequently,* **SInf** *is a full subcategory of* **GInf**.

Proof. If the relations are reflexive then obviously (A'3)≡(A3).

If (A'2) holds and $a_1, a_2 fb$ consider an a such that $a_i \dashv afb$. Then by (S2) $(a_1 \cup a_2) \dashv afb$ and by (A'3) $(a_1 \cup a_2)fb$.

If (A2) holds and $a_1, a_2 fb$ then by (S4) $a_i \leq (a_1 \cup a_2) \dashv (a_1 \cup a_2)$ and $a_i \dashv (a_1 \cup a_2)$. □

2.4. Two special conditions. (1) Scott information systems are the g.i.systems such that $P = \mathfrak{P}(X)$ for some set X, and that we have (S4) and (S5). We will see shortly that, up to isomorphism, this can be reduced to strong inerpolativity (which we will indicate by (SI)).

(2) Another special condition of interest is the *fullness*

(F) $C = \mathsf{K}(P)$ and for each $a \in \mathsf{K}(P)$ there is a b such that $a \dashv b$.

It will play a role in representing some categories of lattices.

2.5. Proposition. *A g.i.system satisfying (SI) is isomorphic to a Scott information system.*

Proof. Set $X = \{a \in P_S \mid a \dashv a\}$ and

$$\overline{S} = (\mathfrak{P}(X), \overline{C}, \dashv\!\dashv)$$

where $A \in \overline{C}$ iff $\bigvee A \in C_S$, and $A \dashv\!\dashv B$ iff $\bigvee A \dashv \bigvee B$. Define $\rho : S \to \overline{S}$ and $\sigma : \overline{S} \to S$ by setting

$$A\rho b \text{ iff } \bigvee A \dashv b, \quad \text{and} \quad a\sigma B \text{ iff } a \dashv \bigvee B.$$

Checking that ρ and σ are approximable maps is straightforward. Now if $A(\rho \circ \sigma)B$ we have an a such that $\bigvee A \dashv a \dashv \bigvee B$ and $A \dashv\!\dashv B$. If $A \dashv\!\dashv B$ interpolate $\bigvee A \dashv a \dashv \bigvee B$ and we have $A\rho a\sigma B$. Thus, $\rho \circ \sigma = \dashv\!\dashv$. Finally, if

$a\sigma B\rho b$ we have $a \dashv \bigvee B \dashv b$ and hence $a \dashv b$, and if $a \dashv b$ we can interpolate $a \dashv c \dashv c \dashv b$ to obtain $a\sigma\{c\}\rho b$. □

3. Locally directed sets

3.1. A *locally directed set* (briefly, *LD-set*) $(X, \lhd)$ is a set with a transitive binary relation such that

(J) each ${\lhd}a = \{x \mid x \lhd a\}$ is directed

(in other words, if $x_1, x_2 \lhd a$ then there is an x such that $x_i \lhd x \lhd a$). For technical reasons, we will, moreover assume that

there is a distinguished element 0, least in $\lhd$.

Note that in particular (J) implies that $\lhd$ is interpolative.

3.2. An *ideal* in an LD-set $(X, \lhd)$ is a subset $J \subseteq X$ such that

(I1) $a \in J$ and $b \lhd a$ imply $b \in J$, and
(I2) J is directed (in $\lhd$).

(recall 1.2: by (I2), in particular, for every $a \in J$ there is a $b \in J$ such that $a \lhd b$).

Define

$$\mathcal{D}(X, \lhd) = \{J \subseteq X \mid J \text{ an ideal}\}.$$

3.2.1. Lemma. *Each* ${\lhd}a$ *is an ideal, and if* $a \lhd b$ *then* ${\lhd}a \subseteq {\lhd}b$.

Proof is immediate. □

3.2.2. Lemma. $\mathcal{D}(X, \lhd)$ *is closed under directed unions.*

Proof. (I1) is preserved in any union. Now if $a, b \in \bigcup\!\!\!\uparrow J_i$ there is a j such that $a, b \in J_j$ and (I2) follows. □

3.2.3. Lemma. *For each* $J \in \mathcal{D}(X, \lhd)$,

$$J = \bigcup\!\!\!\uparrow\{{\lhd}a \mid a \in J\}.$$

Proof. The union is directed: if $a, b \in J$ choose $c \in J$ such that $a, b \lhd c$ by (I2); then ${\lhd}a, {\lhd}b \subseteq {\lhd}c$. As for the equality: if $a \in J$, $a \in {\lhd}b$ for some $b \in J$ by (I2), and the other inclusion follows from (I1). □

3.2.4. Lemma. $J \ll K$ *in* $\mathcal{D}(X, \lhd)$ *iff there is an* $a \in K$ *such that* $J \subseteq {\lhd}a$.

Proof. $\Rightarrow$: $J \ll \bigcup^{\uparrow}\{\lhd a \mid a \in K\}$ and hence there is an $a \in K$ such that $J \subseteq \lhd a$.

$\Leftarrow$: If $J \subseteq \lhd a$ and $a \in K$, and if $K \subseteq \bigvee^{\uparrow} J_i$ then $a \in J_j$ for some j and then $J \subseteq \lhd a \subseteq J_j$. $\square$

3.2.5. Corollary. *$\mathcal{D}(X, \lhd)$ is a continuous domain with zero ($=\lhd 0$).*

3.3. For a continuous domain D with zero set

$$\mathcal{J}(D) = (\{a \in D \mid \exists b \in D,\ a \ll b\}, \ll).$$

By 1.4,

$$\mathcal{J}(D) \textit{ is an LD-set.}$$

3.3.1. Proposition. *$\alpha : D \to \mathcal{D}\mathcal{J}(D)$ and $\beta : \mathcal{D}\mathcal{J}(D) \to D$ defined by $\alpha(x) = \{a \mid a \ll x\}$ and $\beta(J) = \bigvee^{\uparrow} D$ are mutually inverse isomorphisms in* **CDom**.

Proof. The definitions of α and β are correct: as for α see 1.4; as for β, a J directed in $\ll$ is also directed in $\leq$.

α and β are obviously monotone. Hence if we prove they are mutually inverse, we will see that they preserve all existing suprema.

Trivially $\beta\alpha(x) = x$. Now let J be an ideal in $\mathcal{J}(D)$. If $a \in J$ choose a b with $a \ll b \in J$. Then $a \ll b \leq \bigvee^{\uparrow} J$ anf $a \in \alpha\beta(J)$. On the other hand, if $a \ll \bigvee^{\uparrow} J$ then there is a $b \in J$ such that $a \leq b$. Choose a $c \in J$ with $b \ll c$. Then $a \ll c \in J$ and $a \in J$. $\square$

3.4. A *LD-map* $f : (X, \lhd) \to (Y, \lhd)$ is a relation

$$f \subseteq Y \times X$$

such that

(M1) $0f0$,
(M2) $a_1, a_2 fb \quad \Rightarrow \quad \exists a,\ a_i \lhd afb$,
(M3) $a \lhd a'fb$ or $afb' \lhd b \quad \Rightarrow \quad afb$, and if afb there are a', b' such that $a \lhd a'fb' \lhd b$.

Checking that for LD-maps

$$(X, \lhd) \xrightarrow{\ f\ } (Y, \lhd) \xrightarrow{\ g\ } (Z, \lhd)$$

the composition $g \circ f$ is an LD-map, and that the relations $\lhd$ are LD-maps and that $\lhd \circ f = f$ and $f \circ \lhd = f$ is straightforward. The resulting category will be denoted by

LDir.

3.4.1. Up to isomorphism, in an LD-set $(X, \lhd)$ only the elements x for which there is a y with $x \lhd y$ play a role. Set

$$X^{\lhd} = \{x \in X \mid \exists y,\ x \lhd y\}.$$

We have

Lemma. *$(X^{\lhd}, \lhd)$ is an LD-set, and it is isomorphic with $(X, \lhd)$.*

Proof. The first statement is trivial, and the second one follows from a straightforward checking that $f \subseteq X^{\lhd} \times X,\ g \subseteq X \times X^{\lhd}$ defined as the restrictions of $\lhd$ are mutually inverse isomorphisms. $\square$

3.5. For an LD-map $f : (X, \lhd) \to (Y, \lhd)$ define

$$\mathcal{D}f : \mathcal{D}(X, \lhd) \to \mathcal{D}(Y, \lhd)$$

by setting

$$\mathcal{D}f(J) = \bigcup\!\!\uparrow \{\lhd a \mid \exists b \in J,\ afb\}.$$

(The union is really directed: if $b_1, b_2 \in$ and $a_i f b_i$ take a $b \in J$ such that $b_i \lhd b$; then $a_i f b$ and hence there is an a such that $a_i \lhd afb$ and consequently $\lhd a_1, \lhd a_2 \subseteq \lhd a$.)

3.5.1. Lemma. *$a \in \mathcal{D}f(J)$ iff there is a $b \in J$ such that afb.*

Proof. $\Rightarrow$: If $a \lhd a' f b \in J$ for some $b \in J$ we have afb.
$\Leftarrow$: If afb interpolate $a \lhd a' f b$. $\square$

3.5.2. Proposition. *$\mathcal{D}$ is a functor* **LDir** $\to$ **CDom**.

Proof. This is easily checked using 3.5.1. For instance, $a \in \mathcal{D}f(\bigcup\!\!\uparrow J_i)$ iff there is a $b \in \bigcup\!\!\uparrow J_i$ such that afb iff $a \in \bigcup \mathcal{D}f(J_i)$, and the union is directed since $\mathcal{D}f$ is obviously monotone. $\square$

3.6. Theorem. *$\mathcal{D}$ is an equivalence of the categories* **LDir** *and* **CDom**.

Proof. By Proposition 3.3 it suffices to prove that $\mathcal{D}$ is a full embedding. For an $h : \mathcal{D}(X, \lhd) \to \mathcal{D}(Y, \lhd)$ define

$$\widetilde{h} : (X, \lhd) \to (Y, \lhd)$$

by setting

$$a\widetilde{h}b \quad \text{iff} \quad a \in h(\lhd b)$$

$(0 \in h(\lhd 0)$; if $a_1, a_2\widetilde{h}b$ we have $a_i \in h(\lhd b)$ and since $h(\lhd b)$ is an ideal, there is an a such that $a_i \lhd a \in h(\lhd b)$, that is, $a\widetilde{h}b$; if $a \lhd a'\widetilde{h}b$ then $a \lhd a' \in h(\lhd b)$ and $a \in h(\lhd b)$; if $a\widetilde{h}b' \lhd b$ then $a \in h(\lhd b') \subseteq h(\lhd b)$; if $a\widetilde{h}b$ then $a \in h(\lhd b) = h(\bigcup\!\!\uparrow \{\lhd b' \mid b' \lhd b\})$ by 2.3 and if we interpolate

$a \lhd a' \in h(\lhd b)$ we see that there is a b' such that $a' \in h(\lhd b')$ so that $a \lhd a'\widetilde{h}b' \lhd b$).

Now we have, by 5.1,

$$a \in \widetilde{\mathcal{D}f}(J) \quad \text{iff} \quad a \in \mathcal{D}f(\lhd b) \quad \text{iff} \quad \exists b' \lhd b,\ afb' \quad \text{iff} \quad afb,$$

hence $\widetilde{\mathcal{D}f} = f$, and

$$\begin{aligned} a \in \mathcal{D}\widetilde{h}(J) \quad &\text{iff} \quad \exists b \in J,\ a\widetilde{h}b \quad \text{iff} \quad \exists b \in J,\ a \in h(\lhd b) \quad \text{iff} \\ &\text{iff} \quad a \in \bigcup\!\!{\uparrow}\{h(\lhd b) \mid b \in J\} = h(\bigcup\!\!{\uparrow}(\{\lhd b \mid b \in J\}) = h(J) \end{aligned}$$

so that $\mathcal{D}\widetilde{h}(J) = h(J)$. □

4. Representation of some categories of domains

4.1. Some categories of continuous domains. We will use the following notation for some full subcategories of the category **CDom**.

ADom is the category of algebraic domains, and **CDom**$(\top)$ resp. **ADom**$(\top)$ are the subcategories of **CDom** resp. **ADom** generated by the objects with largest elements.

BCDom resp. **BADom** is the subcategory of **CDom** resp **ADom** generated by the domains in which any bounded couple of elements has a supremum.

CLat resp. **ALat** is the category of continuous resp. algebraic lattices ([11],[7]).

4.2. Some conditions in LD-sets.

4.2.1. An LD-set is said to admit *conditional joins* if there is an associative partial operation $x \sqcup y$ defined whenever there is a z such that $x, y \lhd z$, such that

$$x_1 \sqcup x_2 \lhd x \quad \text{iff} \quad x_i \lhd x \text{ for both } i = 1, 2.$$

Setting $\bigsqcup \emptyset = o$ and $\bigsqcup\{x\} = x$ we extend this operation by associativity to

$$\bigsqcup : \{A \in \mathfrak{P}_{\text{fin}}(X) \mid \exists y,\ \forall x \in A, x \lhd y\} \to X$$

4.2.2. $(X, \lhd)$ is said to be *strongly interpolative* resp. *reflexive* if such is the relation $\lhd$.

4.2.3. $(X, \lhd)$ is said to be *directed* if $X^{\lhd}$ (recall 3.4.1) is directed.

4.2.4. Full subcategories of **LDir** determined by some of the conditions above will be indicated by symbols $\sqcup$ for conditional joins, SI for strong interpolativity, R for reflectivity and D for directedness. Thus, for instance, **LDir**($\sqcup$,D) is the category of directed LD-sets admitting conditional joins, **LDir**(SI) is the category of the strongly interpolative LD-sets.

Similar notation will be used for generalized information systems (here, furthermore, F indicates the fullness – recall 2.4).

4.3.1. Lemma. *Let $(X,\lhd)$ admit conditional joins. Then it is isomorphic to the $(C_S,\dashv_S)$-part of a g.i.system S. We can have $P_S=\mathfrak{P}(X^{\lhd})$ and if $X^{\lhd}$ is directed the system S can be made full.*

Proof. Put $\{x_1,\dots,x_n\}\in C$ iff there is an x such that $x_i\lhd x$ for all i and set $\{x_1,\dots x_n\}\dashv\{y_1,\dots y_n\}$ iff $\bigsqcup x_i\lhd\bigsqcup y_j$. Obviously we have (G1) and $\emptyset\dashv\{x_1,\dots x_n\}$. If $\{x^1_1,\dots x^1_{n_1}\},\{x^2_1,\dots x^2_{n_2}\}\dashv\{y_1,\dots y_m\}$ we have $\bigsqcup x^j_i=\bigsqcup x^1_i\sqcup\bigsqcup x^2_i\lhd\bigsqcup y_j$ and hence $\{x^1_1,\dots x^1_{n_1}\}\cup\{x^2_1,\dots x^2_{n_2}\}\dashv\{y_1,\dots y_m\}$. The transitivity is obvious, and if $\bigsqcup x_i\lhd\bigsqcup y_i$ interpolate $\bigsqcup x_i\lhd x\lhd\bigsqcup y_i$ to obtain

$$\{x_1,\dots x_n\}\dashv\{z\}\dashv\{y_1,\dots y_m\}.$$

Now define $f;(X,\lhd)\to(C,\dashv)$, $g:(C\dashv)\to(X,\lhd)$ by setting

$$\{x_1,\dots x_n\}fx\quad\text{iff}\quad\bigsqcup x_i\lhd x,$$

$$xg\{x_1,\dots x_n\}\quad\text{iff}\quad x\lhd\bigsqcup x_i.$$

Checking that f and g are LD-maps is straightforward.

If $\{x_1,\dots x_n\}fxg\{y_1,\dots,y_m\}$ then $\{x_1,\dots x_n\}\dashv\{y_1,\dots,y_m\}$, and if the latter holds interpolate $\bigsqcup x_i\lhd x\lhd\bigsqcup y_j$ to obtain $\{x_1,\dots x_n\}fxg$ $\{y_1,\dots,y_m\}$. If $xg\{x_1,\dots x_n\}fy$ we have $x\lhd y$ and if the latter holds then interpolating $x\lhd z\lhd y$ we obtain $xg\{z\}fy$. $\square$

4.3.2. Proposition. *Let $(X,\lhd)$ be an LD-set. Then the following statement are equivalent.*

1. *$(X,\lhd)$ is isomorphic to an LD-set admitting conditional joins.*
2. *$(X,\lhd)$ is isomorphic to the $(C,\dashv)$ part of a g.i.system.*
3. *$\mathcal{D}(X,\lhd)$ is a BC-domain.*

Proof. (1)⇒(2) is in 4.3.1.

(2)⇒(3): Let $(P,C,\dashv)$ be a g.i.system, and let $\leq$ be the partial order in P. Let J_1,J_2,K be in $\mathcal{D}(C,\dashv)$ and let $J_1,J_2\subseteq K$. Set

$$J=\bigvee\nolimits^{\uparrow}\{\dashv(a_1\vee a_2)\mid a_i\in J_i\}$$

(the definition is correct: such two a_i are both in K and hence there is a b such that $a_1, a_2 \leq b$. Since $\dashv a_i \subseteq \dashv(a_1 \vee a_2)$ (if $b \dashv a_i \leq a_1 \vee a_2$ then $a_i \dashv a_1 \vee a_2$ by (G1)), $J_1, J_2 \subseteq J$. If $J_1, J_2 \subseteq J' \in \mathcal{D}(C, \dashv)$, consider $a_i \in J_i$ and an $a \in J'$ such that $a_1, a_2 \dashv a$. Then $a_1 \vee a_2 \dashv a$ and $\dashv(a_1 \vee a_2) \subseteq \dashv a$; consequently $J \subseteq J'$ and we see that $J = J_1 \vee J_2$ in $\mathcal{D}(C, \dashv)$.

(3)⇒(1): Set $D = \mathcal{D}(X, \lhd)$. Then $\mathcal{J}(D)$, which is by 3.6 and 3.3.1 isomorphic to $(X, \lhd)$, admits a conditional join, namely $a \vee b$. □

4.4.1. Lemma. $J \in \mathcal{D}(X, \lhd)$ is compact iff $J = \lhd a$ for an $a \in X$ such that $a \lhd a$.

Proof. By 3.2.4, J is compact iff there is an a such that $a \in J \subseteq \lhd a$. Since ($a \in J \;\Rightarrow\; \lhd a \subseteq J$) the statement follows. □

4.4.2. Proposition. *The following statements are equivalent:*

(1) $\lhd$ *is strongly interpolative.*
(2) $(X, \lhd)$ *is isomorphic with a reflexive LD-set.*
(3) $\mathcal{D}(X, \lhd)$ *is algebraic.*

Proof. (1)⇒(2): Let $\lhd$ be strongly interpolative. Set $Y = \{x \in X \mid x \lhd x\}$ and consider $(Y, \lhd)$. By strong interpolativity, $(Y, \lhd)$ is an LD-set and it is reflexive. We easily check that the $f \subseteq Y \times X$, $g \subseteq X \times Y$ defined by xfy iff $(x \lhd)x \lhd y$, and xgy iff $x \lhd y (\lhd y)$ are mutually inverse LD-maps (in proving that $g \circ f = \lhd$ we use the interpolativity again).

(2)⇒(3): Let $(X, \lhd)$ be reflexive, $J \in \mathcal{D}(X, \lhd)$. We have, by 3.2.3, $J = \bigcup^{\uparrow}\{\lhd a \mid a \in J\}$. Since $a \lhd a$ we have $a \in \lhd a$ and all the $\lhd\, a$ are compact, by 4.4.1.

(3)⇒(1): Let $\mathcal{D}(X, \lhd)$ be algebraic and $a \lhd b$. By 4.4.1, $\lhd b = \bigcup^{\uparrow}\{\lhd c \mid \lhd c \subseteq \lhd b,\ c \lhd c\}$ and hence there is a c such that $a \lhd c$ and $c \lhd c \lhd b$. □

4.5. Proposition. $\mathcal{D}(X, \lhd)$ *has a largest element iff* $X^{\lhd}$ *is directed in* $\lhd$.

Proof. Let J be largest in $\mathcal{D}(X, \lhd)$. Then for each a, $\lhd a \subseteq J$ and hence $J = X^{\lhd}$ (by 3.2 it cannot be bigger). On the other hand, if $X^{\lhd}$ is directed, it is an ideal, and hence the largest element in $\mathcal{D}(X, \lhd)$.

4.6. Summarizing the facts above we obtain

Theorem. *The functor* $\mathcal{D}$, *providing the equivalence*

(1) **LDir** ≅ **CDom**,

induces, further, the equivalences of categories

(2) **LDir**(D) ≅ **CDom**($\top$),

(3) $\mathbf{LDir}(\mathrm{SI}) \cong \mathbf{LDir}(\mathrm{R}) \cong \mathbf{ADom}$,
(4) $\mathbf{LDir}(\sqcup) \cong \mathbf{GInf} \cong \mathbf{BCDom}$,
(5) $\mathbf{LDir}(\mathrm{D,SI}) \cong \mathbf{LDir}(\mathrm{D,R}) \cong \mathbf{ADom}(\top)$,
(6) $\mathbf{LDir}(\sqcup,\mathrm{D}) \cong \mathbf{GInf}(\mathrm{D}) \cong \mathbf{GInf}(\mathrm{F}) \cong \mathbf{CLat}$
(7) $\mathbf{LDir}(\sqcup,\mathrm{SI}) \cong \mathbf{LDir}(\sqcup,\mathrm{R}) \cong \mathbf{GInf}(\mathrm{SI}) \cong \mathbf{SInf} \cong \mathbf{BADom}$,
(8) $\mathbf{LDir}(\sqcup,\mathrm{D,SI}) \cong \mathbf{LDir}(\sqcup,\mathrm{D,R}) \cong \mathbf{GInf}(\mathrm{D,SI}) \cong \mathbf{GInf}(\mathrm{F,SI}) \cong \mathbf{GInf}(\mathrm{F,R}) \cong \mathbf{ALat}$.

References

[1] S. Abramsky, *Domain Theory and the Logic of Observable Properties,* Doctoral Dissertation, University of London, 1987.
[2] S. Abramsky and A. Jung, *Domain Theory,* in: Handbook of Logic in Computer Science, Vol.3, Clarendon Press, Oxford (1994), 1-168.
[3] B. Banaschewski and A. Pultr, *Scott Information Systems, Frames and Domains,* in: H. Herrlich, H.-E. Porst (Eds), CatMAT 2000, Proc. Conf. Categorical Methods in Algebra and Topology, Mathematik-Arbeitspapiere Nr.54, Bremen 2000, 35-46.
[4] B. Banaschewski and A.Pultr, *Remarks on Information Systems,* submitted for publication (Preprint ITI Series 2001-009).
[5] F. Cagliari and A. Pultr, *A Note on Injective Spaces,* submitted for publication (Preprint ITI Series 2001-017).
[6] M. Erné, *The ABC of order and topology,* in: H. Herrlich and H.-E. Porst (eds.), Category Theory at Work, Heldermann Verlag, Berlin 1991, 57-83.
[7] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, *A Compendium of Continuous Lattices,* Springer-Verlag, Berlin Heidelberg New York 1980.
[8] H. Hulley, *Logical Presentation of Domains,* M.Sc. Thesis, University of Cape Town, 1995.
[9] K.G. Larsen and G. Winskel, *Using information systems to solve recursive domain equations effectively,* in: Mathematical Foundations of Programming Language Semantics, Lecture Notes in Comp. Sci. **173**(1984).
[10] S. Mac Lane, *Categories for the Working Mathematician,* Graduate Texts in Mathematics 5, Springer-Verlag, New York Heidelberg Berlin 1971.
[11] D.S. Scott, *Continuous Lattices,* Springer Lecture notes in Math. **274** (1972), 97-136.
[12] D.S. Scott, *Domains for denotational semantics,* Springer Lecture Notes in Comp. Sci. **140** (1982), 579-613.

Department of Applied Mathematics and ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25
E-mail address: `pultr@kam.ms.mff.cuni.cz`

Dipartimento di Matematica Pura ed Applicata, Università degli Studi di L'Aquila, I 67100 L'Aquila, Via Vetoio, Loc. Coppito
E-mail address: `tozzi@aquila.infn.it`