On Feasible Sets of Mixed Hypergraphs

Daniel Král'*

Abstract

A mixed hypergraph H is a triple $(V, \mathcal{C}, \mathcal{D})$ where V is the vertex set and $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V, $\mathcal{C}$–edges and $\mathcal{D}$–edges. A vertex coloring of H is proper if each $\mathcal{C}$–edge contains two vertices with the same color and each $\mathcal{D}$–edge contains two vertices with different colors. The spectrum of H is a vector $(r_1, \ldots, r_l)$ such that there exists exactly r_k different coloring using exactly k colors and there is no coloring using more than l colors; the feasible set of H is the set of all k's such that $r_k \neq 0$.

We construct a mixed hypergraph with $O(\sum_i \log r_i)$ vertices whose spectrum is equal to $(r_1, \ldots, r_l)$ for each vector of positive integers such that $r_1 = 0$. We further prove that for any fixed finite sets of positive integers $A_1 \subset A_2$ $(1 \notin A_2)$, it is NP–hard to decide whether the feasible set of a given mixed hypergraph is equal to A_2 even if it is promised that it is either A_1 or A_2; this fact has a lot of interesting corollaries (e.g. deciding whether a feasible set of a mixed hypergraph is gap–free is both NP–hard and coNP–hard).

1 Introduction

Graph coloring problems are intensively studied from both the theoretic point view and the algorithmic point of view. A hypergraph is a pair $(V, \mathcal{E})$ where $\mathcal{E}$ is a family of subsets of V of size at least 2; the members of V are called vertices and the members of $\mathcal{E}$ are called edges. A mixed hypergraph

*Department of Applied Mathematics and Institute for Theoretical Computer Science, Charles University, Malostranské náměstí 25, 118 00 Praha 1, Czech Republic. E-mail: kral@kam.ms.mff.cuni.cz.
The author acknowledges partial support by GAČR 201/1999/0242, GAUK 158/1999 and KONTAKT 338/99. Institute for Theoretical Computer Science is supported by Ministry of Education of Czech Republic as project LN00A056.

H is a triple $(V, \mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V of size at least 2; the members of $\mathcal{C}$ are called $\mathcal{C}$–edges and the members of $\mathcal{D}$ are called $\mathcal{D}$–edges. A proper k–coloring c of H is a mapping $c : V \to \{1, \ldots, k\}$ such that there are two vertices with $\mathcal{D}$ifferent colors in each $\mathcal{D}$–edge and there are two vertices with a $\mathcal{C}$ommon color in each $\mathcal{C}$–edge. A proper coloring c is a strict k–coloring if it uses all k colors. A mixed hypergraph is colorable iff it has a proper coloring. Mixed hypergraphs were introduced in [17]. The concept of mixed hypergraphs can find its applications in different areas, e.g. list–coloring of graphs (see [10]), coloring block designs (see [1, 2, 4, 12, 13, 14, 15]), etc. We show a construction described in [10]: Let G be a graph and let L be a function which assigns each vertex a set of colors; a list–coloring c of G with respect to L is proper if $c(v) \in L(v)$ for each vertex v of G and $c(u) \neq c(v)$ for each edge uv of G. Let $\mathcal{L}$ be the union of the lists of all the vertices of G. Consider a mixed hypergraph H with the vertex set $V(G) \cup \mathcal{L}$ and the following edges: a $\mathcal{D}$–edge $\{u, v\}$ for each $uv \in E(G)$, a $\mathcal{D}$–edge $\{x, y\}$ for any $x, y \in \mathcal{L}$ $(x \neq y)$ and a $\mathcal{C}$–edge $\{v\} \cup L(v)$ for each vertex of G. H has a proper coloring iff G has a proper list–coloring.

The feasible set $\mathcal{F}(H)$ of a mixed hypergraph H is the set of all k's such that there exists a strict k–coloring of H. The (lower) chromatic number $\chi(H)$ of H is the minimum number in $\mathcal{F}(H)$ and the upper chromatic number of $\bar{\chi}(H)$ of H is the maximum number in $\mathcal{F}(H)$. The feasible set of H is gap–free (unbroken) iff $\mathcal{F}(H) = [\chi(H), \bar{\chi}(H)]$; we use $[a, b]$ for the set of all the integers between a and b (inclusively). If the feasible set of H contains a gap, we say it is broken. The spectrum of a mixed hypergraph H is the vector $(r_1, \ldots, r_{\bar{\chi}(H)})$ where r_k is the number of different strict k–colorings of H; we consider two colorings different if there does not exist permutation of colors changing one of them to the other. Let $\mathcal{F}$ be a set of positive integers. We say that a mixed hypergraph H is a realization of $\mathcal{F}$ if $\mathcal{F}(H) = \mathcal{F}$; we say that H is one–realization of $\mathcal{F}$ if it is a realization of $\mathcal{F}$ and all the entries of the spectrum of H are either 0 or 1.

A necessary and sufficient condition on a set of positive integers to be a feasible set of a mixed hypergraph was proved in [6]:

Theorem 1 *A set $\mathcal{F}$ of positive integers is a feasible set of a mixed hypergraph iff $1 \notin \mathcal{F}$ or $\mathcal{F}$ is an interval. If $1 \in \mathcal{F}$, then all the mixed hypergraphs with this feasible set contain only $\mathcal{C}$–edges.*

Hence there exists a mixed hypergraph such that its feasible set contains a gap. On the other hand, it was proved that feasible sets of mixed hypertrees (see [8]) and of mixed hypergraphs with maximum degree two (see [9]) are gap–free (feasible sets of mixed hypergraphs with maximum degree three are not gap–free in general). The feasible sets of planar mixed hypergraphs (hypergraphs whose bipartite incidence graphs of their vertices and edges are planar, see [3, 11]) are exactly intervals $[k_1, k_2], 1 \leq k_1 \leq 4, k_1 \leq k_2$ and sets $\{2\} \cup [4, k], k \geq 4$ as proved in [7].

Voloshin in [16] conjectured a sufficient condition on a vector to be a spectrum of a mixed hypergraph (Conjecture 2 in [16]): *Let $n_0, \dots, n_t$ be the sequence of positive integers such that $n_i \geq (n_{i-1} + n_{i+1})/2$ for $1 \leq i \leq l-1$ and $\max\{n_{\lfloor t/2 \rfloor}, n_{\lceil t/2 \rceil}\} = \max_{0 \leq i \leq t}\{n_i\}$, then there exists a mixed hypergraph H such that $\chi(H) + t = \bar{\chi}(H)$ and H allows exactly n_i different strict $(\chi(H) + i)$–colorings ($0 \leq i \leq t$).* Necessary or sufficient conditions on a vector to be a spectrum of a mixed hypergraph were not addressed so far. Since the presence of 1 in the feasible set implies that the mixed hypergraph does not contain any $\mathcal{D}$–edges, we restrict our attention mostly to mixed hypergraphs such that their feasible sets do not contain 1.

We deal with several problems from [16] (Problem 10, 11, Conjecture 2) and [6]. We address the question on the size of the smallest (one–)realization of a given feasible set in Section 2. There are presented two constructions of a mixed hypergraph with a given feasible set $\mathcal{F}$ in [6], but both of them give mixed hypergraphs of sizes which can be exponential both in $\max \mathcal{F}$ and $|\mathcal{F}|$; the second construction from [6] does not even give one–realization of $\mathcal{F}$. We present an algorithmic construction (Theorem 2) which gives a small one–realization for a given feasible set $\mathcal{F}$; the number of vertices of this realization is at most $|\mathcal{F}| + 2 \max \mathcal{F} - 1$ and the number of edges is cubic in the number of vertices.

Theorem 2 from Section 2 can be restated as follows: Let $(r_1, \dots, r_k)$ be a vector such that $r_1 = 0$ and $r_i \in \{0, 1\}$ for $2 \leq i \leq k$; then there exists a mixed hypergraph H such that the spectrum of H is $(r_1, \dots, r_k)$. Note that the condition $r_1 = 0$ is the condition $1 \notin \mathcal{F}$ mentioned earlier. We generalize this theorem in Section 3. We prove that for each vector $(r_1, \dots, r_k)$ of positive integers such that $r_1 = 0$ there exists a mixed hypergraph such that its spectrum is equal to $(r_1, \dots, r_k)$ (Theorem 3). The number of vertices of the mixed hypergraph from Theorem 3 is $2k-2+2\sum_{i=1, r_i \neq 0}^{k}(1+\lfloor \log_2 r_i \rfloor)$ and the number of its edges is cubic in the number of its vertices. Theorem 3 proves Conjecture 2 from [16].

We deal with complexity questions related to feasible sets in Section 4. We prove (Theorem 4) that for any fixed finite sets of positive integers $A_1 \subset A_2$, it is NP–hard to decide whether the feasible set of a given mixed hypergraph H is equal to A_2 even if it is promised that $\mathcal{F}(H)$ is either A_1 or A_2. This theorem has a lot of interesting corollaries: It is NP–complete to decide whether a given mixed hypergraph is colorable (Corollary 2), it is both NP–hard and coNP–hard for a fixed non-empty finite set of positive integers A to decide whether the feasible set of a mixed hypergraph is equal to A (Corollary 3), it is both NP–hard and coNP–hard to decide whether the feasible set of a given mixed–hypergraph is gap–free (Corollary 4) (this result was previously obtained in [9]) and there is no polynomial–time $o(n)$–approximation algorithm for the lower or the upper chromatic number unless $P = NP$ (Corollary 5) where n is the number of vertices of an input mixed hypergraph; we mention there is an $O(n\frac{(\log\log n)^2}{\log^3 n})$–approximation algorithm for chromatic number of ordinary graphs (see [5]).

2 Small Realizations of given Feasible Sets

We present a construction which gives a one–realization with $|\mathcal{F}|+2\max\mathcal{F}-1$ vertices in this section:

Theorem 2 *Let $\mathcal{F}$ be any non–empty finite set of positive integers such that $1 \notin \mathcal{F}$. Then there exists a mixed hypergraph H with at most $|\mathcal{F}| + 2\max\mathcal{F} - 1$ vertices with $\mathcal{F}(H) = \mathcal{F}$ even such that its spectrum contains only 0 and 1. The number of its edges is cubic in the number of its vertices. Moreover, the number of its vertices is at most $|\mathcal{F}|+2\max\mathcal{F}-2$ if $\mathcal{F} \neq \{2\}$.*

Proof: The proof proceeds by induction on $\max\mathcal{F}$. If $2 \notin \mathcal{F}$, then let H' be a one–realization of $\mathcal{F}' = \{i-1 | i \in \mathcal{F}\}$. Let H be a mixed hypergraph H' obtained by adding a vertex x and $\mathcal{D}$–edges $\{x, v\}$ for all $v \in V(H')$; it is clear that proper colorings of H one–to–one correspond to proper colorings of H' (the color of the vertex x has to be different from the color of any other vertex and it does not affect coloring of any edge except for the added $\mathcal{D}$–edges of size two). Hence, H is one–realization of $\mathcal{F}$. The number of vertices of H is at most $1 + |\mathcal{F}'| + 2\max\mathcal{F}' - 1 \leq |\mathcal{F}| + 2\max\mathcal{F} - 2$. It remains to deal with the case that $2 \in \mathcal{F}$. The case that $\mathcal{F} = \{2\}$ is trivial. Thus we assume further $\max\mathcal{F} > 2$.

We define the mixed hypergraph H for $\mathcal{F}$ in this paragraph: Its vertex set is going to be $\{v_2^+, \ldots, v_k^+, v_1^-, \ldots, v_k^-, v_1^{\oplus}\} \cup \{v_i^{\oplus} | i \in \mathcal{F} \wedge i \geq 3\}$ where $k = \max\mathcal{F}$. Let $\mathcal{F}(H) = \{c_1, \ldots, c_k\}$. Let $c_1' = 1, c_2' = c_2, \ldots, c_k' = c_k$.

We add the following edges to H for each $2 \leq l \leq k$:

$$\{v_i^-, v_j^+\} \text{ is a } \mathcal{D}\text{–edge } c'_{l-1} \leq i \leq c'_l \text{ and for } c'_{l-1} < j \leq c'_l \text{ such } i \neq j \quad (1)$$

$$\{v_i^-, v_{c'_{l-1}}^\oplus\} \text{ is a } \mathcal{D}\text{–edge for } c'_{l-1} < i \leq c'_l \quad (2)$$

$$\{v_i^+, v_i^-, v_j^+\} \text{ is a } \mathcal{C}\text{–edge for } c'_{l-1} < i, j \leq c'_l \text{ and } i \neq j \quad (3)$$

$$\{v_i^+, v_i^-, v_{c'_{l-1}}^\oplus\} \text{ is a } \mathcal{C}\text{–edge for } c'_{l-1} < i \leq c'_l \quad (4)$$

$$\{v_i^+, v_i^-, v_j^-\} \text{ is a } \mathcal{C}\text{–edge for } c'_{l-1} < i, j \leq c'_l \text{ and } i \neq j \quad (5)$$

$$\{v_{c'_{l-1}}^\oplus, v_{c'_{l-1}}^-, v_j^+\} \text{ is a } \mathcal{C}\text{–edge for } c'_{l-1} < j \leq c'_l \quad (6)$$

$$\{v_{c'_{l-1}}^\oplus, v_{c'_{l-1}}^-, v_j^-\} \text{ is a } \mathcal{C}\text{–edge for } c'_{l-1} < j \leq c'_l \quad (7)$$

$$\{v_{c'_l}^+, v_{c'_l}^-, v_{c'_l}^\oplus\} \text{ is a } \mathcal{C}\text{–edge} \quad (8)$$

$$\{v_{c'_{l-1}}^\oplus, v_{c'_l}^+, v_{c'_l}^\oplus\} \text{ is a } \mathcal{C}\text{–edge} \quad (9)$$

$$\{v_{c'_{l-1}}^-, v_{c'_l}^-, v_{c'_l}^\oplus\} \text{ is a } \mathcal{D}\text{–edge} \quad (10)$$

$$\{v_i^-, v_j^+, v_j^-\} \text{ is a } \mathcal{D}\text{–edge for } 1 \leq i \leq c'_l, \ c'_{l-1} < j \leq c'_l \text{ and } i \neq j \quad (11)$$

We prove that $H \setminus v_k^\oplus$ (the mixed hypergraph obtained from H by removing $v_k^\oplus$ and all the edges containing $v_k^\oplus$) has the desired properties.

Let H_l be the mixed hypergraph H (for $l \geq 2$) restricted to the vertices $\{v_2^+, \ldots, v_{c'_l}^+, v_1^-, \ldots, v_{c'_l}^-, v_1^\oplus\} \cup \{v_i^\oplus | i \in \mathcal{F} \wedge 3 \leq i < c'_l\}$; the edges of H_l are those edges of H which are fully contained in the vertex set of H_l. We claim the following ($2 \leq l \leq k$):

1. $\mathcal{F}(H_l) = \{c_1, \ldots, c_l\}$
2. Any proper coloring which assigns $v_{c'_l}^+$ and $v_{c'_l}^-$ different colors uses less than c_l colors. Any such colorings gives $v_{c'_{l-1}}^\oplus$ and $v_{c'_l}^+$ the same color and also $v_{c'_{l-1}}^-$ and $v_{c'_l}^-$ the same color (different to the color of $v_{c'_l}^+$).
3. Any proper coloring which assigns $v_{c'_l}^+$ and $v_{c'_l}^-$ the same color uses exactly c_l colors; any such coloring colors vertices $v_1^-, \ldots, v_{c_l}^-$ with mutually different colors. Moreover, the colors of v_i^- and v_i^+ (and $v_i^\oplus$ if it exists) are the same.
4. There exists exactly one proper coloring using exactly λ colors for each $\lambda \in \mathcal{F}(H_l)$.

We prove all these four claims together by induction on l.

We first deal with the case that l is equal to 2. Let c be any proper coloring of H_2. If $c(v_1^\oplus) \neq c(v_1^-)$, then this coloring uses exactly two colors on the vertices $v_1^-, \ldots, v_{c'_2}^-, v_1^\oplus, v_2^+, \ldots, v_{c'_2}^+$. (due to the presence of $\mathcal{C}$–edges (6) and (7)); the presence of $\mathcal{D}$–edges (1) and (2) assures that the vertices are colored as described in the second claim. Let us suppose that $c(v_1^\oplus) = c(v_1^-)$. If $c(v_i^+) \neq c(v_i^-)$ for some $2 \leq i \leq c'_2$, then $c(v_1^\oplus) \neq c(v_1^-)$ due to the presence of $\mathcal{C}$–edges (4) and (5) and $\mathcal{D}$–edges (1) and (2). Thus $c(v_i^+) = c(v_i^-)$ for all $2 \leq i \leq c'_2$. The colors of $c(v_i^-)$ for $1 \leq i \leq c'_2$ are mutually distinct due to the presence of $\mathcal{D}$–edges (1) and (2). Thus any such coloring c uses exactly c'_2 colors for coloring the vertices $v_1^-, \ldots, v_{c'_2}^-, v_1^\oplus, v_2^+, \ldots, v_{c'_2}^+$. This establishes that c uses exactly $c_2 = c'_2$ colors. This together with the beginning of this paragraph gives all the four claims for H_2. We left the straightforward check that both the colorings described in this paragraph are proper to the reader.

Let us prove the claims for H_l ($l \geq 3$) assuming them proved for H_{l-1}. Let c be a proper coloring of H_l. If $c(v_{c'_{l-1}}^-) \neq c(v_{c'_{l-1}}^+)$, then the $\mathcal{C}$–edge (8) and the $\mathcal{D}$–edge (10) together with the second claim assure that $c(v_{c'_{l-1}}^+) = c(v_{c'_{l-1}}^\oplus)$; note that in this case c uses less than c_{l-1} colors to color vertices of H_{l-1}. If $c(v_{c'_{l-1}}^-) = c(v_{c'_{l-1}}^+)$, thus $c(v_{c'_{l-2}}^-) = c(v_{c'_{l-2}}^\oplus) \neq c(v_{c'_{l-1}}^-)$, then the color $c(v_{c'_{l-1}}^\oplus)$ is either $c(v_{c'_{l-1}}^-) = c(v_{c'_{l-1}}^+)$ or $c(v_{c'_{l-2}}^-) = c(v_{c'_{l-2}}^\oplus)$ due to the presence of the $\mathcal{C}$–edge (9) and both is possible. Note that in this case c uses exactly c_{l-1} colors to color vertices of H_{l-1}.

We distinguish two cases (similar to those in the previous but one paragraph): $c(v_{c'_{l-1}}^-) \neq c(v_{c'_{l-1}}^\oplus)$ and $c(v_{c'_{l-1}}^-) = c(v_{c'_{l-1}}^\oplus)$. If $c(v_{c'_{l-1}}^-) \neq c(v_{c'_{l-1}}^\oplus)$, then the same argumentation as used in the previous but one paragraph gives that $c(v_{c'_{l-1}}^-) = \ldots = c(v_{c'_l}^-)$ and $c(v_{c'_{l-1}}^\oplus) = c(v_{c'_{l-1}+1}^+) = \ldots = c(v_{c'_l}^+)$. On the other hand, if $c(v_{c'_{l-1}}^-) = c(v_{c'_{l-1}}^\oplus)$, then the same argumentation as used in the previous paragraph gives that $c(v_{c'_{l-1}}^\oplus) = c(v_{c'_{l-1}}^-)$, $c(v_{c'_{l-1}+1}^+) = c(v_{c'_{l-1}+1}^-) \neq \ldots \neq c(v_{c'_l}^+) = c(v_{c'_l}^-)$ and the colors $c(v_{c'_{l-1}}^-), \ldots, c(v_{c'_l}^-)$ are mutually distinct due to the presence of $\mathcal{D}$–edges (1) and (2) and they are different from colors $c(v_1^-), \ldots, c(v_{c'_{l-1}-1}^-)$ due to the third claim used for H_{l-1} and the presence of $\mathcal{D}$–edges (11). This proves the first, the second and the third claim for H_l. We left the straightforward check that all the described colorings are proper to the reader. As to the fourth claim:

If $c(v^-_{c'_{l-1}}) = c(v^\oplus_{c'_{l-1}})$, then exactly c_{l-1} colors are used to color the vertices of H_{l-1} and new $c_l - c_{l-1}$ colors are used to color the vertices of $v^+_{c'_{l-1}+1}, \ldots, v^+_{c'_l}$ and $v^-_{c'_{l-1}+1}, \ldots, v^-_{c'_l}$ due to the presence of $\mathcal{D}$–edges 11. On the other hand, if $c(v^-_{c'_{l-1}}) \neq c(v^\oplus_{c'_{l-1}})$, there exists unique extension of any proper coloring of H_{l-1} to H_l. This finishes the proof of all the four claims H_l.

One can conclude that $H \setminus \{v^\oplus_k\} = H_l$ has really the desired properties. The bound on the number of edges follows from the fact that each edge has size at most three.

■

The immediate corollary of Theorem 1 and Theorem 2 is following:

Corollary 1 *There is a polynomial algorithm which for a given set $\mathcal{F}$ decides whether it is a feasible set of some mixed hypergraph and if so it outputs a mixed hypergraph H such that $\mathcal{F}(H) = \mathcal{F}$.*

Proof: If $1 \notin \mathcal{F}$, the algorithm uses construction from Theorem 2. If $1 \in \mathcal{F}$ and $\mathcal{F}$ is not interval, then the algorithm outputs that no such mixed hypergraph exists (Theorem 1). If $1 \in \mathcal{F}$ and $\mathcal{F}$ is an interval, then the algorithm outputs a mixed hypergraph consisting of $\max \mathcal{F}$ vertices and no edges.

■

3 Realization of a Given Spectrum

Lemma 1 *Let $\mathcal{F} = \{c_1, \ldots, c_l\}$ be a set of positive integers not containing 1. There exists a mixed hypergraph H^* which is one–realization of $\mathcal{F}$. Moreover, H^* contains $3l$ vertices $w^+_i, w^\oplus_i, w^\ominus_i$ $(1 \leq i \leq l)$ with the following property: Let c be any proper coloring of H^*, then*

- *The vertices $w^+_i, w^\oplus_i, w^\ominus_i$ are colored by c with exactly two colors for each i.*
- *$c(w^\ominus_i) = c(w^+_i) \neq c(w^\oplus_i)$ iff c uses exactly c_i colors.*
- *$c(w^\ominus_i) \neq c(w^+_i) = c(w^\oplus_i)$ iff c does not use exactly c_i colors.*

Proof: We deal first with the case $\mathcal{F} = \{2\}$. The following mixed hypergraph works: $V(H^*) = \{w_1^+, w_1^\oplus, w_1^\ominus\}$ where $\{w_1^\ominus, w_1^+\}$ is the only $\mathcal{C}$–edge of H^* and $\{w_1^\ominus, w_1^\oplus\}$ is the only $\mathcal{D}$–edge of H^*.

Assume that $2 \in \mathcal{F}$; we take care of the case $2 \notin \mathcal{F}$ later. We extend the construction from the proof of Theorem 2. Let H_l be the mixed hypergraph obtained in the construction and let us continue using notation from the proof of Theorem 2. We add vertex v_1^+ together with a $\mathcal{C}$–edge $\{v_1^+, v_1^-\}$ and we add vertex $v_{c'_l}^\oplus$ together with a $\mathcal{C}$–edge $\{v_{c'_{l-1}}^\oplus, v_{c'_l}^\oplus\}$. It is routine to check that the following two claims hold:

- $c(v_{c'_i}^-) = c(v_{c'_i}^+) \neq c(v_{c'_i}^\oplus)$ iff c uses exactly c_i colors.
- $c(v_{c'_i}^+) = c(v_{c'_i}^\oplus)$ iff c does not use exactly c_i colors.

Let $w_i^+ = v_{c'_i}^+$, $w_i^- = v_{c'_i}^-$ and $w_i^\oplus = v_{c'_i}^\oplus$ farther. We add new vertices $w_i^\ominus$ for all $1 \leq i \leq l$ to the mixed hypergraph together with $\mathcal{C}$–edges $\{w_i^\ominus, w_i^+, w_i^\oplus\}$ and $\{w_i^\ominus, w_i^+, w_i^-\}$ for all $1 \leq i \leq l$, $\mathcal{C}$–edges $\{w_i^\ominus, w_i^-, v_1^-\}$ for all $2 \leq i \leq l$ and $\mathcal{D}$–edges $\{w_i^\ominus, w_i^\oplus\}$ for all $1 \leq i \leq l$. We further add a $\mathcal{C}$–edge $\{w_1^\ominus, w_1^-, v_2^-\}$. Let c be a proper coloring of $H_l \cup \{v_1^+, v_{c'_k}^\oplus\}$. If $c(w_i^+) \neq c(w_i^\oplus)$ (and thus $c(w_i^+) = c(w_i^-)$), then the $\mathcal{C}$–edge $\{w_i^\ominus, w_i^+, w_i^\oplus\}$ and the $\mathcal{D}$–edge $\{w_i^\ominus, w_i^\oplus\}$ force the vertex $w_i^\ominus$ to have the color $c(w_i^+) = c(w_i^-)$ (and this extension is possible) — this describes the case when the coloring c uses exactly c_i colors. Let us assume further $c(w_i^+) = c(w_i^\oplus)$. If $c(w_i^+) \neq c(w_i^-)$, then the $\mathcal{C}$–edge $\{w_i^\ominus, w_i^+, w_i^-\}$ and the $\mathcal{D}$–edge $\{w_i^\ominus, w_i^\oplus\}$ force the vertex $w_i^\ominus$ to have the color $c(w_i^-)$ (and this extension is possible). If $c(w_i^+) = c(w_i^\oplus) = c(w_i^-)$, then the $\mathcal{C}$–edge $\{w_i^\ominus, w_i^-, v_1^-\}$ (the $\mathcal{C}$–edge $\{w_1^\ominus, w_1^-, v_2^-\}$ in case $i = 1$) and the $\mathcal{D}$–edge $\{w_i^\ominus, w_i^\oplus\}$ force the vertex $w_i^\ominus$ to have the color $c(v_1^-)$ (the color $c(v_2^-)$); this requires that $c(v_1^-) \neq c(w_i^+) = c(w_i^\oplus) = c(w_i^-)$ ($c(v_2^-) \neq c(w_i^\oplus)$ where $w_i^\oplus = v_1^\oplus$, since i is 1 in this case). The last non–equality is assured by the presence of the $\mathcal{D}$–edge (11) ($\mathcal{D}$–edge (2)) in the construction of Theorem 2. This gives that each coloring c of H_l can be uniquely extended to the constructed mixed hypergraph.

It is straightforward to check all the three properties stated by the lemma: The second and the third one are established due to the presence of a $\mathcal{C}$–edge $\{w_i^\ominus, w_i^+, w_i^-\}$ and a $\mathcal{D}$–edge $\{w_i^\ominus, w_i^\oplus\}$ ($1 \leq i \leq l$) and due to the analogous claims stated in the previous paragraph for $v_{c'_i}^+$ and $v_{c'_i}^\oplus$. The first one is established by the presence of the $\mathcal{C}$–edges $\{w_i^\ominus, w_i^+, w_i^-\}$ and $\mathcal{D}$–edges $\{w_i^\ominus, w_i^\oplus\}$.

The last case that $2 \notin \mathcal{F}$ can be dealt as follows: We construct a mixed hypergraph for $\mathcal{F}' = \{i - k' | i \in \mathcal{F}\}$ and by adding a new vertex x together with $\mathcal{D}$–edges $\{x, v\}$ for all vertices v as in the beginning of the proof of Theorem 2.

■

Lemma 2 *There exists a mixed hypergraph H_m with three special vertices w^+, $w^\ominus$, $w^\oplus$ which satisfies: Let c be any precoloring of w^+, $w^\ominus$ and $w^\oplus$ using two colors such that $c(w^\ominus) \neq c(w^\oplus)$, then:*

- *Any extension of c to a proper coloring of H_m uses no additional colors.*
- *If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then c can be uniquely extended to a proper coloring of H_m.*
- *If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then c can be extended to exactly m different proper colorings of H_m.*

The number of vertices of H_m does not exceed $3 + 2\lfloor \log_2 m \rfloor$.

Proof: The proof proceeds by induction on m. The statement is trivial for $m = 1$. We distinguish two cases:

- **The number m is even.**
  Let H_m be the mixed hypergraph obtained from $H_{m/2}$ by adding a new vertex x, a $\mathcal{C}$–edge $\{w^\oplus, w^\ominus, x\}$ and a $\mathcal{D}$–edge $\{w^\oplus, w^+, x\}$. If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then c can be extended uniquely to $H_{m/2}$ and also to x, since the added edges force that $c(x) = c(c^\ominus)$. If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then c can be extended to $m/2$ different proper colorings to $H_{m/2}$ and it can be extended to c by setting $c(x)$ to either $c(w^\oplus)$ or $c(w^\ominus)$; these extension altogether give m different extensions to the whole H_m.
- **The number m is odd.**
  Let $m = 2t + 1$. Let us consider the mixed hypergraph H_t (with the properties described in the statement of the lemma) and let $w'^+, w'^\oplus, w'^\ominus$ be its contact vertices. We construct H_m as follows: We identify the vertices $w^\oplus$ and $w'^\ominus$, $w^\ominus$ and $w'^\oplus$ and we introduce a new vertex x. Then, we add $\mathcal{C}$–edges $\{w^\oplus, w^\ominus, w'^+\}$ and $\mathcal{C}$–edge $\{w^\oplus, w^\ominus, x\}$ and

$\mathcal{D}$–edges $\{w^\oplus, w^+, w'^+\}$ and $\{w^\ominus, w'^+, x\}$. If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then the added $\mathcal{C}$–edges and $\mathcal{D}$–edges force that $c(w'^+) = c(w^\ominus)$ and $c(x) = c(w^\oplus)$; the coloring c can be then extended uniquely to the remaining vertices of H_t. If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then $c(w'^+)$ can be either $c(w^\oplus)$ or $c(w^\ominus)$: If $c(w'^+)$ is $c(w^\ominus)$, then $c(x)$ has to be $c(w^\oplus)$ and c can be uniquely extended to the remaining vertices of H_t; if $c(w'^+)$ is $c(w^\oplus)$, then $c(x)$ can be either $c(w^\oplus)$ or $c(w^\ominus)$ and c can be extended in t different ways to the remaining vertices of H_t. Thus c can be (if $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$) extended to the remaining vertices H_m in $2t + 1 = m$ different ways.

The bound on the number of vertices of H_m is obviously fulfilled in both the cases.

■

Theorem 3 *Let $(r_1, \ldots, r_k)$ be any vector of non-negative integers such that $r_1 = 0$. Then there exists a mixed hypergraph with at most $2k - 2 + 2\sum_{i=1, r_i \neq 0}^{k}(1 + \lfloor \log_2 r_i \rfloor)$ vertices such that its spectrum is equal to $(r_1, \ldots, r_k)$. Moreover, the number of edges of this mixed hypergraph is cubic in the number of its vertices.*

Proof: Let $\mathcal{F} = \{j | r_j \neq 0\}$ and let H^* be the mixed hypergraph from Lemma 1; we keep the notation of Lemma 1. We apply the following procedure for each $c_i \in \mathcal{F}(H^*)$: We add the copy of $H_{r_{c_i}}$ from Lemma 2 to H^* and we identify vertices w_i^+ and w^+, $w_i^\oplus$ and $w^\oplus$ and $w_i^\ominus$ and $w^\ominus$. It is clear due to Lemma 1 and Lemma 2 that the spectrum of the just constructed mixed hypergraph is $(r_1, \ldots, r_k)$. The bound on the number of vertices easily follows from counting the number of the vertices of H^* and the vertices of $H_{r_{c_i}}$ (and realizing that some of the vertices have been identified). The bound on the number of edges follows from the fact that each edge has size at most three.

■

4 NP–hardness and coNP–hardness

The main theorem of this section is proved in a similar way as Theorem 3 but we use instead of Lemma 2 the following one:

Lemma 3 *Let Φ be a given formula with clauses of size three; let n the number of variables and m the number of clauses of Φ. There exists a mixed hypergraph H_Φ with three special vertices w^+, $w^\ominus$, $w^\oplus$ which satisfies: Let c be any precoloring of w^+, $w^\ominus$ and $w^\oplus$ using two colors such that $c(w^\ominus) \neq c(w^\oplus)$, then:*

- *Any extension of c to a proper coloring of H_Φ uses no additional colors.*
- *If $c(w^\ominus) \neq c(w^+) = c(w^\oplus)$, then c can always be extended to a proper coloring of H_Φ.*
- *If $c(w^\ominus) = c(w^+) \neq c(w^\oplus)$, then c can be extended to a proper colorings of H_Φ iff Φ is satisfiable.*

The number of the vertices of H_Φ is $2n+3$ and the number of its edges is linear in $n+m$.

Proof: Let $x_1, \ldots, x_m$ be the variables of the given formula. Let H_Φ be a mixed hypergraph with vertices $w^+, w^\ominus, w^\oplus, {v_1}^{\mathrm{T}}, {v_1}^{\mathrm{F}}, \ldots, {v_n}^{\mathrm{T}}, {v_n}^{\mathrm{F}}$ and the following edges:

- $\mathcal{C}$–edges $\{w^\oplus, w^\ominus, {v_i}^{\mathrm{T}}\}$ and $\{w^\oplus, w^\ominus, {v_i}^{\mathrm{F}}\}$ for $1 \leq i \leq n$
- $\mathcal{D}$–edges $\{{v_i}^{\mathrm{T}}, {v_i}^{\mathrm{F}}\}$ for $1 \leq i \leq n$
- $\mathcal{D}$–edges $\{w^\ominus, w^+, w_i^X, w_j^Y, w_k^Z\}$ for each clause of the formula containing the variables x_i, x_j and x_k where $X = \mathrm{T}$ if the occurrence of x_i in the clause is positive and $X = \mathrm{F}$ otherwise; Y and Z are set in the same manner

The bounds on the size of H_Φ are clearly fulfilled.

Any extension of precoloring vertices $w^+, w^\ominus, w^\oplus$ such that $c(w^\ominus) \neq c(w^\oplus)$ to the vertices ${v_1}^{\mathrm{T}}, {v_1}^{\mathrm{F}}, \ldots, {v_n}^{\mathrm{T}}, n_n^{\mathrm{F}}$ uses only colors $c(w^\ominus)$ and $c(w^\oplus)$ due to the presence of $\mathcal{C}$–edges $\{w^\oplus, w^\ominus, {v_i}^{\mathrm{T}}\}$ and $\{w^\oplus, w^\ominus, {v_i}^{\mathrm{F}}\}$ for $1 \leq i \leq n$. If $c(w^\ominus) \neq c(w^+)$, then all the $\mathcal{D}$–edges corresponding to the clauses of Φ are properly colored already by the precoloring and thus assigning all the vertices ${v_i}^{\mathrm{T}}$ the color $c(w^\oplus)$ and all the vertices ${v_i}^{\mathrm{F}}$ the color $c(w^\ominus)$ gives a proper extension of c.

Let us assume in the rest of the proof that $c(w^\ominus) = c(w^+)$. The color $c(w^\oplus)$ represents true and the color $c(w^\ominus)$ represents false in our construction; the presence of $\mathcal{D}$–edges $\{{v_i}^{\mathrm{T}}, {v_i}^{\mathrm{F}}\}$ assures that each variable and its negation have opposite values (the value of x_i is represented by the color of ${v_i}^{\mathrm{T}}$). The presence $\mathcal{D}$–edges $\{w^\ominus, w^+, v_i^X, v_j^Y, v_k^Z\}$ assures that each clause contains at least one true literal (a vertex colored by the color $c(w^\oplus)$. Hence, c can be extended to H_Φ iff there is a satisfying assignment of Φ.

■

Theorem 4 *Let $A_1 \subset A_2$ be two finite non-empty subsets of $\{2, 3, \ldots\}$. It is NP–hard to decide whether the feasible set of a given mixed hypergraph H is equal to A_2 even if it is promised that $\mathcal{F}(H)$ is either A_1 or A_2.*

Proof: We present a reduction from the well–known NP–complete problem 3SAT. Let Φ be a given formula and H_Φ the mixed hypergraph from Lemma 3. Let H^* be the mixed hypergraph from Lemma 1 applied for the set $A_2 = \{c_1, \ldots, c_l\}$; let $A_2 \setminus A_1 = \{c_{i_1}, \ldots, c_{i_{l'}}\}$. We create $|A_2| - |A_1| = l'$ copies of H_Φ and we identify the vertices $w^\ominus, w^+, w^\oplus$ of the j-th copy with the vertices $w^\ominus_{c_{i_j}}, w^+_{c_{i_j}}, w^\oplus_{c_{i_j}}$ of H^*; let H be the obtained mixed hypergraph. H has a strict k–coloring for $k \in A_1$ since any strict k–coloring of H^* can be extended to the copies of H_Φ due to Lemma 3 (note that $c(w^\ominus_{c_{i_j}}) \neq c(w^+_{c_{i_j}}) = c(w^\oplus_{c_{i_j}})$ for $1 \leq j \leq l'$ for any strict k–coloring of H^* where $k \in A_1$). On the other hand, H has a strict k–coloring for $k \in A_2 \setminus A_1$ iff Φ is satisfiable: a strict k–coloring of H^* for $k = c_{i_j}$ can be extended to the j–th copy of H_Φ iff Φ is satisfiable due to Lemma 3 since it holds that $c(w^\ominus_{c_{i_j}}) = c(w^+_{c_{i_j}}) \neq c(w^\oplus_{c_{i_j}})$ for any such strict k–coloring of H^*. Note that the number of vertices of H is at most $3 \max A_2 + 2|A_2 \setminus A_1|n$ and the number of its edges is cubic in the number of its vertices.

■

Several interesting corollaries almost immediately follow:

Corollary 2 *It is NP–complete to decide whether a given mixed hypergraph H is colorable.*

Proof: This problem clearly belongs to the class NP. It is enough to set $A_1 = \emptyset$ and $A_2 = \{2\}$ in Theorem 4 to get the result.

■

Corollary 3 *Let A be a fixed finite subset of $\{2, 3, \ldots\}$. It is coNP–hard to decide whether the feasible set of a given mixed hypergraph H is equal to A. If $A \neq \emptyset$, then this problem is NP–hard, too.*

Proof: The coNP–hardness is established by setting $A_1 = A$ and A_2 to a proper superset of A omitting 1 in Theorem 4; the NP–hardness by setting A_2 to A and A_1 to a proper subset of A.

■

Corollary 4 *It is both NP–hard and coNP–hard to decide whether the feasible set of a given mixed hypergraph H is gap–free even for H with $\bar{\chi}(H) = 4$.*

Proof: The NP–hardness is established by setting $A_1 = \{2, 4\}$ and $A_2 = \{2, 3, 4\}$ in Theorem 4; the coNP–hardness by setting $A_1 = \{4\}$ and $A_2 = \{2, 4\}$.

■

Corollary 5 *There does not exist a polynomial–time $o(n)$–approximation algorithm for the lower or the upper chromatic number of a mixed hypergraph where n is the number of vertices unless $P = NP$.*

Proof: Suppose that there exists polynomial–time $f(n)$–approximation algorithm for the lower chromatic number where $f(n) \in o(n)$ and n is the number of vertices of a given mixed hypergraph. Let Φ be a given formula with clauses of size three with N variables. Choose k such $k < 2f(3k+2N)$; it exists since $f(n) \in o(n)$. Let H be the mixed hypergraph from the construction of Theorem 4 for $A_1 = \{k\}$ and $A_2 = \{2, k\}$. It is NP–complete due to Theorem 4 to decide whether the feasible set is A_2 if it is promised that it is either A_1 or A_2. The approximation algorithm for the lower chromatic number outputs a number which is less then k iff the feasible set of the input mixed hypergraph is A_2. Thus the existence of the polynomial–time $o(n)$–approximation algorithm implies $P = NP$. The non–existence (unless P=NP) of a polynomial–time $o(n)$–approximation algorithm for the upper chromatic number can be proved similarly.

■

5 Conclusions

There exists a mixed hypergraph whose feasible set is $\mathcal{F}$ for any set $\mathcal{F}$ of positive integers which does not contain 1; we proved that there exists a mixed hypergraph whose spectrum is $(r_1, \ldots, r_k$ for any vector $(r_1, \ldots, r_k)$ of positive integers such that $r_1 = 0$. The number of the vertices of the smallest mixed hypergraph which is a realization of a given set $\mathcal{F}$ has been substantially decreased from exponential to linear in $\max \mathcal{F}$. But the following question has not been answered: What is the number of vertices of the smallest mixed hypergraph whose feasible set is equal to a given set $\mathcal{F}$? Or even, what is the number of vertices of the smallest mixed hypergraph whose spectrum is equal to a given spectrum $(r_1, \ldots, r_k)$? The answer to any of these questions probably requires some very fine analysis.

We have not dealt with mixed hypergraphs containing only $\mathcal{C}$–edges in this paper. It is clear that if $r_1 \neq 0$ (this is equivalent to the fact that a mixed hypergraphs contains only $\mathcal{C}$–edges), then $r_1 = 1$. It can be further proved that $r_2 = (2^n - 2)/2$ for some n; this follows from the fact that $\mathcal{C}$–edges of size two can be contracted without affecting the spectrum and any two–coloring of a mixed hypergraph without $\mathcal{C}$–edges of size two and without $\mathcal{D}$–edges is proper. This leads to the following problem: What are necessary and sufficient conditions on a vector $(r_1, \ldots, r_k)$ of positive integers (such that $r_1 = 1$) in order to be a spectrum of a mixed hypergraph?

Acknowledgement

The author would like to thank Jiří Sgall for suggesting an easy way how to prove Lemma 2, Jan Kratochvíl for helpful discussions and Zdeněk Dvořák for careful reading a preliminary version of this paper.

References

[1] C. Colbourn, J. Dinitz, A. Rosa: Bicoloring Triple Systems, Electronic J. Combin. 6# 1, paper 25, 16 pages.

[2] Ch. J. Colbourn, A. Rosa: Triple Systems, Clarendon Press, Oxford (1999), sect. 18.6. Strict colouring and the upper chromatic number, 340–341.

[3] Z. Dvořák, D. Král': On Planar Mixed Hypergraphs, Electronic J. Combin. 8 (1), 2001, #R35.

[4] T. Etzion and A. Hartman: Towards a large set of Steiner quadruple systems, SIAM J. Discrete Math. 4 (1991), 182–195.

[5] M. M. Halldórsson: A still better performance guarantee for approximate graph coloring, Inform. Process. Lett. 45 (1993), 19–23.

[6] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. B. West: Chromatic spectrum is broken, 6th Twente Workshop on Graphs and Combinatorial Optimization, 26–28 (1999), H. J. Broersma, U. Faigle and J. L. Hurink (eds.), University of Twente (1999), 231–234.

[7] D. Kobler, A. Kündgen: Gaps in the chromatic spectrum of face–constrained plane graphs, Electronic J. Combin. 8 (1) (2001), #N3.

[8] D. Král', J. Kratochvíl, A. Proskurowski, H.-J. Voss: Coloring mixed hypertrees, Proceedings 26th Workshop on Graph-Theoretic Concepts in Computer Science, LNCS vol. 1928 (2000), 279–289.

[9] D. Král', J. Kratochvíl, H.-J. Voss: Complexity Note on Mixed Hypergraphs, Proceedings 26th International Symposium on Mathematical Foundations of Computer Science, LNCS vol. 2136 (2001), 474–486.

[10] J. Kratochvíl, Z. Tuza, M. Voigt: New trends in the theory of graph colorings: Choosability and list coloring, in: Contemporary Trends in Discrete Mathematics (from DIMACS and DIMATIA to the future), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 49, American Mathematical Society, Providence, RI (1999), 183–197.

[11] A. Kündgen, E. Mendelsohn, V. Voloshin: Colouring planar mixed hypergraphs, Electronic J. Combin. 7, 2000, #R60.

[12] H. Lefmann, V. Rödl, and R. Thomas: Monochromatic vs. multicolored paths, Graphs Combin. 8 (1992), 323–332.

[13] L. Milazzo: On upper chromatic number for SQS(10) and SQS(16), Le Matematiche L (Catania, 1995), 179–193.

[14] L. Milazzo and Zs. Tuza: Upper chromatic number of Steiner triple and quadruple systems, Discrete Math. 174 (1997), 247–259.

[15] L. Milazzo and Zs. Tuza: Strict colorings for classes of Steiner triple systems, Discrete Math. 182 (1998), 233–243.

[16] V. Voloshin: On the upper chromatic number of a mixed hypergraph, Australasian Journal of Combinatorics 11 (1995), 25–45.

[17] V. Voloshin: The mixed hypergraphs, Computer Science Journal of Moldova 1 (1993), 45–52.