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Preface

It has become a tradition of DIMATIA to organize one day meetings of Czech and Slovak graph theorists. These meetings, called DIMATIA Graph Theory Days, help the Czech and Slovak graph theory community to keep informed about the recent results and research conducted by our researchers, as well as personal meetings play its important role in organizing mutual contacts and planning further activities.

This time, however, DIMATIA Graph Theory Day 5 was really a special one. It was special because of the large number of both Czech and foreign participants. And it was special because it was organized on a special occassion. On April 13th, 2001, Jarik Nešetřil was celebrating his 55th birthday.

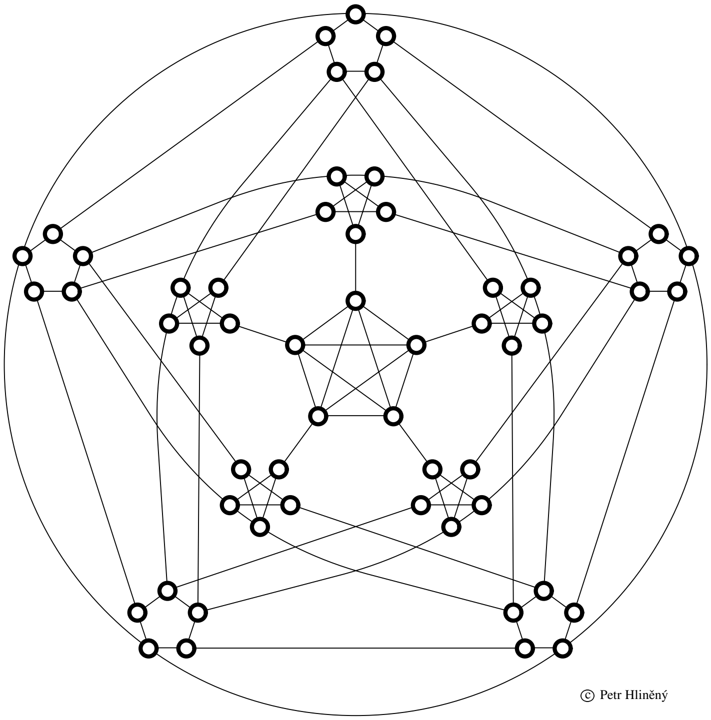
The idea to organize a meeting to celebrate this event initiated in January. A more specific plan for the form of the celebration came from the discussions among the two of us and Jirka Matoušek. We were aware that the number 55 is not round enough for a real international conference, but its symmetry inspired us to dedicate to it a one day meeting. We thought this would be a good opportunity to thank Jarik for everything he has done for us, for other Prague students and colleagues and for the Czech graph theory community as a whole. Because all of us have profited in some way from Jarik's professional and organizational skills. We have learned a lot attending and later supervising the Combinatorial Seminar and annual Spring Schools in Combinatorics, all of us are enjoying the wonderful scientific atmosphere at DIMATIA and ITI.

The idea of organizing a small meeting with 3 to 5 foreign participants seemed very natural. From the very beginning we decided to keep the foreign participants secret to Jarik, making their presence a suprise and kind of a birthday present for him. With this simple idea in mind, we made the first steps already in January. Everybody who ever organized a conference knows the counts - if you want 3 invited speakers to come, you have to negotiate with at least 6. If you want 5, ask 10. This explains how an intendedly small meeting turned into a busy day packed with very nice and interesting talks. Perhaps we should not say it was unexpected, but it definitely was not usual that out of 10 invited distinguished foreign speakers, all of them were enthusistic about the idea and 9 of them really came.

The actual preparation of DGT5 brought unexpected excitement. Hoping to keep the participants secret to Jarik till the last minute, we have announced the program to an 'ongoing full day problem session', and in order to make sure that Jarik does not leave the country for some other conference, we asked him to chair this problem session. Knowing only about Czech (and by far not all) participants, he became uncertain if the program will last for the whole day, and started inviting more people on his own. There is no need in describing the horror that swept over us when one of our French participants wrote to us that he had been invited by Jarik as well. It was perhaps unavoidable that an information leak occured, and by the time of the meeting Jarik knew about two of the foreign participants. He probably expected more, and so he was not surprised to see 4 foreigners in the memorable S6 classroom by the time the meeting started. (By the way, this classroom has never been so full before.) But the group of other 4 friends including Moshe Rosenfeld who flew in from Seattle has definitely caught him by surprise, exactly as planned. And Hans Jürgen Prömel arriving two hours later directly from a meeting of his rectorate was the cherry on the cake.

Thanks to the foreign participants and all speakers we have had a very nice and worthwhile meeting, with excellent and inspiring talks and discussions, and also with a tasty cake and toasts. We would like to thank our PhD students and our secretaries for organizational help, Aleš Pultr and Věra Trnková from Charles University for their talks, the foreign participants for coming over for just one day to join the celebration, and last but not least Jarik Nešetřil for giving us the opportunity and reason to organize DGT5 (and for far more).

Jan Kratochvíl   Pavel Valtr



The official logo of the Day prepared by Petr Hliněný.

Two problems on planar graphs

Éric Sopena

About ten years ago, I was attending an ALTEC conference on Algorithms in Budapest. At this time, I was puzzled by the problem of determining the oriented chromatic number of oriented planar graphs. Jarik Nešetřil was attending the same conference and I approached him for the first time, thinking that he could be attracted to my problem. Beyond my expectations, Jarik was extremely enthusiastic about it, and we spent most of the remaining free time of the conference discussing planar graphs. This was the starting point of a very fruitful collaboration, together with my colleague André Raspaud, regularly stimulated by Jarik's numerous questions which allowed us to consider many new interesting problems related to oriented colourings.

Ten years after, the original problem is still open and I am particularly pleased to include it in this volume in honour of Jarik's 55th birthday. I have also included a more recent problem concerning the size of k-dominating sets in planar graphs.

1 The oriented chromatic number of oriented planar graphs

Our first problem deals with oriented graphs, that is digraphs having no loops and no opposite arcs. If G is an oriented graph, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of arcs. An *oriented k-colouring* of an oriented graph G is a mapping $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that

(i) if $xy \in E(G)$ then $c(x) \neq c(y)$, and

(ii) if $xy, zt \in E(G)$ then $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$.

With every oriented k-colouring c of G one can associate a digraph H_c with vertex set $V(H_c) = \{c(x),\ x \in V(G)\}$ and arc set $E(H_c) = \{c(x)c(y),\ xy \in E(G)\}$. Thanks to conditions (i) and (ii), H_c is an oriented graph. The oriented k-colouring c can be viewed as a homomorphism (that is an arc-preserving vertex mapping) from G to H_c. Similarly, every homomorphism of G to an oriented graph H on k vertices can be viewed as an oriented k-colouring of G, using the vertices of H as colours. Oriented colourings have been introduced in [7] and studied by several authors (see [9] for a general overview). In particular, various problems related to planar graphs are discussed in [1], [2] and [6].

The *oriented chromatic number* $\vec{\chi}(G)$ of an oriented graph G is defined as the smallest k such that G admits an oriented k-colouring or, equivalently, as the smallest order of an oriented graph H such that G admits a homomorphism to H. The oriented chromatic number $\vec{\chi}(\mathcal{F})$ of a family $\mathcal{F}$ of oriented graphs is then defined as the largest oriented chromatic number among the oriented chromatic numbers of its members ($\vec{\chi}(\mathcal{F})$ can thus be infinite). Our first problem is the following:

Problem 1.
Determine the oriented chromatic number of the family of oriented planar graphs.

Let us denote by $\mathcal{O}$ the family of oriented outerplanar graphs and by $\mathcal{P}$ the family of oriented planar graphs. It has been proved in [8] that $\vec{\chi}(\mathcal{O}) = 7$ and in [7] that $\vec{\chi}(\mathcal{P}) \leq 80$.

From the definition of oriented k-colourings, it is not difficult to observe that if xyz is a directed 2-path in G ($xy, yz \in E(G)$) then $c(x) \neq c(y) \neq c(z) \neq c(x)$ for every k-colouring c of G. In other words, any two vertices that are linked in G by a directed path of length 1 or 2 must be assigned distinct colours. Let G_1 be the oriented graph having one vertex and no arc and G_i, $i \geq 2$, be the oriented graph obtained by taking two disjoint copies of G_{i-1}, a new vertex x_i, and adding all the arcs from the vertices of the first copy towards x_i and all the arcs from x_i towards the vertices of the second copy (see Figure 1). By construction, and considering the observation above, we get $\vec{\chi}(G_i) = 2^i - 1$ for every $i \geq 1$. Note in particular that the graph G_3 is outerplanar, so that $\vec{\chi}(\mathcal{O}) \geq 7$, and that the graph G_4 is therefore planar, so that $\vec{\chi}(\mathcal{P}) \geq 15$. We have proved in [10] that $\vec{\chi}(\mathcal{P}) > 15$. Recently, T.H. Marshall [5] improved this result by showing that $\vec{\chi}(\mathcal{P}) > 16$. Concerning the family of oriented planar graphs, we thus have:

Theorem 1. $17 \leq \vec{\chi}(\mathcal{P}) \leq 80$.

Every improvement of these bounds seems, at least up to now, to be particularly challenging!

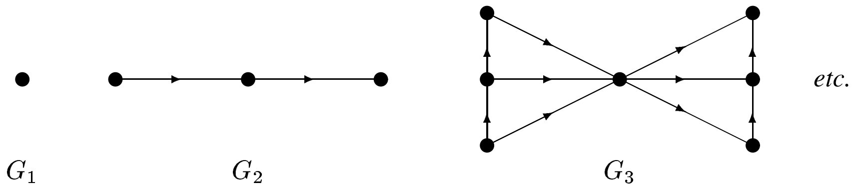


Figure 1: A sequence of oriented graphs G_i, $i \geq 1$, with $\vec{\chi}(G_i) = 2^i - 1$

Let us say that an oriented graph U is *$\mathcal{F}$-universal* for some family $\mathcal{F}$ of oriented graphs if every graph $F \in \mathcal{F}$ admits a homomorphism to U. If the family $\mathcal{F}$ has oriented chromatic number at most k, a $\mathcal{F}$-universal oriented graph on k vertices does not necessarily exist. (Consider for instance the family $\mathcal{G}_k$ of all oriented graphs on k vertices; the oriented chromatic number of $\mathcal{G}_k$ is clearly at most k but there exists no $\mathcal{G}_k$-universal oriented graph on k vertices since such a graph would contain all tournaments on k vertices as subgraphs). However, if we consider the family $\mathcal{P}$ of oriented planar graphs (or, similarly, the family $\mathcal{O}$ of oriented outerplanar graphs) such a universal graph exists. To see that, suppose to the contrary that for every oriented graph U on k vertices there exists some oriented planar graph P_U which does not map homomorphically to U. Denote by ℓ the number of oriented graphs on k vertices and let $\mathcal{G}_k = \{U_1, U_2, \ldots, U_\ell\}$. Consider the oriented graph P obtained by taking one copy of each graph P_{U_1}, P_{U_2}, …, P_{U_ℓ} and gluing them by identifying one vertex of each of them into a unique vertex. The graph P thus obtained is clearly planar and therefore there exists some oriented graph $U = U_i$ in $\mathcal{G}_k$ such that $P \longrightarrow U$. Considering the restriction to P_{U_i} of such a homomorphism, we get a homomorphism from P_{U_i} to U_i which contradicts our assumption.

All the above-mentionned results concerning the oriented chromatic number of oriented planar graphs have been obtained by considering such universal graphs. One way to improve Theorem 1 is thus either to exhibit some $\mathcal{P}$-universal oriented graph having less than 80 vertices, or to prove that no oriented graph on 17 (or more) vertices can be $\mathcal{P}$-universal.

2 k-dominating sets in planar graphs

We now consider undirected graphs. Let k be a non-negative integer; we say that a set $S \subseteq V(G)$ *k-dominates* a graph G if every vertex x in G is at distance at most k from the set S (the distance $d_G(x,S)$ from x to S is given by $d_G(x,S) = \min\{d_G(x,s),\ s \in S\}$). Using this terminology, a dominating set of G is a 1-dominating set of G. We are interested in the minimum size of such k-dominating sets. For that, we define the parameter

$$\gamma_k(G) \ = \ \min\{|S|,\ S \text{ } k\text{-dominates } G\}$$

for every graph G. Let us denote by $D(G)$ the *diameter* of G, that is the maximum length of a shortest path in G. We have the following conjecture [3]:

Conjecture 1. *There exists some constant c_0 such that for every planar graph G and every $k \geq D(G)/2$, $\gamma_k(G) \leq c_0$.*

Our second problem is therefore the following:

Problem 2.
Prove (or disprove) Conjecture 1.

Observe that if a set S k-dominates a graph G then it also k'-dominates G for every $k' \geq k$. Therefore, considering the case $k = \lceil \frac{D(G)}{2} \rceil$ in Conjecture 1 is enough.

Consider the tree $T_{n,p}$, $n \geq 2$, $p \geq 2$, obtained from the star $K_{1,n}$ by replacing each edge by a path of length p. We clearly have $D(T_{n,p}) = 2p$. Since every leaf in $T_{n,p}$ is at distance p from the central vertex, every $(p-1)$-dominating set of $T_{n,p}$ must contain one vertex per branch. Therefore, $\gamma_{p-1}(T_{n,p}) \geq n$. This shows that the condition $k \geq D(G)/2$ in Conjecture 1 is necessary in order to get a k-dominating set of bounded size.

Recall that a *center* in a tree T is a vertex whose maximal distance to other vertices is minimum (every tree has at most two centers). If c is a center of T, we clearly have $d_T(c,x) \leq D(T)/2$ for every vertex $x \in V(T)$ and thus $\gamma_{\lceil \frac{D(T)}{2} \rceil} = 1$ for every tree T.

In case of outerplanar graphs, it is proved in [3] that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 2$ for every outerplanar graph G (considering the cycle on four vertices shows that this bound is tight).

For planar graphs, MacGillivray and Seyffarth [4] proved that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 3$ if $D(G) = 2$ (and this bound is tight) and that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 10$ if $D(G) = 3$. In [3], we have proved that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 23$ if $D(G) = 4$ and that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 41$ if $D(G) = 7$. The only general result we have concerning k-dominating sets of bounded size for planar graphs is the following:

Theorem 2 ([3]). *For every planar graph G, for every* $\varepsilon > 0$, *for every* $k > (\frac{5}{7} + \varepsilon)D$, $\gamma_k(G) \leq \frac{3}{\varepsilon} + 6$.

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Some problems and conjectures related to local properties of graphs

Zdeněk Ryjáček

A graph G is *claw-free* if G does not contain a copy of the claw $K_{1,3}$ as an induced subgraph. For a vertex x of a graph G, the subgraph of G induced by the set of neighbors $N_G(x)$ of x is called the *neighborhood* of x (in G). We say that x is *locally connected* if its neighborhood is a connected graph. A locally connected vertex with a noncomplete neighborhood is called *eligible* and the graph G'_x, obtained from G by adding to the neighborhood of an eligible vertex x all missing edges, (i.e. such that $N_{G'_x}(x)$ induces in G'_x a complete graph), is called the *local completion of* G *at* x. The basic observations concerning the local completion operation are given in the following statement (in which $c(G)$ denotes the circumference of G).

Proposition 1. *[5]. Let G be a claw-free graph and let x be an eligible vertex of G. Let $N'_x = \{uv \mid u, v \in N_G(x), uv \notin E(G)\}$ and let G'_x be the graph with vertex set $V(G'_x) = V(G)$ and with edge set $E(G'_x) = E(G) \cup N'_x$. Then*

(i) *the graph G'_x is claw-free,*

(ii) $c(G'_x) = c(G)$.

It is not difficult to show that a graph which is obtained from a claw-free graph G by recursively performing the local completion operation as long as there is at least one eligible vertex, is uniquely determined and is again claw-free. This graph is called the *closure of* G and is denoted by $\mathrm{cl}(G)$.

Since $\mathrm{cl}(G)$ is claw-free and has no eligible vertices, $\mathrm{cl}(G)$ is the line graph of a triangle-free graph. We thus have the following theorem.

Theorem 2. *[5]. Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is well-defined (i.e., uniquely determined by G),*

(ii) *there is a triangle-free graph H such that $\mathrm{cl}(G)$ is the line graph of H,*

(iii) $c(\mathrm{cl}(G)) = c(G)$.

Specifically, G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian. On the other hand, it is known that there are nonpancyclic graphs with pancyclic closure [1]. The graph in Figure 1 is an example of such a nonpancyclic graph with complete (and hence pancyclic) closure. This situation gives rise to the following questions.

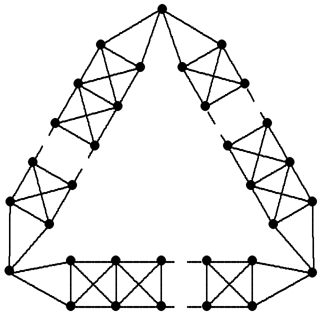

Figure 1

Problem 1. *Determine the maximum number of cycle lengths that can be missing in a claw-free graph on n vertices with complete closure.*

Conjecture 1. *Let c_1, c_2 be fixed constants. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, where $3 \leq i \leq c_1$ and $n - c_2 \leq i \leq n$.*

It is easy to see that a claw-free graph with complete closure on at least 4 vertices can miss neither a C_3 nor a C_4. The main result of [6] shows that such a graph G cannot be missing a cycle of length $n-1$; however, the proof of this result is difficult and cannot be iterated.

A graph G is *locally connected* if every its vertex is locally connected. Theorem 2 immediately implies the following previous result (originally by Oberly and Sumner [3]).

Theorem 3. *[3]. Every connected, locally connected claw-free graph on at least three vertices is hamiltonian.*

The local connectivity concept can be weakened by modifying the concept of vertex neighborhood. We say that an edge xy is a *neighboring edge* of a vertex v if $x \neq v \neq y$ and x or y is adjacent to v. The edge induced subgraph on the set of all neighboring edges of v will be called the *neighborhood of the second type* of v in G and denoted by $N_G^2(v)$ (this concept was introduced by Sedláček [7]). A graph G is said to be N_2*-locally connected* if $N_G^2(v)$ is connected for every $v \in V(G)$. Clearly, every locally connected graph is N_2-locally connected.

It is known that there are 2-connected N_2-locally connected nonhamiltonian claw-free graphs (to obtain an analogue of Theorem 3, an additional assumption is needed - see [4]). However, no such 3-connected graph is known.

Conjecture 2.
Every 3-connected N_2-locally connected claw-free graph is hamiltonian.

Theorem 3 was strengthened by Clark [2] as follows.

Theorem 4. *[2] Every connected, locally connected claw-free graph on at least three vertices is vertex pancyclic.*

Let $g(G)$ denote the girth (i.e. the length of a shortest cycle) of G. We say that G is *weakly pancyclic* if G contains cycles of all lengths ℓ for $g(G) \leq \ell \leq c(G)$. Theorem 4 motivates the following question.

Conjecture 3.
Every connected, locally connected graph is weakly pancyclic.

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Problems in Extremal Graph Theory

Miklós Simonovits

1 Introduction

This extended abstract tries to cover my lecture in Prague, on the 5th Graph Theory Day, (March 12, 2001) connected also to Jarik Nešetřil's birthday. The lecturers were encouraged to be informal, speak about some of their favorite problems and give also the background of the problem. I have selected three topics. Many related results and problems can be found, e.g., in [1], [11], [14].

Notation. Given a graph the first subscript usually denotes the number of vertices, e.g., G_n is always a graph on n vertices. The number of edges of G is $e(G)$; $\chi(G)$ is the chromatic number of G. Given two graphs G and H, we shall denote by $G \otimes H$ the graph created by joining each vertex in G to each one in H.

2 Product Conjecture

Given a family $\mathcal{L}$ of forbidden graphs, a theorem of Erdős [4], and myself [9] states that if S_n is an extremal graph for $\mathcal{L}$ then one can delete and add $o(n^2)$ edges to it to get a Turán graph $T_{n,p}$, where $p+1$ is the minimum chromatic number in $\mathcal{L}$. The question is: when is it enough to add edges to a p-chromatic graph (i.e., not to delete at all?).

Fix an integer p. We shall say that H_n has a product form if

$$H_n = U_{n_1} \otimes \ldots \otimes U_{n_p} \quad \text{where each} \quad n_i > \frac{n}{2p}.$$

Problem 1 (Informal form). *Is it true that under mild conditions the extremal graphs for $\mathcal{L}$ are of product form?*

Conjecture 1 (Simplified form). *Assume that L is a $p+1$-chromatic graph for which, if we color it in an arbitrary way with $p+1$ colors, then any two* *color classes span at least a cycle. (I.e. they do not span a tree or a forest.) Then for every sufficiently large* n *there exist extremal graphs* S_n *of product form.*

In such cases $n_i = n/p + o(n)$. Maybe, not only some but all extremal graphs are of product form? The meaning of this conjecture is that in cases when the answer is YES, the problem can be reduced to *degenerate* extremal graph problems, i.e. to problems where the maximum is $o(n^2)$. Some counterexample is constructed for the "tree-case" in [10].

3 Minimum degrees in 4-critical graphs

Let us call a graph G_n k-edge-color-critical, (or here, k-critical for short) if it is k-chromatic but deleting any edge, it becomes $k-1$-chromatic. For $k = 3$ the description of k-critical graphs is trivial, since they are just the odd cycles; for $k \geq 4$ their structure may become really involved. There are many results in this field. The investigation started with Erdős and Dirac and Gallai and one of the most well-known researcher of the field is Bjarne Toft.

Erdős asked if a k-critical graph can have many edges. Dirac answered that YES, $C_\ell \times C_\ell$ is 6-critical and has $\approx \frac{n^2}{4}$ edges, assumed that ℓ is odd. For $k \geq 6$ this method always works. For $k = 4, 5$, the question remained open for a while, then Bjarne Toft (who was a PhD student of Dirac) came up with a beautiful construction.

Construction 1 (Toft, Special case, [18]). *Let* m *be odd,* $n = 4m$ *and form 4 classes of* n *vertices,* A, B, C, D. *Form a cycle* C_m *on* A *and also on* D, *a one-factor between* A *and* B *and between* C *and* D *and join* B *to* C *completely.*

This is a 4-critical graph G_n with $e(G_n) \approx \frac{n^2}{16}$. Unfortunately (?) half of its vertices have degree 3.

I was working partly in this field those days and had a construction (to kill a conjecture of Gallai on the mimimum degrees in k-critical graphs). My construction was superseded by a construction of Brown and Moon, [3], so I decided to forget mine. Then Toft came around with his construction and we spoke about it with Bjarne (and with Erdős). Finally (semiindependently) we both combined his and my methods to construct (among others) infinite sequences of 4-critical graphs with minimum degree $c \cdot n^{1/3}$.

Problem 2 (Erdős). *Is it true that if G_n is 4-chromatic but deleting any vertex of G_n we get a 3-chromatic graph then the minimum degree of G_n is $o(n)$?*

The problems on critical graphs have also many connections to extremal graph theory, see, e.g., [13] and to the applications of algebraic methods in hypergraph extremal problems, see Lovász [8]. See also the book of T. Jensen and B. Toft [7].

4 A Ramsey-Turán problem

The Ramsey-Turán problems were motivated by some application of Turán's graph theorem. The applications were started by Turán, and the systematic investigation of Ramsey-Turán theory by Vera T. Sós [16].

Problem 3. *Given a "sample graph", L, and a function $\phi(n)$ what is the maximum of $e(G_n)$ under the condition that $L \not\subseteq G_n$ and $\alpha(G_n) \leq \phi(n)$?*

Problem 4 (V. T. Sós). *Characterize the graph parameters which determine if*

$$\max e(G_n) = o(n^2)$$

or not, under the conditions that $L \not\subseteq G_n$ and $\alpha(G_n) = o(n)$.

It was a surprise to learn that for K_4 this maximum is $\frac{n^2}{4} + o(n^2)$, [17],[2] One of the simplest unsolved problems in this field is

Problem 5 (Erdős-T. Sós). *Is it true that if G_n is a graph sequence for which $K(2,2,2) \not\subseteq G_n$ and $\alpha(G_n) = o(n)$, then $e(G_n) = o(n^2)$?*

See e.g. our survey with Vera T. Sós [15].

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Problems and conjectures on parametrized H-coloring

Josep Díaz

Recall that given two directed or undirected graphs G and G' an *homomorphism* of G into G' is a map ha $\theta : V(G) \rightarrow V(G')$ with the property that $\{v, w\} \in E(G) \Rightarrow \{\theta(v), \theta(w)\} \in E(G')$. The *$H$-coloring problem* is the problem of checking whether, for fixed H, there exists an homomorphism (H-coloring) from an input graph G to H.

It is known that if H is bipartite or it has a loop the H-coloring problem can be trivially solved in polynomial time, but in the case that H is loop less and not-bipartite the problem is known to be NP-complete [HN90]. An interesting generalization of the H-coloring problem is the *list* H-coloring problem where each vertex of G carries a list of the vertices of H where it is allowed to be mapped [FH98, FHH99]. In [DST01] we consider the following parametrized version of the H-coloring problem:
Set up a weighting $K = \{k_j | j \in C\}$ of $C \subseteq V(G)$ with non negative integers, we say that an input graph G has a (H, C, K)-coloring if there exists an H-homomorphism $\chi : V(G) \rightarrow V(H)$ such that $\forall v \in C, |\chi^{-1}(v)| = k_v$. Denote (H, C, K) a *partial weighted assignment*. If we additionally assign to each vertex of G a list permissible images when we have a more general version of the (H, C, K)-coloring problem that we call *list* (H, C, K)-coloring. We can consider the integers in K to be small fixed constants, which constitutes a parameterization of the above problems problems. A weighted assignment (H, C, K) is a *weighted extension* of a graph F if $H - C = F$.

In [DST01] we prove that if F is a graph where the F-coloring is NP-complete then for any weighted extension (H, C, K) of F the (H, C, K)-coloring is also NP-complete. On the other hand, if F-coloring is in P then *there exist* a weighted extension (H, C, K) of F so that the (H, C, K)-coloring is in P.

In the same reference we also prove that if F is a graph where the list F-coloring is in P then for any weighted extension (H, C, K) of F, the (H, C, K)-coloring is also in P and from the other hand, we provided some examples where the F-list coloring is NP-complete and F has a weighted extension (H, C, K) such that (H, C, K)-coloring is also NP-complete.

The above results seem to imply that there are graphs that for different weighted extensions produce parameterized coloring problems with different complexities. It seems to be a hard problem to achieve a dichotomy distinguishing those parameterized assignments (H, C, K) of a given graph F for which the (H, C, K)-coloring problem is NP-complete or in P. We conjecture the following.

Conjecture 1. *For any graph F such that the list F-coloring is NP-complete, there is a weighted extension (H, C, K) of F such that the (H, C, K)-coloring is also NP-complete.*

All of our results on NP-completeness indicate that this frontier depends not only on $H - C$ but also on the structure imposed by the weighted vertices. However, we observe that in all our complexity results – positive or negative – this dichotomy does not depend on the choice of the numbers in K when K is positive.

Another conjecture, let (H, C, K) be a weighted assignment. We call it *compact* when each connected component H_i of H satisfies one of the following:

(1) $E(H_i - C) = \emptyset$,

(2) $H_i[C]$ is a non-empty reflexive clique with all its vertices adjacent with one looped vertex of $H_i - C$, or

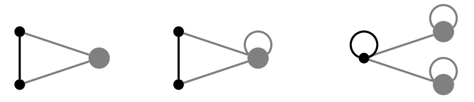
(3) $V(H_i) \cap C = \emptyset$ and H_i contains at least one looped vertex.

Conjecture 2. *For any partial weighted assignment (H, C, K), if (H, C, K) is compact then the (H, C, K)-coloring problem, with the numbers in K as parameters, is in FPT, otherwise it is W[1]-hard.*

In support of our dichotomy conjecture we recall that the parameterized independent set problem, that is known to be W[1]-complete [DF99] falls in this area.

We mention that our second conjecture is quite general to express several open problems in parameterized complexity such as

1. (Parameter: k) Does G contain an independent set S, where $|S| = k$, and such that $G[V(G) - S]$ is bipartite? (This problem can be seen as a parameterization of 3-coloring where some color should be used exactly k times.)



(H_1, C_1, K_1) (H_2, C_2, K_2) (H_3, C_3, K_3)

Figure 2: Three weighted extensions conjectured to be W[1]-hard when K is parameterized

2. (Parameter: k) Is there a set $S \subseteq V(G)$, where $|S| = k$ and $G[V(G) - S]$ is bipartite?
3. (Parameters: k, l) Does G contain $K_{k,l}$ as subgraph?

This problems correspond to the the parameterized colorings given in figure 2. The (H_3, C_2, K_3)-problem is equivalent with asking if the complement of G contains K_{k_1,k_2} as a subgraph.

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Graph semantics of the first order language of the clone theory

Věra Trnková

Let us start with the definitions. In categorical description, an *abstract clone* (called an *algebraic theory* in [2], [3]) is a small category C, the set of all objects of which are all finite powers a^n, $n \in \omega = \{0, 1, 2, \dots\}$, of a "basic" object a, i.e.

$$Obj\ C = \{a^0, a^1 = a, a^2, \dots\}$$

(we may suppose $a^i \neq a^j$ for $i \neq j$); moreover, for every n, an n-tuple of product projections

$$\pi_i^{(n)} : a^n \longrightarrow a$$

is specified and enumerated by elements $i \in n = \{0, \dots, n-1\}$. Notice that the zero-th power a^0 of a is just a terminal object of the category C, i.e. there is precisely one morphism $a^n \longrightarrow a^0$ in C, for every $n \in \omega$.

If $f_0, \dots, f_{n-1}$ is an n-tuple of C-morphisms $a^m \longrightarrow a$, we denote

$$f_0 \dot{\times} \cdots \dot{\times} f_{n-1}$$

their *fibre product*, i.e. the unique C-morphism $f : a^m \longrightarrow a^n$ for which

$$\pi_i^{(n)} \circ f = f_i \quad \text{for all} \quad i \in n.$$

Since a^n is supposed to be the categorical product of n copies of a, the above equations determine a bijection between all the C-morphisms $a^m \longrightarrow a^n$ and all the n-tuples of C-morphisms $a^m \longrightarrow a$. Hence C is fully determined by the sets $C(a^m, a)$ of all C-morphisms with the domain a^m and codomain a,$m \in \omega$, and by the following data:

- in each $C(a^m, a)$, the m-tuple $\pi_0^{(m)}, \dots, \pi_{m-1}^{(m)}$;
- for each n-tuple $f_0, \dots, f_{n-1} \in C(a^m, a)$ and $g \in C(a^n, a)$, the morphism $g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1}) \in C(a^m, a)$.

Choosing these data as the fundamental ones, one gets an *algebraic description of an abstract clone* (modified by [4], [5]): it is an ω-sorted universal algebra (i.e. heterogeneous in the sense of [1])

$$\mathbb{A} = (\ \{A_n\}_{n\in\omega}\ ,\ \{S^n_m\}_{m,n\in\omega}\ ,\ \{e^n_i\}_{i\in n\in\omega}\ )$$

where A_n is the carrier (= underlying set) of the n-th sort, S^n_m are operations

$$S^n_m : A_n \times (A_m)^n \longrightarrow A_m$$

and e^n_i are nullary operations of the sort n, i.e. $e^n_i \in A_n$, subject by the following axioms:

$$(ru) \qquad S^n_n(f; e^n_0, \dots, e^n_{n-1}) = f \quad \text{for every } n \in \omega\ ;$$

$$(lu) \qquad S^n_m(e^n_i; f_0, \dots, f_{n-1}) = f_i \quad \text{for every } n, m \in \omega, i \in n\ ;$$

$$(a) \qquad \begin{array}{c} S^m_p(S^n_m(g; f_0, \dots, f_{n-1}); h_0, \dots, h_{m-1}) = \\ = S^n_p(g; S^m_p(f_0; h_0, \dots, h_{m-1}), \dots, S^m_p(f_{n-1}; h_0, \dots, h_{m-1})) \\ \text{for all } m, n, p \in \omega. \end{array}$$

If C is an abstract clone with a basic object a in the categorical description, its algebraic description is obtained through putting

$$(*) \qquad \begin{array}{r} A_n = C(a^n, a), \\ e^n_i = \pi^{(n)}_i, \\ S^n_m(g; f_0, \dots, f_{n-1}) = g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1})\ . \end{array}$$

The "terrible" axiom (a) expresses the associativity of the composition in C.

The axioms (lu) and (ru) reflect the fact that $1_{a^n} = \pi^{(n)}_0 \dot{\times} \cdots \dot{\times} \pi^{(n)}_{n-1}$ is the left and the right unit of the composition in C.

Conversely, if

$$\mathbb{A} = (\ \{A_n\}_{n\in\omega}\ ,\ \{S^n_m\}_{m,n\in\omega}\ ,\ \{e^n_i\}_{i\in n\in\omega}\ )$$

satisfying (a), (lu), (ru) is given, we form an abstract clone C in the categorical description as follows: the unique "freedom" is the choice of the basic object a; then we use the formulas $(*)$ for the definition of $C(a^n, a)$ and

$\pi_i^{(n)}$; $C(a^n, a^m)$ is the set of all m-tuples of elements of $C(a^n, a)$, written in the form $f_0 \dot{\times} \cdots \dot{\times} f_{m-1}$ (if $m = 0$, there is precisely one 0-tuple, hence $C(a^n, a^0)$ is a singleton), and the categorical composition $\circ$ is defined so that for $h = h_0 \dot{\times} \cdots \dot{\times} h_{m-1} \in C(a^p, a^m)$ and $f = f_0 \dot{\times} \cdots \dot{\times} f_{n-1} \in C(a^m, a^n)$ we set

$$f \circ h \; = \; S_p^m(f_0; h_0, \ldots, h_{m-1}) \; \dot{\times} \; \cdots \; \dot{\times} \; S_p^m(f_{n-1}; h_0, \ldots, h_{m-1}).$$

The axiom (a) guarantees that the composition is associative, (lu), (ru) and the formulas $(*)$ guarantee that $e_0^n \dot{\times} \cdots \dot{\times} e_{n-1}^n$ is just the left and the right unit; the definitions and the axioms (a), (ru), (lu) also guarantee that a^n with the e_i^n's as product projections is the categorical product of n copies of a.

We conclude that the categorical description and the algebraic description of abstract clones are equivalent (up to isomorphism).

The algebraic description of abstract clones permits us to say immediately what is *the first order language of the clone theory.* Since abstract clones are (ω-sorted) algebras of a fixed (heterogeneous) signature, the first order language of the clone theory is just the first order language of these algebras: it has ω sorts of variables (variables of the n-th sort being denoted by $x^{(n)}, y^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots$), operational symbols S_n^m of type $m \times n^m \longrightarrow n$ and constants e_i^n with $i \in n \in \omega,\ m \in \omega$.

Then the terms of the n-th sort are constructed by the following rules:

- each variable of the n-th sort is a term of the n-th sort;
- each e_i^n is a term of the n-sort;
- $S_n^m(t; t_0, \ldots, t_{m-1})$ is a term of the n-th sort whenever t is a term of the m-th sort and $t_0, \ldots, t_{m-1}$ are terms of the n-th sort

(and there are no other terms of the n-th sort than those obtained by the above rules). Atomic formulas have the form $t_1 = t_2$ for terms t_1, t_2 of *the same sort.* The formulas are formed in the usual way by means of the logical connectives $\vee, \wedge, \neg, \longrightarrow, \longleftrightarrow$ and the quantifiers $\forall, \exists$.

This gives the *syntax* of the language: a sentence is valid syntactically if it can be deduced from the above axioms (a), (ru), (lu) and from the logical tautologies. Every category $\mathcal{K}$ with finite products endowes the language by a semantics: for every non-initial object a of $\mathcal{K}$ (in concrete categories, the initial object is usually the object with the empty underlying set), let us denote by $\mathcal{K}[a]$ the full subcategoty of $\mathcal{K}$, the set of all objects of which consists of finite powers of a (each in one copy). Then a sentence is valid in the semantics given by $\mathcal{K}$ iff it is satisfied in every clone $\mathcal{K}[a]$.

The author investigates the semantics given by Top , the category of all topological spaces, see [7]. For graph-theoretists, this language offers a large field of problems to examine the semantics of this language given by the most current categories of graphs, namely

$\mathcal{K}_1$ = the category of all directed graphs,
$\mathcal{K}_2$ = the category of all reflexive directed graphs,
$\mathcal{K}_3$ = the category of all undirected graphs,
$\mathcal{K}_4$ = the category of all reflexive undirected graphs.

In all the cases, the morphisms are all compatible maps and the products are the categorial products in the category in question.

Are all four semantics distinct? Clearly, the sentence

$$(\exists x^{(1)})(\forall y^{(1)})(S_1^1(x^{(1)};y^{(1)}) = x^{(1)})$$

[informally: for a graph G, there is always a morphism $x : G \to G$ such that $x \circ y = x$ for every morphism $y : G \to G$] is satisfied in all clones of $\mathcal{K}_2$ and $\mathcal{K}_4$ but not in all the clones of $\mathcal{K}_1$ and $\mathcal{K}_3$ so that these semantics are distinct. Also it is not difficult to see that the semantics of $\mathcal{K}_2$ and $\mathcal{K}_4$ are distinct but it is not clear whether also semantics of $\mathcal{K}_1$ and $\mathcal{K}_3$ are distinct.

The author investigates more complicated formulas of the language for the semantics given by Top and by the dual (=opposite) category $Top^{\,op}$. The last one can be regarded as the semantics given by Top in the dual language of the co-clone theory, see [8]. In the last one, it is no problem to express the connectedness for spaces: clearly, a topological space X is connected if and only if every continuous mapping of X into the coproduct $X + X$ of two copies of X factors through one of the two coproduct embeddings. This property can be expressed by a formula of the language of the co-clone theory; the same formula of the language of the co-clone theory gives the connectedness of graphs (in all $\mathcal{K}_1 - \mathcal{K}_4$). Is this formula semantically equivalent to a formula of the monoid theory (i.e. a formula in which appear only the variables of the first sort and only S_1^1, e_0^1)? Clearly *not*, for all $\mathcal{K}_1 - \mathcal{K}_4$: the graphs $G_1 = (\{0,1\}, R_1)$ and $G_2 = (\{0,1\}, R_2)$ with $R_1 = \{(0,0),(1,1)\}$ and $R_2 = R_1 \cup \{(0,1),(1,0)\}$ have the same endomorphism monoid but G_2 is connected and G_1 is not.

The dual formula (for co-connectedness) in the language of the clone theory is the sentence

$$(\forall x^{(2)})(\exists y^{(1)})((x^{(2)} = S_2^1(y^{(1)};\pi_0^{(2)})) \vee (x^{(2)} = S_2^1(y^{(1)};\pi_1^{(2)})))$$

[informally: every morphism $G \times G \to G$ factors through the 0-th or the 1-st projection; let us mention explicitly that $\times$ means *the categorial product* in the category in question; since products in $\mathcal{K}_1 - \mathcal{K}_4$ behave in distinct ways, it will probably influence the validity of the above sentence]. Is this sentence semantically equivalent to a sentence of the monoid theory in some of the semantics given by $\mathcal{K}_1 - \mathcal{K}_4$? The proof of the negative answer would require to find two graphs (in $\mathcal{K}_i$) with the equivalent endomorphism monoids such that one of them is co-connected (in the category $\mathcal{K}_i$) but the other one is not. To prove the positive answer, one has to find a formula $\mathcal{S}$ in the language of the monoid theory such that a graph in $\mathcal{K}_i$ is co-connected (in $\mathcal{K}_i$) if and only if it satisfies $\mathcal{S}$.

There is a lot of similar questions.

The semantics of the first order language of the clone theory given by Top forms the contents of the monograph [6]. A book about graph semantics could appear if *you* write it.

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Jarik's four Birthday Problems

Moshe Rosenfeld

1 Introduction

Problems are the fuel that feeds mathematical explorations. I agree with M. Simonovits, that it is important to expose problems repeatedly, in various forms, because for many of these problems, frequently a new idea, sometimes a revolutionary new idea is needed in order to solve them. In this note, I chose to discuss 4 problems, each explores an interaction of combinatorics with another area. Doufám že Jarik bude první, kdo to vyřeší.

2 Higher dimensional Vizing Theorem.

Vizing's classical theorem [5] asserts that for a simple graph, with maximum degree $\Delta(G)$, the chromatic index $\chi_1(G)$, that is the smallest number of colors needed to color the edges of G so that edges sharing a vertex have distinct colors satisfies:

$$\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1. \tag{1}$$

From a pure abstract geometrical point of view, Vizing's theorem can be rephrased as follows:

Let G be a 1-dimensional simplicial complex such that every 0-dimensional simplex is contained in at most $\Delta(G)$ 1-simplices. Then the 1-simplices of G can be colored in $\Delta(G) + 1$ colors so that two 1-simplices that intersect in a 0-simplex have distinct colors.

A natural generalization to higher dimensions (Vizing's generalized problem) might read like:

Let G_d be a d-dimensional simplicial complex such that every (d-1)-dimensional simplex is contained in at most $\Delta(G_d)$ d-simplices. Then the d-simplices of G_d can be colored in $\Delta(G_d) + d$ colors so that two d-simplices that intersect in a (d-1)-simplex have distinct colors.

We ask whether Vizing's generalized problem holds for every positive integer d. That is for a simplicial d-comlex G_d is it true that:

$$\Delta(G_d) \leq \chi_d(G_d) \leq \Delta(G_d) + d. \qquad (2)$$

For $d = 2$ the question is whether any 3 uniform hypergraph G_2, in which the number of edges containing any pair of vertices $\{x, y\}$ is $\leq \Delta(G_2)$, can be colored by $\Delta(G_2) + 2$ colors so that two edges (triples) that share two vertices will be assigned distinct colors.

The following examples indicate that the answer (at least for d = 2) is most likely affirmative.

1. Let T_n be the complete 3 uniform hypergraph on the vertices $\{0, 1, \ldots, n-1\}$. Note that for this hypergraph $\Delta(T_n) = n - 2$. It is easy to verify that the coloring $\chi_2(i, j, k) = i + j + k \pmod{n}$ is a proper coloring of the edges of T_n by $n = \Delta(T_n) + 2$ colors. For $n = 4, 5, 6$ this scheme gives an optimal coloring (the case $n = 6$ was observed by Doug West). For most n we can do better. We first note that in general, if T is a 3-unifrom hypergraph then $\chi_2(T) \geq \Delta(T)$. If it was possible to color the $\binom{n}{3}$ edges of T_n by $n-2$ colors then each monochromatic class would determine a Steiner Triple System on an n-set. This means that this can only be done for $n \equiv 1, 3 \pmod{6}$. Indeed for these values ($n \geq 9$), it is possible to partition T_n into disjoint Steiner Triple Systems [1] (Chapter 15: Large sets and partitions). That is $\chi_2(T_n) = n - 2 = \Delta(T_n)$ for these hypergraphs. For $n = 0, 2 \pmod{6}$, we get $\chi_2(T_n) = n - 1 = \Delta(T_n) + 1$ by coloring triples of T_{n+1} by n+1 colors as above, and deleting one vertex and all edges (triples) containing it. Clearly, this is best possible. For $n = 4, 5 \pmod{6}$ the question whether $\chi_2(T_n) = n + 1$ is open. Tuvi Etzion [2] shows that $\chi_2(T_{11}) = 10$. At any rate, for all n, $\chi_2(T_n) \leq \Delta(T_n) + 2$.

2. Let B_{3n} be the complete tripartite 2-complex. That is the vertices of B_{3n} consist of 3 disjoint n-sets A, B, C and every triple $\{a_i, b_j, c_k\}$ where $a_i \in A, b_j \in B, c_k \in C,\ 0 \leq i, j, k < n$ forms an edge. Clearly $\Delta(B_n) = n$. Again, the coloring scheme $\chi(\{a_i, b_j, c_k\}) = i + j + k \pmod{n}$ is a proper coloring and thus $\chi_2(B_n) = \Delta(B_n)$.

3. Our last example exhibits an even closer relation with Vizing's theorem.

Let G be a simple graph with $\Delta(G) = k$. Define the 3-uniform hypergraph H_G as follows:
$V(H_G) = V(G)$
$\{x, y, z\} \in E(H_G)$ iff the subgraph induced by $\{x, y, z\}$ in G determines exactly two edges.

It is easy to see that $\Delta(H_G) \leq 2(k-1)$. Let $\phi : E(H_G) \to E(L(G))$ be the mapping that associates with every triple the unique edge in the line graph of G, connecting the two edges detremined in G by the triple. Clearly, if two triples share an edge, then their corresponding edges in $L(G)$ will share a vertex there. If the girth of $G \geq 5$, the edges of $L(G)$ account for all possible interesection of two triples in H_G. Thus a proper coloring of the edges of $L(G)$ will induce a proper coloring of the triples of H_G. Since $\Delta(L(G)) \leq 2(k-1)$ by Vizing's theorem, $\chi_1(L(G)) \leq 2k-1$ and thus $\chi_2(H_G) \leq 2k-1 = \Delta(G_H)+1$. Note that we have one color to spare. I conjecture that for all graphs G the corresponding 3-uniform hypergraph H_G has chromatic index $\chi_2(H_G) \leq \Delta(H_G)+2$. This could be established for instance by proving that it is possible to color the edges of the line graph $L(G)$ by $2\Delta(G)$ colors so that the 4 edges of every induced 4-cycle have distinct colors.

3 Equi-partite graphs.

Definition A graph G of order $2n$ is called ***equi-partite*** if $\forall A \subset V(G)$, $|A| = n$ the subgraph spanned in G by A is isomorphic to the subgraph spanned by $V(G) \setminus A$.

The cycles C_4 and C_6, the 3-cube, $K_{2n}, K_{n,n}, nK_2, K_{n,n} - nK_2$ (the complete bipartite graph less than a perfect matching) and their complements are all equi-partite. Clearly, if G is equi-partite then so is its complement $\overline{G}$.

Problem 1: *Does the list $K_{2n}, K_{n,n}, nK_2$ and their complements include all equi-partite graphs ?*

Conjecture: *If the graph G is equipartite then $\forall A \subset V(G),\ |A| = \frac{|V(G)|}{2}$ there is an automorphism $\psi_A \in Aut(G)$ that maps A onto $V(G) \setminus A$*

If the conjecture holds, it can be used to prove Problem 1. The proof will go along the following lines: first it implies that $Aut(G)$ acts transitively on the $2n$ vertices of G. There are $\binom{2n}{n}$ sets. For each set there is an automorphism between the set and its complement. A single automorphism can "acommodate" relatively few sets, thus the automorphism group must be "very large". This can be used to show that it acts 2-homogeneously on half the vertices of G. But this means that this set must be either a complete subgraph or an empty subgraph. While this proof outline is not trivial, it will give an affirmative answer to problem 1. It might be possible that proving Problem 1 directly might be easier. This will give us the conjecture as a corollary.

4 Seidel matrices with smallest eigenvalue -4.

A set of n lines through the origin in R^d is **θ-equiangular** if the (smaller) angle between any pair of ditinct lines is θ. For instance, the four diagonal of the regular cube in R^3 is a set of 4 $\arccos\frac{1}{3}$-equiangular lines. In [3] it was shown that for all $k = 2n + 1, \theta_k = \arccos\frac{1}{k}$, there are m θ_k-equiangular lines in R^d for all dimensions $d \geq k$ and some $m > d$. Frequently, m can be much greater than d. For instance, in R^6 there are 16 $\arccos\frac{1}{3}$-equiangular lines. The situation is very different for $k = 2n$. In this case there can be at most $d+1$-equiangular lines in R^d and these can exist only when $d \equiv 2n \pmod 4$. An attempt to find $(4d + 1)$ $\arccos\frac{1}{4}$-equiangular lines in R^{4d} yielded a dismal success. We were able to construct these elusive sets of lines only in dimensions $4, 8, 12, ..., 48$. For instance, the five lines connecting the origin in R^4 to the five vertices of the regular 4-simplex form 5 $\arccos\frac{1}{4}$-equiangular lines.

Definiton The *Seidel* matrix of a graph G of order n is the $n \times n$ matrix S where $S_{i,i} = 0,\ S_{i,j} = -1$ if $ij \in E(G)$ and $S_{i,j} = 1$ if $ij \notin E(G)$.

As was pointed out by Seidel in his many explorations of equiangular lines [4], the existence of $m > d$ $\arccos\theta$-equiangular lines in R^d is equivalent to finding a graph G of order m, whose Seidel matrix G has smallest eigenvalue $-\frac{1}{\theta}$ with multiplicity $m - d$.

Conjecture $\forall k \geq 1$ *there are* $4k + 1$ $\arccos\frac{1}{4}$*-equiangular lines in* R^{4k}

To prove this conjecture, one needs to construct graphs of order $4k + 1$ whose Seidel matrices have smallest eigenvalue -4 (one does not have to worry about the multilicity, this eigenvalue will be simple). While it is not difficult to construct Seidel matrices with eigenvalue -4 it does not look easy to ascertain that it will be the smallest eigenvalue.

5 Odd distances in the plane.

What is the smallest number of colors needed to color the points of the plane so that points at unit distance will have distinct colors? This is Nelson's classical problem. It is known that at least 4 colors are needed. Hadwiger showed a coloring by 7 colors using an appropriate hexagonal tiling of the plane.
It is not possible for all 6 distances detrmined by 4 points in the plane to be odd integers. This leads to the following natural generalization of Nelson's problem:

Problem A: *Can the points in the plane be colored by a finite number of colors so that two points at odd integral distance will be assigned distinct colors?*

I do not know a configuration that requires 5 colors. Clearly, the points on a line can be colored by two colors avoiding monochromatic pairs at an odd integral distance and so do the lattice points $\{(n, n)\}$. A simpler problem that may help solve Problem A is:

Problem B: *Can the points of any circle in the plane be colored by 3 colors so that two points at odd integral distance will be assigned distinct colors?*

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On the Maximum Size of (p,Q)–free Families

Miklós Ruszinkó

Let p be a positive integer and let Q be a subset of $\{0, 1, \ldots, p\}$. Call p sets $A_1, A_2, \ldots, A_p$ of a ground set X a (p, Q)-system if the number of sets A_i containing x is in Q for every $x \in X$. In hypergraph terminology, a (p, Q)-system is a hypergraph with p edges such that each vertex x has degree $d(x) \in Q$. A family of sets $\mathcal{F}$ with ground set X is called (p, Q)-free if no p sets of $\mathcal{F}$ form a (p, Q)-system on X. We address the Turán type problem for (p, Q)-systems: $f(n, p, Q)$ is defined as $\max |\mathcal{F}|$ over all (p, Q)-free families on the ground set $[n] = \{1, 2, \ldots, n\}$.

We study the behavior of $f(n, p, Q)$ when p is fixed (allowing 2^{p+1} choices for Q) while n tends to infinity. The new results of this paper mostly relate to the middle zone where $2^{n-1} \leq f(n, p, Q) \leq (1 - c)2^n$ (in this upper bound c depends only on p). This direction was initiated by Paul Erdős who asked about the behavior of $f(n, 4, \{0, 3\})$. In addition we give a brief survey on results and methods (old and recent) in the low zone (where $f(n, p, Q) = o(2^n)$) and in the high zone (where $2^n - (2 - c)^n < f(n, p, Q)$). The results were obtained by Zoltán Füredi, András Gyárfás and Miklós Ruszinkó.

Lower Bounds on Sets Supporting Fáry Drawings

Hubert de Fraysseix, Patrice Ossona de Mendez

1 Fáry Drawing on a Grid

It is well known that any planar graph of order n has a Fáry drawing on a grid of size $O(n) \times O(n)$ [1] [2]. In [1], it was shown that a grid of size $\lfloor\frac{2n}{3}\rfloor \times \lfloor\frac{2n}{3}\rfloor$ (where each dimension denotes the number of coordinates) is necessary to draw any planar graph of order n (with prescribed outer face). At this time, experimentations made us conjecture that the minimum grid size allowing Fáry drawing with prescribed outer face is $(1+\lfloor\frac{2n}{3}\rfloor) \times (1+\lfloor\frac{2n}{3}\rfloor)$ for planar graphs of order n and $(1+\lfloor\frac{n}{2}\rfloor)\times(1+\lceil\frac{n}{2}\rceil)$ for 4-connected planar graphs of order n.

We shall explicit here some easy lower bounds for the grid size needed to draw specific families of planar graphs of order n with prescribed outer face. These bounds are easy consequences of the following simple geometric lemma :

Lemma 1. *Let $\mathcal{F}$ be a family of straight lines and let X be a bounded set of points, such that:*

- *every point in X belong to some line in $\mathcal{F}$,*
- *no two lines in $\mathcal{F}$ intersect in the convex hull of X,*
- *X is not included in a single line in $\mathcal{F}$.*

Then, there exists two lines in $\mathcal{F}$ which are tangent to the convex hull of X

Theorem 2. *The number $f(n)$ of coordinates needed in each direction of a grid allowing to draw (with straight lines and prescribed outer face) is, depending on the family of planar graphs of order n:*

$$f(n) = \begin{cases} \lceil\frac{n}{3}\rceil + \lfloor\frac{n}{3}\rfloor, & \textit{for planar graphs} \\ \lceil\frac{n+1}{4}\rceil + \lfloor\frac{n+1}{4}\rfloor, & \textit{for } 4\textit{-connected planar graphs} \\ \lceil\frac{n}{4}\rceil + \lfloor\frac{n}{4}\rfloor, & \textit{for bipartite planar graphs} \\ \lceil\frac{n+2}{5}\rceil + \lfloor\frac{n+2}{5}\rfloor, & \textit{for cyclically } 5\textit{-connected planar graphs} \\ \lceil\frac{n+3}{6}\rceil + \lfloor\frac{n+3}{6}\rfloor, & \textit{for } 3\textit{-regular } 3\textit{-connected planar graphs} \end{cases}$$

Remark that these bounds have to be divided by 2 if the outer face is not prescribed.

We conjecture that these bounds may be achieved, up to an additive constant. Moreover, we conjecture that there exists a universal constant C, such that, for any n, there exists a grid of area $Cn \log n$ which is allows to draw a series-parallel graph of order n with no prescribed embedding.

2 Drawing on Stars

Using a Breadth First Search numbering, it is easy to draw an internally triangulated outer-planar graph with straight lines, the vertices belonging to the union of three convergent halve lines.

From Lemma 1, we obtain that the number of convergent halve lines needed to draw a graph of order n is at least a linear function in n, for any of the families of planar graphs considered in Theorem 2.

On one hand, it is easily shown that a set of 3 convergent straight lines in $\mathbb{R}^d$ does not allow to draw any planar graph, as a drawing would imply a drawing on the plane. On the other hand, it is not obvious if 4 convergent halve lines in $\mathbb{R}^3$ allows to draw any planar graph or not.

In higher dimension, it is also easy to prove (using orientation arguments) that a graph of order n drawn on a family of k convergent halve-lines (of $\mathbb{R}^d$) have at most $kn - \binom{k+1}{2}$ edges.

3 Universal Grid

We conjecture that there exists a universal set U_n for planar graphs of order n of size roughly $n \log n$, that is: any planar graph of order n has a Fáry drawing with vertices on U_n.

The set U_n is the vertex set of the geometric graph G_n built by induction. Let G_n^0 be the triangle $\{v_1, v_2, v_3\}$ and let $X_0 = \{v_1, v_2, v_3\}$. Then, for any $i \leq \log_2 n$, the geometric graph G_n^i is obtained from G_n^{i-1} as follows: Start with X_i equal to X_{i-1}. For any interior face, add a vertex v in its center, add three straight paths of length $\frac{n}{2^i}$ from v to the three vertices of the face belonging to X_{i-1} and add v to X_i.

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Graphs and Orders on Integers – Old Results and New Questions

Hans Jürgen Prömel

The celebrated result of van der Waerden (1928) on arithmetic progressions says that for every 2-coloring of every sufficiently large $n = n(s,t) = \{0,\ldots,n-1\}$ there exists either an arithmetic progression of length s which is monochromatic in color 0 or there exists an arithmetic progression of length t which is monochromatic in color 1. It is easy to see that for every n there are colorings of $\binom{n}{2}$, i.e., of the pairs in n, such that there is neither an arithmetic progression of length 3 which is monochromatic in color 0 nor there is an arithmetic progression which is monochromatic in color 1. In other words, there exist graphs on n such that every arithmetic progression of length 3 simultaneously contains a pair which is joined by an edge and contains another pair which is not joined by an edge.

Erdős (1995) raised the question what can be said if one considers only special graphs on n, e.g. graphs which do not contain a clique of a given size. Then indeed it is true that one can always find an arithmetic progression of given length which is completely contained within an independent set. More precisely: In every K_s-free graph on every sufficiently large $n = n(s,t) = \{0,\ldots,n-1\}$ there exists an independent set which contains an arithmetic progression of length t. This result was proved by Gunderson, Leader, Prömel and Rödl (2001).

The proof of the theorem strongly relies on similar results for graphs on combinatorial cubes. As it turns out these results are closely related to results on the canonical orders of combinatorial cubes which were obtained already at the beginning of the 1980s by Jarik Nešetřil together with Rödl, Voigt and the author.

Tic-Tac-Toe or Is the dual of an algebraic matroid algebraic?

Winfried Hochstättler

1 Introduction

We present a matroid that has some interesting properties related to the theory of algebraic matroids.

Algebraic matroids, although already known from the very beginning of matroid theory [7] are far less understood than matroids that are linearly representable over some field. This is illustrated by the fact that it is still not known whether the dual of an algebraic matroid is algebraic or not, although this problem has been posed in several prominent places.

Definition 1. *Let* k *be a field and* $k \subseteq K$ *a field extension. A finite subset* $\{v_1, \ldots, v_k\} \subseteq K$ *is called* algebraically dependent *over* k *if there exists a non-trivial polynomial* $0 \neq p(x_1, \ldots, x_k) \in k[x_1, \ldots, x_k]$ *such that*

$$p(v_1, \ldots, v_k) = 0.$$

The set K *is called* algebraically independent*, otherwise.*

Algebraic independency satisfies the axioms of matroid theory (see e.g. 6.7.1 in [6]).

Definition 2. *A matroid* $M = (E, \mathcal{I})$ *on a finite set* E *is called* algebraic*, if there exists a field extension* $k \subseteq K$ *a subset* $E' \subseteq K$ *and a bijection* $\sigma : E \to E'$ *such that for all* $I \subseteq E$

$$I \in \mathcal{I} \Leftrightarrow \sigma(I) \text{ is algebraically independent over } k.$$

Tt has been shown by Pfiff (1969) that algebraic matroids are a superclass of the class of linear matroids and by Ingleton (1971) that this superclass is proper (see [6] 6.7.10). It took until 1975 to show that the Vámos matroid is an example of a non-algebraic matroid [4].

2 Combinatorial Properties of Algebraic Matroids

The key lemma in the proof that the Vámos matroid is non-algebraic was generalized by A. Dress and L. Lovász to the "series reduction theorem" [3].

Definition 3. *Let* $M = (E, \mathcal{I})$ *be a matroid on a finite set* E *and* $S \subseteq A \subseteq E$. *Then* S *is* in series in A *if contracting* $A \setminus S$ *turns* S *into a circuit.*

Theorem 4 (Dress-Lovász 1987). *Let* $M = (E, \mathcal{I})$ *be an algebraic matroid represented by a set* $E' \subseteq K$ *over a field k,* S' *in series in* $A' \subseteq E'$ *and* K *algebraically closed. Then there exists* $\beta \in K$ *such that* $\forall T' \subseteq A' \setminus S'$:

$$S' \cup T' \text{ is algebraically dependent}$$
$$\Updownarrow$$
$$\beta \cup T' \text{ is algebraically dependent.}$$

From this it is seen that the Vámos matroid is non-algebraic as follows:

The Vámos matroid V is defined on an eight point set $\{a, a', b, b', c, c', d, d'\}$ by its bases. All four point sets are bases except for the five four point planes $\{\{a, a', b, b'\}, \{a, a', c, c'\}, \{b, b', c, c'\}, \{b, b', d, d\}, \{c, c', d, d'\}\}$. In particular $\{a, a', d, d'\}$ is independent. Assume V were algebraically represented over a field extension $k \subseteq K$. Since $\{a, a'\}$ is in series in $\{a, a', b, b', c, c'\}$ there exists a β_1 in the algebraic closure of K which lies on the intersection of the two "lines" $\beta_1 \in \overline{b, b'} \cap \overline{c, c'}$. We denote the closure of $V \cup \beta_1$ by overlining the sets. We also have $\beta_1 \in \overline{a, a', b, b'} \cap \overline{a, a', c, c'} = \overline{a, a'}$. By symmetry there also exists $\beta_2 \in \overline{a, a'} \cap \overline{b, b'} \cap \overline{d, d'}$. Since β_1 and β_2 lie in the intersection of the same lines they must be parallels. Thus the two lines $\overline{a, a'}$ and $\overline{d, d'}$ have an intersection of rank at least one, contradicting the independence of the set $\{a, a', d, d'\}$.

All proofs of non-algebraicity, known to the author, apply the series reduction theorem, postulate additional points and derive a contradiction.

3 The Tic-Tac-Toe Matroid

A matroid that already has all the points required by Theorem 4 is called *pseudomodular* [2]. The Tic-Tac-Toe matroid presented below is pseudomodular and its dual is non-algebraic.

The Tic-Tac-Toe matroid (see Figure 3) is defined as a rank five matroid on the points $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Any five points are independent except for the eight "five point hyperplanes" $\{1, 2, 3, 4, 7\}$, $\{1, 2, 3, 5, 8\}$, $\{1, 2, 3, 6, 9\}, \{4, 5, 6, 1, 7\}, \{4, 5, 6, 3, 9\}, \{7, 8, 9, 1, 4\}, \{7, 8, 9, 2, 5\}, \{7, 8, 9, 3, 6\}$. These are the sets that have the shape of an L or a T in Figure 3. Note, that $\{4, 5, 6, 2, 8\}$ is independent.

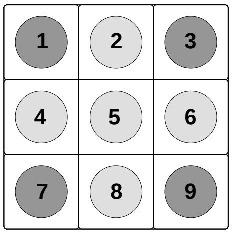

Figure 3: The Tic-Tac-Toe Matroid

It is proven in [1] that this matroid is pseudomodular. We present a proof that its dual is non-algebraic following [5]. This fact had been observed before, independently by M. Alfter and by B. Lindström. Note, that the dual consists of "three prisms in a triangle with one broken hyperplane $\{1, 3, 7, 9\}$". In particular $\{5, 8, 6, 9\}$, $\{4, 7, 6, 9\}$ and $\{4, 7, 5, 8\}$ form such a prism as well as $\{2, 5, 3, 6\}$, $\{1, 4, 3, 6\}$ and $\{1, 4, 2, 5\}$. Assume this matroid were algebraic. According to the series reduction theorem there exist $a \in \overline{4, 7} \cap \overline{5, 8} \cap \overline{6, 9}$ and $a' \in \overline{1, 4} \cap \overline{2, 5} \cap \overline{3, 6}$. Since $\{a, a'\}$ is a subset of $\overline{1, 4, 7}$ as well as of $\overline{2, 5, 8}$ and $\overline{3, 6, 9}$, the eight points $\{1, 2, 3, 7, 8, 9, a, a'\}$ form a Vámos matroid, contradicting algebraic representability.

4 Conclusion

Is the Tic-Tac-Toe matroid algebraic? Known techniques do not suffice to prove it is not. If it where algebraic, this would settle the old question whether the dual of an algebraic matroid is algebraic to the negative. Up to now, we did not have any better idea, than to search for algebraic coordinates with trial and error. Unfortunately we did not get beyond error.

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On Exact Algorithms for the Closest String Problem

Rolf Niedermeier

CLOSEST STRING is one of the core problems in the field of consensus word analysis with particular importance for computational biology.

> Given k strings of same length and a positive integer d, find a "closest string" s in the sense that none of the given strings has Hamming distance greater than d from s.

CLOSEST STRING is NP-complete. In biological practice, however, d is usually very small.

We can show how to solve CLOSEST STRING in linear time for constant d (the exponential growth in d is bounded by $(2d)^d$) and arbitrary k. We can extend this result to the closely related problems d-MISMATCH and DISTINGUISHING STRING SELECTION. Moreover, we can give a linear time algorithm for CLOSEST STRING when $k = 3$ and d is arbitrary. It remains open to generalize this result to arbitrary constant k. Also, the generalization CLOSEST SUBSTRING of CLOSEST STRING deserves further investigation.

The practical usefulness of our findings is substantiated by some experimental results.

A long paper is available upon request. (Joint work with Jens Gramm (Tübingen) and Peter Rossmanith (München).)

The 3-Flow Conjecture and Related Problems

Martin Kochol

The graphs considered here are all finite, unoriented, and loopless. Multiple edges are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. With each edge of G there are associated two distinct *arcs* (see [4]). Arcs on distinct edges are distinct. If an arc on an edge is denoted by x the other is denoted by x^{-1}. If the ends of an edge e of G are vertices u and v, one of the arcs on e is said to be *directed* from u to v and the other one is directed from v to u. Let $D(G)$ denote the set of arcs on G. If $v \in V(G)$, then the set of arcs of G directed out from v is denoted by $\omega_G^+(v)$.

Let A be an additive Abelian group. By an *A-chain* in G we mean a mapping φ from $D(G)$ to A such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. We say that φ is *nowhere-zero* if $\varphi(x) \neq 0$ for every arc x of G. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that

$$\partial\varphi(v) \;=\; \sum_{x \in \omega_G^+(v)} \varphi(x) \qquad (v \in V(G))$$

is called the *boundary* of φ. An A-chain φ in G is called an *A-flow* in G if $\partial\varphi(v) = 0$ for every vertex v of G. By a *nowhere-zero k-flow* in G we mean a nowhere-zero $\mathbb{Z}$-flow in G satisfying $|\varphi(x)| < k$ for every arc x of G. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in [2], [8], [10]. The only difference is that instead of a fixed (but arbitrary) orientation of a graph G we use the set $D(G)$ as a domain for a flow.

By Tutte [9], a graph has a nowhere-zero k-flow iff it has a nowhere-zero $\mathbb{Z}_k$-flow (see, e. g., [2], [8], [10]). The 3-flow (resp. 5-flow) conjecture of Tutte is that every bridgeless graph without 3-edge-cuts (resp. every bridgeless graph) has a nowhere-zero 3-flow (resp. 5-flow). It is easy to check that the 3-flow conjecture reduces to 4-edge-connected graphs.

An edge cut is called *trivial* if it is the set of edges incident to a single vertex. In [7] we have proved the following statement.

Theorem 1. *The following statements are pairwise equivalent.*

(1) Every bridgeless graph without a 3*-edge-cut has a nowhere-zero* 3*-flow.*

(2) Every 5*-edge-connected graph has a nowhere-zero* 3*-flow.*

Note that, by [6], every 5-edge-connected graph has a nowhere-zero 3-flow if and only if every 5-regular 5-connected simple hamiltonian graph has a nowhere-zero 3-flow.

A *circular k-flow* φ in a graph G is a nowhere-zero $\mathbb{Z}_k$-flow in G such that $\varphi(x) \in \{\pm 1\}$ for every $x \in D(G)$. The circular flow conjecture of Jaeger [1] is that every $4t$-edge-connected graph has a circular $(2t+1)$-flow. For $t = 1$ this is equivalent to the 3-flow conjecture. For $t = 2$ Jaeger [1] proved that the 5-flow conjecture holds true if every 9-edge-connected graph has a circular 5-flow (see [1], [6], [10] for more details). Thus the 3- and 5-flow conjectures are implied by the following one (see [7]).

Conjecture 1. *Every* $(4t+1)$*-edge-connected graph has a circular* $(2t+1)$*-flow.*

Since Conjecture 1 is equivalent to the circular flow conjecture for $t = 1$, it would be of some interest to study whether these two conjectures are equivalent for every $t \geq 2$.

Let A be an additive Abelian group and φ be a nowhere-zero A-flow in a graph G. Then by [4], [5], $\sum_{v \in V(G)} \partial\varphi(v) = 0$. We say that G is A-*connected* if for every mapping $b : V(G) \to A$ satisfying $\sum_{v \in V(G)} b(v) = 0$ there exists a nowhere-zero A-chain φ in G such that $\partial\varphi(v) = b(v)$ for every vertex v of G. This concept was introduced by Jaeger, Linial, Payan, and Tarsi [3], where is also proved that every bridgeless (4-edge-connected) graph is $\mathbb{Z}_6$-connected ($\mathbb{Z}_4$-connected). There is a conjecture from [3], that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. Thus, by Theorem 1, the 3-flow conjecture is a special case of this conjecture. Note that there are known 4-connected graphs wich are not $\mathbb{Z}_3$-connected (see [3], [10]).

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A Pisier type problem

Jiří Matoušek

The following problem is motivated by abstract analogues of the celebrated (p, q) theorem of Alon and Kleitman, but it is also related to Pisier-type problems. Let G be a graph. Let a blowup of a vertex v mean replacing the vertex v in G by a clique and connecting each vertex of the clique to all the vertices of G to which v was connected. Let us say that a graph G has the (α, β)-clique property (where $\alpha, \beta \in (0, 1)$ are some real numbers) if every k-vertex subgraph of G with at least $\alpha\binom{k}{2}$ edges contains a clique of size βk, $k = 1, 2, \ldots$. Is the following true? For every $\alpha, \beta \in (0, 1)$ there exists $\beta' \in (0, 1)$ such that if G is a graph with the (α, β)-clique property then every graph H obtained from G by repeated vertex blowups has the (α, β')-clique property. Maybe the parameter α is superfluous; one should probably first understand the case with $\alpha = 0$.

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