A dualistic approach to bounding the chromatic number of a graph

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Abstract

We give a new and more direct proof of the characterization theorem for finitary homomorphism dualities of directed graphs. This result may be viewed as a characterisation of Gallai-Hasse-Roy type theorems. We exhibit

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infinitely many examples where this general setting improves the bounds for chromatic number of graphs and we relate this to extremal problems for oriented graphs.

1 Introduction

In this paper we deal mostly with oriented graphs. However the whole paper is motivated by coloring problems for undirected graphs and we use oriented graphs as a tool for estimating chromatic number (and variants like oriented chromatic number) of undirected graphs. To this problem we apply the homomorphism techniques developed mainly in [9].

The connection between chromatic number and orientations is not new and goes back to [3, 11]. These pioneering works provided a name for the result although both of these papers are anticipated by M. Hesse [4] where she proved the same thing (in the algebraic language of category theory). Another influential connection between orientetions ad chromatic number is given by [6] but that goes in a different direction (flows and matroids). For our purposes Gallai-Hesse-Roy result takes the following compact form:

Theorem 1 *For any directed graph G the following holds:*

$$P_k \not\to G \Longleftrightarrow G \to T_k$$

The undefined notions have the following meaning: P_k denotes the directed path of length k (i.e. with $k+1$ vertices), T_k denotes the transitive tournament with k vertices and $G \to H$ ($G \not\to H$, respectively) denotes the existence (and non-existence) of a homomorphism $G \to H$.

It may be seen easily that for undirected graph this has the following consequence

Corollary 2 *For an undirected graph G the following statements are equivalent:*

1. $\chi(G) \leq k$ *(which is equivalent to $G \to K_k$);*
2. *There exists an orientation $\vec{G}$ of G such that $\vec{G} \to T_k$;*
3. *There exists an orientation $\vec{G}$ of G such that $P_k \not\to \vec{G}$.*

This particular result was one of the starting points for the discovery of the following result which also gave a new name to this type of results. We state the theorem for oriented graphs only:

Theorem 3 *(Homomorphism Dualities) ([9])*

For every oriented tree T there exists an oriented graph D_T (called the dual *of T) such that the following holds:*

$$T \not\to G \Longleftrightarrow G \to D_T.$$

This paper is motivated by efforts to use this more general theorem for estimating various extremal problems for graphs. There are two main obstacles to this project. The first is that the description of the graphs D_T is complicated and ineffective even in small examples. The second obstacle is more subtle and we introduce two numbers first.

For every directed acyclic graph G we can associate number $h(G)$ -*height* of G- as the longest oriented path P_k with $P_k \to G$. By Theorem 1 $h(G) = \min\{k; G \to T_k\}$. For oriented graph G define also the *algebraical length* $al(G)$ of G as minimal k such that there exists a homomorphism $G \to P_k$. (If there is no such k we put $al(G) = \infty$.) Obviously every tree T has a finite algebraical length and $al(T)$ is simply the number of "levels" which we need in a "leveled" drawing of T. It follows that $T \not\to G$ implies $P_{al(T)} \not\to G$ and thus we have an obvious upper bound for all graphs G with $T \not\to G$:

$$\chi(G) \leq al(T).$$

It is not clear whether we can beat this bound by applying Theorem 3. This is the second obstacle.

In this paper we show that both of these problems may be resolved. First, we give a new proof (independent of [9]) of Theorem 3 which yields an explicite description of graph D_T for any oriented tree. Although the construction is simple and much easier then the proof given in [9] the following is still open:

Problem 1 *Can one choose D_T of a polynomial size (in $|V(T)|$)?*

Note that up to homomorphism equivalence the dual D_T is uniquely determined. However the exponential construction of D_T which we give for a tree T may have a much smaller *core* (i.e. the minimal retract). In the full generality of the paper [9] one needs exponential large size dual models however for graphs (or hypergraphs of a fixed arity) the problem is still open. Problem 1 is the simplest case.

In some cases one can find core of the dual D_T. This is so in Theorem 1. As a further example consider for the orientation P' of the path of length 5 which we get from the directed path of length 5 by reversing the middle arc the dual $D_{P'}$ is the only orientation of a cycle of length 5, which does not contain directed path of length 3, see [5]. Recently, authors found explicite description of all cores of D_T

for all trees with $al(T) = 3$, [10]. In all these cases we have $al(P') = 3$ and also $\chi(D_{P'}) = 3$ and thus duality does not help.

Let us consider the next example: P'' is the orientation of the directed path of length 6 which we get from the directed path of length 6 by reversing of the third arc. Then one can show that the dual $D_{P''}$ has the following form: the vertices are $1, 2, \ldots, 6$ with edges $(1,2), (2,3), (1,4), (4,5), (5,6), (1,3), (1,5), (2.6)$. It is easy to color this graph by 3 colors: $c(i) = i$ for $i = 1, 2, 3$, $c(4) = 2, c(5) = 3, c(6) = 1$ ($D_{P''}$ is uniquelly 3-colorable). Now duality helps: it is $al(P'') = 4$ but $\chi(D_{P''}) = 3$. In Proposition 4 we give infinitely many such examples.

In Section 4 we include some further remarks which relate duality theorems to further conjectures and exrtremal for directed graphs.

2 General construction of duals

Let $\vec{T}$ be a directed tree. We define its dual $D_{\vec{T}}$ by

$$V(D_{\vec{T}}) = \{f : V(\vec{T}) \mapsto V(\vec{T}) : (u, f(u)) \in A(\vec{T}) \text{ or } (f(u), u) \in A(\vec{T}) \text{ for all } u \in V(\vec{T})\},$$
$$A(D_{\vec{T}}) = \{(f, g) : \text{ for all } (u, v) \in A(\vec{T}), \text{ we have } f(u) \neq v \text{ or } g(v) \neq u\}.$$

We now prove that this construction of $D_{\vec{T}}$ has the properties claimed by Theorem3.

Proof of Theorem 3. We first show that $\vec{T} \not\to D_{\vec{T}}$. Suppose that $\phi : \vec{T} \mapsto D_{\vec{T}}$ is a homomorphism, and denote $f_u = \phi(u)$ for all $u \in V(\vec{T})$. Select $u_0 \in V(\vec{T})$ and define $(u_n)_{n>0}$ recursively by $u_n = f_{u_{n-1}}(u_{n-1})$. Then u_n is a neighbour of u_{n-1} for all $n > 0$. Let n be the smallest index such that $u_n = u_m$ for some $m < n$. Since $\vec{T}$ is a tree, we must then have $n = m + 2$. We then have $f_{u_m}(u_m) = u_{m+1}$ and $f_{u_{m+1}}(u_{m+1}) = u_m$, but this contradicts the definition of $D_{\vec{T}}$: Since u_m and u_{m+1} are neighbours, f_{u_m} and $f_{u_{m+1}}$ should be neighbours hence we should have $f_{u_m}(u_m) \neq u_{m+1}$ or $f_{u_{m+1}}(u_{m+1}) \neq u_m$. This shows that $\vec{T} \not\to D_{\vec{T}}$, hence for any directed graph $\vec{G}$ such that $\vec{G} \to D_{\vec{T}}$, we have $\vec{T} \not\to \vec{G}$.

It remains to show that if $\vec{T} \not\to \vec{G}$, then we can define a homomorphism $\phi : \vec{G} \to D_{\vec{T}}$. We will use the following definitions:

- Let u be a vertex of $\vec{T}$ and v a neighbour of u. We denote $\vec{T}_{u,v}$ the maximal subtree of $\vec{T}$ which contains u and v but not any other neighbour of u.
- We define a labelling of the arcs of $\vec{T}$ by characterising the arcs of label n recursively as follows: The leaves of $\vec{T}$ have label 1. For $n > 1$, the leaves of

the subtree of $\vec{T}$ obtained by removing all arcs of label smaller than n have label n.

Let x be a vertex of $\vec{G}$. Since $\vec{T} \not\to \vec{G}$, every vertex u of $\vec{T}$ has a neighbour v for which there exist no homomorphism $\psi : \vec{T}_{u,v} \mapsto \vec{G}$ such that $\psi u = x$. Select such a vertex with maximal label, and call it $f_x(u)$. This defines a map $f_x : V(T) \mapsto V(T)$ which belongs to the vertex set of $D_{\vec{T}}$. We will show that $\phi : \vec{G} \mapsto D_{\vec{T}}$ defined by $\phi(x) = f_x$ is a homomorphism.

Let (x, y) be an arc of $\vec{(}G)$. We need to show that (f_x, f_y) is an arc of $D_{\vec{T}}$, that is, for every arc (u, v) of $\vec{T}$, we have $f_x(u) \neq v$ or $f_y(v) \neq u$. Let (u, v) be an arc of $\vec{T}$. Note that either the labels of all other arcs incident to u are smaller than that of (u, v), or the labels of all other arcs incident to v are smaller than that of (u, v). In the first case, $\vec{T}_{v,u}$ admits a homomorphism to $\vec{G}$ mapping v to y, whence $f_y(v) \neq u$. In the second case, $\vec{T}_{u,v}$ admits a homomorphism to $\vec{G}$ mapping u to x, whence $f_x(u) \neq v$. Therefore, (f_x, f_y) is an arc of $D_{\vec{T}}$, and ϕ is a homomorphism. ■

This proof together with the correspondence between gaps and dualities and by the density proof for oriented graphs presents simplest way how to prove the full version of the Duality Characterization Theorem [9], see Theorem 5 below.

3 Almost directed paths

In this section, we consider the directed paths $P_{m,n}$ with m forward edges followed by one backward edge followed by n forward edges, where $m \geq 3$ and $n \geq 2$. The algebraic length of $P_{m,n}$ is $m + n - 1$, hence by the Gallai-Hasse-Roy theorem, if a graph G admits an orientation $\vec{G}$ such that $P_{m,n} \not\to \vec{G}$, then $\chi(G) \leq m + n - 1$. We can improve this bound by 1:

Proposition 4 *Let G be a graph which admits an orientation $\vec{G}$ such that $P_{m,n} \not\to \vec{G}$. Then $\chi(G) \leq m + n - 2$.*

Proof. Let $\vec{G}$ be an orientation of G such that $P_{m,n} \not\to \vec{G}$. For every vertex u of G, we associate the vector (a_u, b_u) where a_u is the length of the longest path of $\vec{G}$ ending at u, and b_u is the length of the longest path of $\vec{G}$ starting at u. Note that $a_u + b_u \leq m + n - 2$ for every vertex u. Also, if (u, v) is an arc of $\vec{G}$, then $a_v < m - 1$

or $b_u < n-1$. We define the colouring $c : V(G) \mapsto \{0, 1, \ldots, m+n-3\}$ by

$$c(u) = \begin{cases} 0 \text{ if } (a_u, b_u) = (m+n-2, 0) \\ a_u \text{ if } a_u + b_u \geq n \text{ and } a_u \neq m+n-2 \\ a_u + 1 \text{ if } a_u + b_u < n \end{cases}$$

Let (u, v) be an arc of $\vec{G}$. Then $b_u \geq 1$ whence $c(u) = a_u$ or $a_u + 1$. If $(a_v, b_v) = (m+n-2, 0)$, then $c(v) = 0$, but we cannot have $c(u) = 0$, because it would imply $a_u = 0$ and $b_u \geq n$ whence $P_{m,n} \to \vec{G}$. If $(a_v, b_v) \neq (m+n-2, 0)$, then $c(v) = a_v$ or $a_v + 1$. Note that $a_v > a_u$ and $b_v < b_u$. Therefore the only possible trouble is that we have $c(u) = a_u + 1 = a_v = c_v$. But then $n \leq a_v + b_v \leq a_u + b_u < n$ which is impossible. Therefore c is a proper colouring of G with $m+n-2$ colours. ■

This bound is sharp if $\min\{m, n\} = 2$. In this case the tournament T_m or T_n has the chromatic number meeting the bound in Proposition 4. Another example is the path $P_{3,3}$. In this case the graph which we get from K_4 by a single operatio of Hájos join has chromatic number 4 and it can be oriented in such a way so that it does not contain a homomorphic image of $P_{3,3}$. However it is not clear whether the bound in Proposition 4 is sharp in general.

4 Some extremal problems

For (directed or undirected) graphs G, H we write $G \leq H$ if there exists a homomorphism $G \to H$. This defines a quasiorder called *coloring* (or homomorphism) *order*. The coloring order has a very complicated structure, see e.g. [9, 7] for results in this direction. Using this order the main result (for graphs) of [9] may be stated easily. First let us introduce one more notation: Given a finite set of connected graph $\mathcal{F}$ denote by $\mathrm{Forb}_h(\mathcal{F})$ the class of all graphs G for which hold

$$F \in \mathcal{F} \Rightarrow F \not\leq G$$

Thus $\mathrm{Forb}_h(\mathcal{F})$ contains those graphs which do not contain any $F \in \mathcal{F}$ nor any of its homomorphic images. For $\mathcal{F} = \{C_{2k+1}\}$ these are just graphs with the odd girth $> 2k+1$.

Theorem 5 *([9]) The class* $\mathrm{Forb}_h(\mathcal{F})$ *has the greatest element (with respect to the coloring order* $\leq$*) if and only if* $\mathcal{F}$ *is a finite set of trees.*

Our Theorem 3 is a special case of Theorem 5: the result is claimed for a single tree and in one direction only. (However we have a simpler proof for this.) We say that a class $\mathcal{C}$ of graphs is *bounded* by a graph H if $G \leq H$ for all $G \in \mathcal{C}$ (note that it is possibly $H \notin \mathcal{C}$). We complement Theorem 5 by characterizing the bounded classes $\mathrm{Forb}_h(\mathcal{F})$ of oriented graphs.

Corollary 6 *For a class* $\mathrm{Forb}_h(\mathcal{F})$ *of directed graphs the following two statements are equivalent:*

1. $\mathrm{Forb}_h(\mathcal{F})$ *is bounded;*

2. $\mathrm{Forb}_h(\mathcal{F})$ *is bounded by a simple oriented graph (i.e. without digons) and thus the* oriented chromatic number *of graphs from* $\mathrm{Forb}_h(\mathcal{F})$ *is bounded.*

3. *For an* $F \in \mathcal{F}$ *there exists* n *with* $F \leq P_n$.

Proof. Clearly 2. ⇒ 1.. 1. ⇒ 3. follows by contradiction: If there exists an $F \in \mathcal{F}$ which is not homomorphic to an oriented path then every homomorphic image of F contains an orientation of a cycle. Thus if G is any orientation of a graph with girth $> \max\{|V(F_i); F_i \in \mathcal{F}\}$ then $G \in \mathrm{Forb}_h(\mathcal{F})$ and high chromatic graphs G show that $\mathrm{Forb}_h(\mathcal{F})$ is not a bounded class. 3. ⇒ 1. and 3. ⇒ 2. is now clear as the class $\mathrm{Forb}_h(\mathcal{F})$ is a subclass of $\mathrm{Forb}_h(\{P_n\})$ which is bounded by a simple oriented graph by virtue of Theorem 3. ■

Note that the situation changes for more restricted classes of graphs when we consider e.g minor closed classes, [8]. For undirected graphs the Theorems 3, 5 are easy as the only bounded class is the class of discrete graphs (so all above results are true for both directed and undirected graphs but the essence of the results is for oriented graphs). Yet the above results are reminiscent to results and problems for (undirected or directed) graphs when we change the definition of classes $\mathrm{Forb}_h(\mathcal{F})$: Let $\mathrm{Forb}(\mathcal{F})$ ($\mathrm{Forb}_i(\mathcal{F})$, respectively) denotes the class of all undirected graphs G which do not contain any $F \in \mathcal{F}$ as a subgraph (as an induced subgraph, respectively).

The following is then a reformulation of the classical Erdős result ([2] and [7] for this formulation):

Theorem 7 *For a finite set* $\mathcal{F}$ *of connected graphs the following are equivalent:*

1. $\mathcal{F}$ *contains a tree;*

2. $\mathrm{Forb}(\mathcal{F})$ *is a bounded class.*

It is interseting to note for the oriented graphs we cannot add the third condition about bounded oriented chromatic number as it is easy to find counterexamples. One can also formulate *Gyarfás - Sumner Conjecture* in this setting:

Problem 2 *(Gyarfás - Sumner) Is the class* $\mathrm{Forb}_i(\mathcal{F}) \cap \mathrm{Forb}(K_k)$ *bounded for every* $\mathcal{F}$ *and* k*.*

Note that this last conjecture does not hold for oriented chromatic number (so the bound has to have some digons even for the case of simple oriented graphs).

Let us finish this paper with a result related to extremal problems, see [1] for an introduction to graph theory extremal problems. As an easy application of our main theorem 5 we are going to show that the forbidding a set of oriented tree gives an exact solution to corresponding extremal problem. Towards this end let us define for a set $\mathcal{F}$ of oriented trees $ext_h(\mathcal{F}, n)$ *hom-extremal number* as the maximal number of arcs of an oriented graph (V, E) with $|V| = n$ and $(V, E) \in \mathrm{Forb}_h(\mathcal{F})$. The extremal problems induced by homomorphisms were studied by Sidorenko [12] in his important study of continuous versions of Turán's theorem (and other extremal result). As a consequence of Theorem 3 we shall prove an exact result for all instances covered by Theorem 3:

Theorem 8 *For any finite set* $\mathcal{F}$ *of oriented trees holds*

$$ext_h(\mathcal{F}, n) = ext(k, n)$$

where $k = \max\{h(F); F \in \mathcal{F}\}$*. Recall that* $h(F)$ *is the maximal number of edges of a monotone of* F *and* $ext(k, n)$ *is the Turán number for* n *vertices and clique of size* k*; asymptotically ext(k, n) equals to* $\frac{n^2}{2}(1 - \frac{1}{k-1})$*.*

Proof. Let $F \in \mathcal{F}$ be a tree with the smallest height. Recall that the height $h(F)$ is both the maximal length of an oriented path in F and $1 + h(F)$ is also the minimal size of transitive tournament T_k for which there exists a homomorphism $F \to T_k$. Thus we have $F \not\to T_{h(F)}$ and thus by the duality $P_{h(F)} \to D_{\mathcal{F}}$. Thus $D_{\mathcal{F}}$ contains transitive tournament $T_{h(F)}$ as a subgraph. On the other hand $D_{\mathcal{F}}$ does not contain any tournament T' with more than $h(F)$ vertices again by the Duality Theorem as clearly $F \not\to D_{\mathcal{F}}$. (Perhaps this needs a bit of explanation: a tournament T' is either transitive and then it contains $T_{h(F)+1}$ and the existence of homomorphism follows from the definition of the height of F. The other possibility is that T' is not transitive but then it contains a directed cycle and we have again $F \to T'$.) Finally, as every graph $G \in \mathrm{Forb}(\mathcal{F})$ has a homomorphism into $D_{\mathcal{F}}$ we see that the symmetrization of the extremal graph does not contain complete graph K_{k+1} while it may contain any complete k-multipartite graph. Thus we apply Turán's Theorem. ■

Finally, let us remark that as we can prove duality theorem for relational systems we have some exact extremal results even there. This will appear elsewhere.

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