On maximal finite antichains in the homomorphism order of directed graphs

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Abstract

We show that the pairs $\{T, D_T\}$ where T is a tree and D_T its dual are the only maximal antichains of size 2 in the category of directed graphs endowed with its natural homomorphism ordering.

1 Introduction

Let G, H be directed graphs. A *homomorphism* from G to H is an arc-preserving map ϕ from the vertex set of G to that of H. We write $G \to H$ if there exist a homomorphism from G to H, and $G \not\to H$ otherwise. G and H are called *homomorphically equivalent* if $G \to H$ and $H \to G$; we then write $G \leftrightarrow H$. A directed graph G is called a *core* if it is not homomorphically equivalent to any directed graph with fewer vertices. The relation $\to$ induces a natural order on the classes of homomorphically equivalent graphs, called the "skeleton" of the category of finite directed graphs. Thus the standard order-theoretic terminology can be applied here: If $G \not\to H$ and $H \not\to G$ then G and H are called *incomparable*; we then write $G||H$. The pair $\{G, H\}$ is called a *maximal* 2-*antichain* if G and H are incomparable and every directed graph is comparable to G or H. Though this is a purely order-theoretic definition, we will see that these maximal 2-antichains are related to an important algorithmic phenomenon.

A couple (P, D) of directed graphs is called a *duality* if for every directed graph G, we have $P \to G$ if and only if $G \not\to H$. This relationship is denoted by the equation

$$P \to = \not\to D \tag{1}$$

where $P \to$ denotes the class of graphs admitting a homomorphism from P and $\not\to D$ the class of graphs not admitting a homomorphism to D. Here, "P" and "D" stand respectively for "primal" and "dual": A greedy search for a homomorphism from a graph G to D would require $|D|^{|G|}$ steps, but (1) allows to replace it by a greedy search for a homomorphism from P to G in $|G|^{|P|}$ steps. Thus the existence of duality (1) implies that the problem of deciding whether a given directed graph G admits a homomorphism into D is solvable in polynomial time. The dualities in the category of directed graphs are characterised in [3, 7] (see also [4, 5]):

Theorem 1 ([3, 7]) *Given a directed graph P, there exists a directed graph D_P such that (P, D_P) is a duality pair if and only if P is homomorphically equivalent to an orientation of a tree.*

The bottom of the skeleton of the category of directed graphs consists of the directed paths P_0, P_1 and P_2, where P_i is the directed path with i edges. It is not hard to see that any other directed graph is comparable to all three of these paths.

As we shall see below any other core tree T satisfies $T||D_T$. As any other directed graph is comparable to T or D_T, $\{T, D_T\}$ is a maximal 2-antichain. We will show that in fact the maximal 2-antichains are characterised in this way.

Theorem 2 *The maximal 2-antichains in the category of directed graphs are precisely the pairs $\{T, D_T\}$ where T is a core tree different from P_0, P_1, P_2 and D_T is its dual.*

The characterisation of larger maximal antichains does not seem so simple: For instance, given two incomparable trees T_1, T_2 with duals D_1, D_2, one can find the anichain $\{T_1 \cup T_2, D_1, D_2\}$ as well as $\{T_1, T_2, D_1 \times D_2\}$. More possibilities arise when considering larger families of pairwise incomparable trees. And it is not clear either whether all maximal antichains are related to trees in a similar fashion.

The situation is even more complex with relational systems of a given type Δ: [7] contains the characterisation of all duality pairs for all such relational systems. As above these duality theorems induce (with a few exceptions) maximal antichains. However a result similar to Theorem 2 is not valid as we have infinitely other situations not covered by duality theorems. Let us consider just the simplest example of relational systems of type $\Delta = (2, 2)$ which we may view as Blue-Red colored oriented graphs (with homomorphisms preserving the colors). Then the systems $(\{0, 1\}, \{(0, 1)_B\})$ (i.e. the single blue arc) and $(\{0, 1\}, \{(0, 1)_R\})$ (i.e. the single red arc) are not a duality pair while it is easy to see that they constitute a maximal antichain. There are 2 more such exceptional pairs.

Note also that there are infinitely many quintuplets of the form $\{(\{0, 1, 2, 3\}, \{(0, 1)_R, (2, 3)_B\}), T_B, D_{T_B}, T_R, D_{T_R}\}$. All these quintuplets are maximal 5-antichains for 2-coloured graphs. But perhaps there is a hope that for a given type Δ, Theorem 2 admits only finitely many exceptions, which correspond to the "irregularities" at the bottom of the lattice.

Let us also note that the problem is interesting and hard for infinite graphs. It has been proved in [6] that for every countable infinite graph G, G not equivalent to K_1, K_2, K_ω, there exists a graph H such that $G||H$. There are also infinitely many maximal such antichains, however, as pointed in [6], all maximal antichains seem to contain a finite graph.

2 Gaps in the category of directed graphs

The initial observation towards a proof of Theorem 2 is the following:

Lemma 3 *Let $\{G, H\}$ be a maximal antichain. Then one of the following holds:*

- *For any directed graph G' such that $G \times H \to G' \to G$ we have $G' \leftrightarrow G \times H$ or $G' \leftrightarrow G$.*

- *For any directed graph H' such that $G \times H \to H' \to H$ we have $H' \leftrightarrow G \times H$ or $H' \leftrightarrow H$.*

Proof. Let G', H' be such that $G \times H \to G' \to G$ and $G \times H \to H' \to H$. Then the disjoint union K of G' and H' is comparable to G or to H. If $K \to G$, then $H' \to G$ whence $H' \to G \times H$. Similarly, $K \to H$ implies $G' \to G \times H$. On the other hand, $G \to K$ implies $G \to G \times (G' \cup H) \to G'$, and similarly $H \to K$ implies $H \to H'$. ■

Following [7], a couple (X, Y) of directed graphs is called a *gap* if $X \to Y$, $Y \not\to X$ and for every K such that $X \to K \to Y$ we have $K \leftrightarrow X$ or $K \leftrightarrow Y$. According to the previous lemma, if $\{G, H\}$ is a maximal antichain, then $(G \times H, G)$ or $(G \times H, H)$ is a gap. The gaps in the category of directed graphs are characterised as follows:

Theorem 4 ([7] Theorems 2.8, 3.1) *A couple (X, Y) is a gap in the category of directed graphs if and only if there exists a tree T such that $T \times D_T \to X \to D_T$ and $Y \leftrightarrow X \cup T$.*

Note that this result brings us closer to the proof of Theorem 2: Let $\{G, H\}$ be a maximal antichain. Then without loss of generality $(G \times H, H)$ is a gap, hence there exists a directed tree T such that $H \leftrightarrow (G \times H) \cup T$. We then have $T \not\to G$ whence $(D_T \times T) \to (G \times H) \to G \to D_T$. We now need to show that $H \leftrightarrow T$ and $G \leftrightarrow D_T$. To carry out this task we will need a greater knowledge of the bottom part of the category of directed graphs.

3 Duals of thunderbolts

The *algebraic length* of a tree T is the least n such that T admits a homomorphism to P_n. The cores with algebraic length at most 2 are just P_0, P_1 and P_2. We define the *thunderbolt* $T_n, n \geq 1$ as follows:

- The vertices of T_n are the integers from 0 to $2n-1$, along with two additional vertices A and B.
- The arcs of T_n are $(2i, 2i+1), 0 \leq i \leq n-1$, $(2i, 2i-1), 1 \leq i \leq n-1$, $(A, 0)$ and $(2n-1, B)$.

Note that the thunderbolts have algebraic length 3. For any tree T with algebraic length $n \geq 3$ and any homomorphism $\phi : T \mapsto P_n$, any path of shortest length between a vertex of $\phi^{-1}(0)$ and $\phi^{-1}(3)$ is a thunderbolt. Nešetřil and Zhu citenz have shown that the thunderbolts $T_n, n \geq 1$ are precisely the cores trees with algebraic length 3. We have $T_1 \leftrightarrow P_3$ and $T_{n+1} \to T_n$ for all n. Therefore, for every tree T with algebraic length at least 3, there exists some n_0 such that $T_n \to T$ for all $n \geq n_0$. Thus, the bottom of the category of directed graphs consists of the paths P_0, P_1, P_2, above which we find the infinite decreasing chain $\{T_n : n \geq 1\}$.

Let D_n be the graph obtained from T_{n+1} after removing the vertices A, B and identifying the vertices 0 and $2n+1$ to a new vertex labelled 0. Then D_n is an orientation of the $(2n+1)$-cycle with a unique directed path with two edges, namely $\{2n, 0, 1\}$. We have $T_n \not\to D_n$, since any homomorphism from T_n to D_n would have to identify both vertices 0 and $2n-1$ of T_n to the vertex 0 of D_n. The path in T_n connecting 0 to $2n-1$, alternating in forward edges and backward edges, would then have to be mapped to the rest of the cycle, but the cycle turns out to be just a bit too short. In fact, this argument is the basis of the following result:

Lemma 5 *D_n is the dual of T_n.*

Proof. Let G be a directed graph. If $T_n \to G$, then $G \not\to D_n$ since $T_n \not\to D_n$. It remains to show that if $T_n \to G$, then there exists a homomorphism $\phi : G \mapsto D_n$.

A vertex v of G is called a *middle point* of G if both its indegree and its outdegree are at least one. Let M be the set of middle points of G. Note that any vertex of G that is not in M is either a source or a sink. We define

$$f : V(G) \mapsto \{0, 1, 2, \ldots, \infty\}$$

by letting $f(u)$ be the length of a shortest path that starts in M and ends in u, alternating in forward and backward arcs (with the first arc being a forward arc). We can now define the homomorphism $\phi : G \mapsto D_n$ by

$$\phi(u) = \begin{cases} f(u) \text{ if } 0 \leq f(u) \leq 2n, \\ 2n \text{ if } f(u) > 2n \text{ and } u \text{ is a source}, \\ 0 \text{ if } f(u) > 2n \text{ and } u \text{ is a sink}. \end{cases}$$

Clearly $(\phi(u), \phi(v))$ is an arc of D_n whenever (u, v) is an arc of G and $f(v) > 0$. In the case where $f(v) = 0$ we must have $f(u) \geq 2n$, otherwise we could define a homomorphism from T_n to G. Hence $\phi(u) = 2n$ and ϕ is a homomorphism.

■

Corollary 6 *Let T be a tree with algebraic length at least 3. Then $\{T, D_T\}$ is a maximal antichain.*

Proof. If T has algebraic length at least 3, then there exists some n such that $T_n \to T$. By Lemma 5 we then have $T \not\to D_n$ whence $D_n \to D_T$. Therefore, D_T contains an odd cycle and $D_T \not\to T$. By the definition of dualities, we also have that $T \not\to D_T$ (since $D_T \to D_T$) and any other directed graph is comparable to T or D_T. Therefore $\{T, D_T\}$ is a maximal antichain. ■

4 Proof of Theorem 2

Lemma 7 *Let T be a tree with algebraic length at least 3. Then for every directed graph X such that $D_T \times T \to X \to D_T$ and $D_T \not\to X \not\to D_T \times T$,* *there exists a directed graph Y such that $D_T \times T \to Y \to D_T$ and $X||Y$.*

Proof. Since T has algebraic length at least 3, there exists some n such that $T_n \to T$. We then have

$$T_{n+1} \to (T_n \times D_n) \to (T \times D_T) \to X.$$

In particular this shows that $X \not\to P_2$; in fact we may even assume that no connected component of X admits a homomorphism to P_2. It is still possible for X to be bipartite, but in this case for $m > |V(X)|$ we have $X \not\to D_m$ (since removing any vertex from D_m results in a graph which admits a homomorphism to P_2), and $D_m \not\to X$ (since X is bipartite). Therefore $X||(D_m \cup (D_T \times T))$.

Suppose that X is nonbipartite. As seen in [7, 8], for any undirected graph W with odd girth larger than $|V(X)|$ and chromatic number larger than $|V(X)|^{|V(D_T)|}$, we have $X \not\to (W \times D_T)$ and $(W \times D_T) \not\to X$. The classical theorem of Erdős [1] guarantees the existence of such a graph W. We then have $(T \times D_T) \to (W \times D_T) \to D_T$, and $X||(W \times D_T)$. ∎

Now, all that is needed to complete the proof of Theorem 2 is to apply this result at the point where we left it at the end of Section 2. We know that if $\{G, H\}$ is a maximal antichain, then without loss of generality there exists a directed tree T such that $D_T \times T \to G \to D_T$ and $H \leftrightarrow (G \times H) \cup T$. Of course, T has algebraic length at least 3. The interval from $D_T \times T$ to D_T is isomorphic to the interval from T to $D_T \cup T$, with the natural lattice-theoretic isomorphisms between the two.

Now, if $G \not\leftrightarrow D_T$ then by the previous result there exists Y between $D_T \times T$ and D_T such that $Y||G$. We have $T \not\to Y$ whence $H \not\to Y$; and $Y \not\to H$ (for otherwise we would have $Y \to D_T \times H \leftrightarrow G \times H \to G$). Therefore G, H and Y are pairwise incomparable, contradicting the fact that $\{G, H\}$ is a maximal antichain.

Similarly, if $H \not\leftrightarrow T$, then $G \times H \not\leftrightarrow D_T \times T$, hence there exists Y between $D_T \times T$ and D_T such that $Y||(G \times H)$. We then find that G, H and $(Y \cup T)$ are pairwise incomparable, again contradicting the fact that $\{G, H\}$ is a maximal antichain. ∎

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