Hajós' theorem for list colorings of hypergraphs*

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October 24, 2001

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Abstract

A well-known theorem of Hajós claims that every graph with chromathic number greater than k can be constructed from disjoint copies of the complete graph K_{k+1} by repeated application of three simple operations. This classical result has been extended in 1978 to colorings of hypergraphs by C. Benzaken and in 1996 to list-colorings of graphs by S. Gravier. In this note, we capture both variations to extend Hajós' theorem to list-colorings of hypergraphs.

*This research was partially supported by the PROTEUS Project 00874RL.

†Supported in part by the Ministry of Science and Technology of Slovenia, Research Project Z1-3129.

[1]Supported in part by the Ministry of Education of Czech republic, Project LN00A056.

1 Introduction

In 1961, Hajós [5] gave a constuction of the graphs that are not k-colorable. The constuction use the following simple operations:

(1) Add a new vertex or edge.

(2) Let G_1, G_2 be two vertex-disjoint graphs, and a_1b_1 and a_2b_2 be edges in G_1 and G_2, respectively. Make a graph G from $G_1 \cup G_2$ by deleting the edge a_ib_i from G_i (for $i = 1, 2$), identifying a_1 and a_2 (the resulting vertex is called a_1a_2), and adding a new edge b_1b_2 (see the Figure 1).

(3) Identify two non-adjacent vertices.

Theorem 1.1 (Hajós) *Every non-k-colorable graph can be constucted by operations* (1)-(3) *from disjoint copies of the complete graph* K_{k+1}.

This classical result has been extended to colorings of hypergraphs by Benzaken [1, 2] and to list-colorings of graphs by Gravier [4]. In this note we capture both variations to extend Hajós' theorem to list-colorings of hypergraphs. However, Zhu[8] gave an analogue of Hajós' theorem for the circular chromathic number. Recently, the classical result was extended by Mohar [6] in three slightly different ways to colorings and circular colorings of edge-weighted graphs (enhancing the channel assignment problem as well). Moreover, it is mentioned in [6] that one of these extensions sheds some new light of the fact that today no nontrivial application of Hajós' theorem is known.

2 Hajós' theorem for list colorings of hypergraphs

In a hypergraph $\mathcal{H}$, the set of vertices and the set of hyperedges are denoted by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. Given a hypergraph $\mathcal{H}$, a *k-coloring* of the vertices of $\mathcal{H}$ is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that for every hyperedge e of $\mathcal{H}$ there exists two vertices $x, y \in e$ with $c(x) \neq c(y)$, or shortly $|c(e)| \geq 2$. A hypergraph $\mathcal{H}$ is *k-colorable* if it admits a k-coloring, and the *chromatic number* of $\mathcal{H}$ is the smallest integer k such that $\mathcal{H}$ is k-colorable.

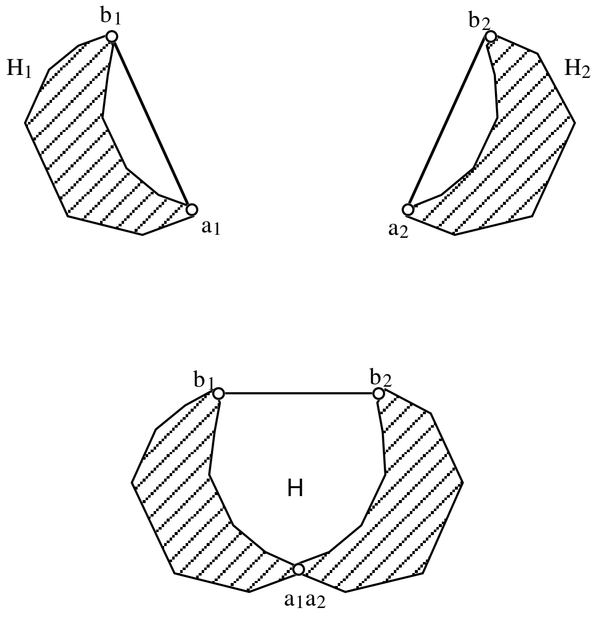

Figure 1: Operation (2).

Vizing [7] and independently Erdős, Rubin, and Taylor [3] introduced the concept of list colorings. This concept could be naturaly extended to hypergraphs in the following way. Suppose that each vertex v is assigned a list $L(v)$ of possible colors; we then want to find a vertex-coloring c such that $c(v) \in L(v)$ for all $v \in V(\mathcal{H})$. In the case where such a coloring exists we will say that the hypergraph $\mathcal{H}$ is *L-colorable*; we may also say that c is an *L-coloring* of $\mathcal{H}$. Given an integer k, the hypergraph $\mathcal{H}$ is called *k-choosable* if it is L-colorable for every assignment L that satisfies $|L(v)| \geq k$ for all $v \in V(\mathcal{H})$. Finally, the *choice number* or *list-chromatic number* $\chi_l(\mathcal{H})$ of $\mathcal{H}$ is the smallest k such that $\mathcal{H}$ is k-choosable.

Concernig the problem of coloring the hypergraphs, without lose of generality, we can restrict ourselfs on hypergraphs with the Sperner property, i.e. no hyperedge contains (as a subset) another hyperedge in a hypergraph. Indeed, if we have a coloring c of a hypergraph $\mathcal{H}$ and e, f are hyperedges of $\mathcal{H}$ with $e \subseteq f$, then condition $|c(e)| \geq 2$ implies that $|c(f)| \geq 2$. In all of our constructions given bellow, by deleting the superflous hyperedges of the new constructed hypergraph, we may assume that it has the Sperner property.

In order to obtain Hajós' theorem for list colorings of hypergraphs, we will use the following operations:

(H1) Add a new hyperedge e (possibly with new vertices) in a hypergraph $\mathcal{H}$. The new hypergraph is denoted by $\mathcal{H} \vee e$.

(H2) Let $\mathcal{H}_1, \mathcal{H}_2$ be two vertex-disjoint hypergraphs, and e_1 and e_2 be hyperedges in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Also, let $a_1 \in e_1$ and $a_2 \in e_2$. Make a hypergraph $\mathcal{H}$ from $\mathcal{H}_1 \cup \mathcal{H}_2$ by deleting the edge e_i from $\mathcal{H}_i$ (for $i = 1, 2$), identifying a_1 and a_2 (the resulting vertex is called $a_1 a_2$), and adding a new hyperedge $e_1 \cup e_2$.

(H3) If $\mathcal{H}$ is not L-colorable for some assignment with $|L(x)| \geq k$ for each $x \in V(\mathcal{H})$, then identify two vertices u and v of $\mathcal{H}$ with $L(u) = L(v)$ into a new vertex uv.

Remark that if $\mathcal{H}_1$ and $\mathcal{H}_2$ have the Sperner property then the hypergraph obtained from $\mathcal{H}_1$ and $\mathcal{H}_2$ by operation (H2) has also the Sperner property. Regarding the operation (H3), every hyperedge e which contains u or v is replaces by $e \setminus \{u, v\} \cup \{uv\}$.

Theorem 2.1 *A hypergraph $\mathcal{H}$ can be constructed with operations* (H1)-(H3) *from disjoint copies of any non-k-choosable bipartite graph if and only if $\chi_l(\mathcal{H}) \geq k \geq 2$.*

Proof. Note that introducing a new vertex or hyperedge in a given hypergraph could not decrease the choice-number. The same holds if we identify two (non-)adjacent vertices under the assumption of opertation (H3).

Now, we will show that the class of non-k-choosable hypergraphs is closed under operation (H2). We use the same notation as in its description. For $i = 1, 2$, since $\mathcal{H}_i$ is not k-choosable, there exists an assignment L_i with $|L_i(v)| = k$ for all $v \in V(\mathcal{H}_i)$ and such that $\mathcal{H}_i$ is not L_i-colorable. We may assume that $L_1(a_1) = L_2(a_2)$ by a suitable permutation of the colors.

Now, we create a list assignment L of $V(\mathcal{H})$ by setting $L(v) = L_i(v)$ in $v \in V(\mathcal{H}_i)$. We claim that $\mathcal{H}$ is not L-colorable. Indeed, suppose that there is an L-coloring c of $\mathcal{H}$. Then, $|c(e_1 \cup e_2)| \geq 2$ and $c(a_1a_2) = c(a_1) = c(a_2)$, we infer that either $|c(e_1)| \geq 2$ or $|c(e_2)| \geq 2$. Therefore, c is either an L_1-coloring of $\mathcal{H}_1$ or an L_2-coloring of $\mathcal{H}_2$, a contradiction. Since $|L(v)| = k$ for all $v \in V(\mathcal{H})$, this shows that $\mathcal{H}$ is not k-choosable.

Thus, the if part of the theorem is established. To prove the only part, we will prove first that every non-k-choosable hypergraph can be obtained by (H1)-(H3) starting with (hyper)graphs from the family of complete multipartite graphs.

So, assume that it is false and that there exists a counterexample $\mathcal{H}$. Then, there exists an assignment L with $|L(v)| = k$ for all $v \in V(\mathcal{H})$ such that $\mathcal{H}$ is not L-colorable.

Define a relation $\preceq$ on the hypergraphs which set of vertices is $V(\mathcal{H})$, in the following way:

$$\mathcal{H}_a \preceq \mathcal{H}_b \quad \text{if and only if} \quad \forall e_a \in E(\mathcal{H}_a)\ \exists e_b \in E(\mathcal{H}_b) \quad \text{such that} \quad e_b \subseteq e_a.$$

Obviously, $\preceq$ is a transitive and reflexive relation. By the Sperner property, it follows that this relation is also antisymmetric. So, it is a partial ordering. We say that $\mathcal{H}_b$ is *bigger* than $\mathcal{H}_a$ (regarding the relation $\preceq$). Note that if $\mathcal{H}_a$ is non-k-choosable then $\mathcal{H}_b$ is also non-k-choosable.

If $\chi_l(\mathcal{H}) = \infty$ then it contains a hyperedge with precisely one vertex. In that case, starting with K_{k+1} use (H3) to construct a single-vertex hyperedge, and afterwards use (H1) to obtain $\mathcal{H}$. Note that by preserving the Sperner property, we can use (H1) to introduce isolated vertices if necessary.

Now, we may assume that the choice-number of $\mathcal{H}$ is finite. We may also assume that $\mathcal{H}$ is as big as possible hypergraph regarding $\preceq$ (which is still not constructible). Thus, every bigger hypergraph than $\mathcal{H}$ is constructible.

In what follows, we will prove that for any independent sets I_1, I_2 with non-empty intersection, the set $I_1 \cup I_2$ is also independent. (Recall that a set is independent if it contains no hyperedge as a subset.) Consider the hypergraphs $\mathcal{H} \vee I_1$ and $\mathcal{H} \vee I_2$. Since I_1, I_2 are independent, we infer that $\mathcal{H} \preceq \mathcal{H} \vee I_i$ and $\mathcal{H} \neq \mathcal{H} \vee I_i$ for $i = 1, 2$. So, it follows that these two hypergraphs can be constructed from complete multigraphs by operation (H1)-(H3).

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two vertex-disjoint copies of $\mathcal{H} \vee I_1$ and $\mathcal{H} \vee I_2$, respectively. For every vertex x from $\mathcal{H}$, we denote by x_1 and x_2 its counterparts in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Let $a \in I_1 \cap I_2$. Now, use the same notation as in (H2) with I_1, I_2 playing the roles of e_1, e_2, we constuct a new hypergraph $\mathcal{H}^*$. Define an assignment L^* on $\mathcal{H}^*$ by setting $L^*(v_i) = L(v)$ for each $v \in V(\mathcal{H})$ and each $i = 1, 2$. Observe that $\mathcal{H}^*$ is not L^*-colorable. Finally, using the operation (H3), identify vertices x_1, x_2 from $\mathcal{H}^*$ for each vertex x of $\mathcal{H}$. It is easy to see that the new constructed hypergraph is isomorphic with $\mathcal{H}$ if and only if the set $I_1 \cup I_2$ is not independent. Therefore, if $I_1 \cup I_2$ is not an independent set, we obtain a construction of $\mathcal{H}$, which is a contradiction. So, the property for independet sets is established.

From this property, it easily follows that the relation $\sim$ on vertices of $\mathcal{H}$ defined as

$$a \sim b \qquad \text{if and only if} \qquad \{a\} \cup \{b\} \text{ is independent set,}$$

is an equivalent relation. In particular this means that $\mathcal{H}$ is a complete multipartite graph.

In [4], it was proven that using only rules (1) and (3), from any non-k-choosable bipartite complete, we can construct every non-k-choosable multipartite graph. To achieve the proof of the theorem, it is sufficient to observe that similarly using only rules (H1) and (H3), from any non-k-choosable bipartite graph we can construct every non-k-choosable bipartite complete graph. $\square$

Theorem 2.1 shows that, for fixed k, any minimal graph (for the subgraph relation) in the class of non-k-choosable bipartite graph form a basis for the non-k-choosability of hypergraphs.

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