Heavy paths, light stars, and big melons

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Abstract

A graph H is defined to be light in the family $\mathcal{H}$ of graphs if there exists a finite number $w(H,\mathcal{H})$ such that each $G \in \mathcal{H}$ which contains H as a subgraph, contains also a subgraph $K \cong H$ such that the sum of degrees (in G) of vertices of K (that is, the weight of K in G) is at most $w(H,\mathcal{H})$. In this paper we study the conditions related to weight of fixed subgraph of plane graphs which can enforce the existence of light graphs in families of plane graphs. For the families of plane graphs and triangulations whose edges are of weight $\geq w$ we study the necessary and sufficient conditions for lightness of certain graphs according to values of w.

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1 Introduction and preliminaries

Let $\mathcal{H}$ be a family of graphs and let H be a connected graph. Denote by $w(H,\mathcal{H})$ the smallest integer such that each graph $G \in \mathcal{H}$ containing a subgraph isomorphic with H, contains also a subgraph K such that K is isomorphic with H (or shortly, $K \cong H$) and

$$\sum_{v \in V(K)} d_G(v) \leq w(H,\mathcal{H}).$$

The sum on left side is called the *weight* of subgraph K in G. If such a finite number does not exist or there is no graph in $\mathcal{H}$ which contains H as a subgraph, then we set $w(H,\mathcal{H}) = +\infty$.

We say that a graph H is *light* in the family $\mathcal{H}$ provided $w(H,\mathcal{H})$ is finite, otherwise we call it *heavy*. The integer $w(H,\mathcal{H})$ is called the *weight* of H in the family $\mathcal{H}$. Denote by $\mathcal{L}(\mathcal{H})$ the set of graphs which are light in $\mathcal{H}$. Thus, if $\mathcal{H} = \emptyset$ then also $\mathcal{L}(\mathcal{H}) = \emptyset$, i.e. every graph is heavy in the empty class of graphs.

It is well known that every plane graph contains a vertex of degree at most 5. Kotzig [13] stated that each 3-connected plane graph contains an edge of weight at most 13 and at most 11 in the case of absence of 3-vertices, the bounds being sharp. This result was generalized in many directions; namely, it served as starting point for looking for subgraphs of small weight in plane graphs.

Borodin [2] extended this theorem by showing that every simple planar graph with minimum degree ≥ 3 has also an edge of weight ≤ 13. Fabrici and Jendrol' [7] proved that the only light graphs in the family of all 3-connected plane graphs are paths; this holds also for the family of all 3-connected plane graphs of minimum degree 4 (see [6]) and of minimum face size 4 (see [8]). In the family of plane graphs of minimum degree 5, there are light graphs other than paths [1, 9, 10, 15]. The survey of results on light graphs in various families of plane, projective plane, and higher genera graphs can be found in papers [11, 12].

Recently, there were studied light graphs in families of plane graphs with prescribed minimum vertex degree ≥ 4 and edge weight ≥ 9 [16]. These conditions can be viewed as weakening of requirement of minimum vertex degree ≥ 5. Following this motivation, we will study the light graphs in families of plane graphs with minimum edge weight at least w, $7 \leq w \leq 13$. In the second section, we show that the minimum vertex degree and minimum edge weight restrictions are - in certain sense - the only conditions which may force the existence of non-trivial light structures in the family of plane graphs with minimum vertex degree ≥ 3. The similar restrictions over conditions $\delta \geq 4$ and $\delta = 5$ are studied as well. Third section contains the necessary and sufficient conditions for lightness of some small cycles, paths, and stars according to the value of minimum edge weight; these results generalize the ones in [1, 9, 16].

Throughout this paper, we consider connected plane graphs without loops or multiple edges. Denote by P_n and C_n the path and the cycle on n vertices, respectively. By $d(v)$ we denote the degree of a vertex v in a graph, i.e. the number of edges incident with v. Vertices of degree n will be called *n-vertices*. By $r_k(f)$ we denote the number of k-vertices incident with the face f (endvertices of each cut edge incident with f are counted twice). Thus, $r(f) = \sum_{i\geq 1} r_i(f)$ is the *size* of the face f. Faces of size n are called *n-faces*, and 3-faces are also called *triangles*. By $N(x)$ we denote the set of all neighbours of a vertex x. If an edge uv is incident with k 3-faces, then we say that u and v are *k-adjacent* (where $k \in \{0, 1, 2\}$).

The *edge weight* of a graph G is just $w(K_2, G)$, i.e. the minimum of the sum of degrees of the endvertices of an edge in G. We will denote by $\mathcal{P}$ and $\mathcal{T}$ the class of planar graphs and its subclass of triangulations, respectively. A near-triangulation is a plane graph which all faces (except maybe the outerface) are triangles.

For proving that a certain graph is heavy in a class of graphs, occasionally we will use the following construction. Let G be a connected plane graph on at least 3 vertices and let u, v be two vertices lying on the outerface of G. By $(G, u, v; n)$-*melon* (or simply, *melon*) we denote the graph constructed in the following way: take n copies (*slices*) of G, identify all vertices corresponding to u and all vertices corresponding to v in all copies. In addition, if u and v are adjacent in G, then delete the multiple edges in the melon in order to obtain a simple graph. Two vertices resulted from this identification are called *poles*, the graph $G - u - v$ is the *pulp*. In our constructions melons are always planar graphs.

For proving the heaviness of a graph in a class of triangulations, we will use similar construction. Let G be a near-triangulation and let u and v be two vertices lying on the outerface of G. Take a n-wheel W_n (with nave z and fixed oriented cycle C of its spokes) and n copies of G. For each edge e of C, identify starting vertex of e with vertex u and ending vertex of e with vertex v of a copy of G; then delete all edges of original C. Let $f = [x_1 \dots x_k]$ be the outerface of so obtained graph. Insert into f three new vertices a, b, c and add new edges $ax_1, \dots, ax_{\lfloor \frac{k}{3} \rfloor + 1}$, $bx_{\lfloor \frac{k}{3} \rfloor + 1}, \dots, bx_{\lfloor \frac{2k}{3} \rfloor + 1}$,

$cx_{\lfloor \frac{2k}{3} \rfloor +1}, \ldots, cx_k,\ cx_1$. In addition, connect z with all non-adjacent vertices which are on a same face with z in order to obtain a near-triangulation. The resulting graph G' is called the $(G, u, v; n)$-*tripod*, vertices a, b, c are called *tripoles*, the graph $G' - a - b - c - z$ is also called the *pulp* of the tripod.

¿From other side, for proving the results concerning existence of light graphs in families of plane graphs, the Discharging method is used. In each proof of this type, we consider a hypothetical counterexample G with vertex set $V(G)$ and face set $F(G)$. We assign initial charge to every vertex $v \in V(G)$ and every face $f \in F(G)$ of the graph G in the following way:

$$c(v) = \alpha\, d(v) - 6 \quad \text{and} \quad c(f) = (3 - \alpha)\, r(f) - 6, \tag{1}$$

where α is some prescribed number. Hence, we can rewrite Euler's formula in the following form:

$$\sum_{v \in V(G)} c(v) + \sum_{f \in F(G)} c(f) = -12.$$

Thus, the total sum of the charge of the vertices and faces of G is negative. We will redistribute the charge of the vertices and faces of G by applying some rules without changing the total sum of all charges. Denote by $c^*(x)$ the charge of a vertex or a face x after applying the rules. It will be also called the *final charge* of x. In the proof of each claim, we will prove that each face and vertex of G has a nonnegative final charge. Since the total sum must be negative, it will be a contradiction and end of the proof.

2 Conditional lightness versus melon slices

Investigating a relation between existence of light graphs and requirements for a family of plane graphs, we can restrict to the family of planar graphs in which each copy of one or more prescribed graphs has large weight, which may imply that some graphs are getting light in this family. More precisely, let W be a set of pairs (G, w) where G is a graph and w is an integer. We will call W a *list of weight constraints*. If $\mathcal{G}$ is a class of graphs, then denote by $\mathcal{G}(W)$ the class of all graphs G from $\mathcal{G}$ such that for every pair $(H, w) \in W$, we have that every subgraph of G isomorphic to H has a weight $\geq w$. From this starting point, the next problem naturally arises:

Problem 2.1 *For a given list of weight constraints W, find all light graphs in $\mathcal{P}(W)$.*

Considering such a general problem, perhaps one should deal first with some basic properties of W and $\mathcal{P}(W)$. Without loss of generality we may assume that a graph can appear in the list W only once.

Claim 2.1 *The class of graphs $\mathcal{P}(W)$ is either empty or infinite.*

Proof. If $\mathcal{P}(W) \neq \emptyset$, then let G be a graph from $\mathcal{P}(W)$ and let n be an integer greater than any weight from W. We construct a new graph H from G in the following way: if $G \cong K_1$, then let $H \cong K_{1,n}$. Otherwise, $G \ncong K_1$, and in this case take n copies of G, select one vertex from each copy and identify all these vertices in order to obtain H. It is easy to see that the new constructed graph H belongs to $\mathcal{P}(W)$. Hence, we easily infer that $\mathcal{P}(W)$ is an infinite class of graphs. $\square$

Claim 2.2 *If (K_1, w) with $w \geq 3$ does not appear in W, then K_1 is the only possible light graph in $\mathcal{P}(W)$.*

Proof. If $\mathcal{P}(W) = \emptyset$ then by the definition every graph is heavy in $\mathcal{P}(W)$. Otherwise, take a graph $G \in \mathcal{P}(W)$ and m being an integer greater than any weight from the list W. If either (K_1, w) with $w \geq 3$ is missing in W or $(K_1, 1) \in W$, then for each vertex x of G, take m new vertices and connect them to x. If $(K_1, 2) \in W$, for each edge e of G, take m 3-paths and identify their endvertices with endvertices of e. Now, the resulting graph belongs to $\mathcal{P}(W)$ and each of its subgraphs, different from K_1, is of weight $\geq m$. $\square$

¿From the other side, if (K_1, w) with $w \geq 6$ appears in W, then $\mathcal{P}(W) = \emptyset$, and so $\mathcal{L}(\mathcal{P}(W)) = \emptyset$. More generally, we can rewrite it in the following way:

Claim 2.3 *Let $(G, w) \in W$ and $W^* = W \backslash \{(G, w)\}$. If the weight of G in $\mathcal{P}(W^*)$ is less than w, then $\mathcal{P}(W)$ is an empty set.*

By Claim 2.3 and Borodin's extension of the Kotzig theorem (see the previous section) it follows that $\mathcal{P}(\{(K_1, 3), (K_2, 14)\}) = \emptyset$.

Let $\mathcal{S}(W)$ be the class of all slices G (with poles u, v) such that for every $(H, w) \in W$ we have that every subgraph in the pulp $G - u - v$ isomorphic with H, has weight $\geq w$ in G. Note that by the definition each slice is a graph on ≥ 3 vertices. Finally, let $\mathcal{M}(W)$ be the set of connected graphs which are contained in the pulp of every slice from $\mathcal{S}(W)$.

Claim 2.4 *Every light graph in $\mathcal{P}(W)$ belongs to $\mathcal{M}(W)$. In particular, the set of light graphs $\mathcal{L}(\mathcal{P}(W))$ is finite.*

Proof. Suppose that there exists a graph G which is light in $\mathcal{P}(W)$ but is not a member of $\mathcal{M}(W)$. Then, G is not contained in the pulp of some slice S from $\mathcal{S}(W)$. Note that $G \ncong K_1$, and hence it has an edge. Let u, v be the poles of S.

For a given integer m we construct a plane graph $H \in \mathcal{P}(W)$ in the following way: Let M be an $(S, u, v; m)$-melon. Then, H is obtained from G by identifying endvertices of each edge of G with poles of a copy of M (if necessary delete the multiple edges to obtain a simple graph). Since each isomorphic copy of G in H contains a vertex of degree $\geq m$, it is a contradiction to assumption that G is light.

Since $\mathcal{M}(W)$ is a finite set, it follows that $\mathcal{L}(\mathcal{P}(W))$ is also a finite set.

□

Observe that the family restrictions which are not based on lists of weight constraints may lead to infinite sets of light graphs – in the family of all 3-connected planar graphs the only light graphs are paths, see [7].

By Claim 2.4, for light graphs in $\mathcal{P}(W)$ we should look in $\mathcal{M}(W)$. By Claim 2.1, $\mathcal{S}(W)$ is an infinite set of graphs. From other side, $\mathcal{M}(W)$ is a finite set; so it is enough to consider only finite number of slices to determine $\mathcal{M}(W)$. Note that a graph from $\mathcal{M}(W)$ does not need to be light in $\mathcal{P}(W)$. For example, if $W = \{(K_1, 4)\}$ then the 3-cycle C_3 belongs to $\mathcal{M}(W)$, but C_3 is heavy, just consider the double pyramid.

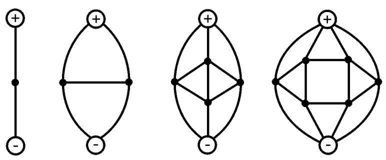

Figure 1: The slices for cases $\delta = 2, 3, 4$

In what follows, we will consider some special cases of Problem 2.1. For the sake of simplicity, let $\mathcal{P}_d = \mathcal{P}(\{(K_1, d)\})$, and similarly define $\mathcal{S}_d$ and

$\mathcal{M}_d$. From the first two slices from Figure 2, we easily obtain that $\mathcal{M}_2 = \{K_1\}$ and $\mathcal{M}_3 = \{K_1, K_2\}$. Moreover $\mathcal{M}_2$ and $\mathcal{M}_3$ coincide with light graphs in $\mathcal{P}_2$ and $\mathcal{P}_3$, respectively. ¿From the other two slices of the same figure, we obtain that $\mathcal{M}_4$ is contained in the set of graphs comprised of $K_{1,3}^+$, C_4 and all their connected subgraphs (the graph $K_{1,3}^+$ can be constructed from $K_{1,3}$ by connecting two non-adjacent vertices). Next result implies that $\mathcal{M}_4$ is precisely this set of graphs, but from an example above, we have $\mathcal{M}_4 \neq \mathcal{L}(\mathcal{P}_4)$. In other words, the following proposition claims that the pulp of a slice from $\mathcal{S}_4$ contains $K_{1,3}^+$ or C_4 as a subgraph.

Proposition 2.2 *Let G be a plane graph with outerface f. Suppose that f contains at most two 2-vertices, all other vertices of f are of degree ≥ 3 and all inner vertices of G are of degree ≥ 4. Then, G contains $K_{1,3}^+$ or C_4.*

Proof. By contradiction. Let there exist a counterexample G. Assume that G has l 2-vertices, k 3-vertices, and s vertices of degree ≥ 4. Observe that G is a graph of girth at least 5. By (1) with $\alpha = \frac{9}{5}$, we have

$$\sum_{v \in V(G)} \left(\frac{9}{5} d(v) - 6 \right) + \sum_{f \in F(G)} \left(\frac{6}{5} r(f) - 6 \right) = -12.$$

Thus

$$l\left(\frac{9}{5} \cdot 2 - 6\right) + k\left(\frac{9}{5} \cdot 3 - 6\right) + s\left(\frac{9}{5} \cdot 4 - 6\right) + \left(\frac{6}{5}(k+l) - 6\right) \leq -12.$$

Since $l \leq 2$, we easily obtain that

$$\frac{3}{5}k + \frac{6}{5}s < 0,$$

which is a contradiction. $\square$

Considering the double pyramid, we infer that C_3, $K_{1,3}$, $K_{1,3}^+$, and C_4 are heavy in $\mathcal{P}_4$. By Theorem 3.4, it follows that P_4 (and so P_3) are light in $\mathcal{P}_4$. Thus, $\{P_1, P_2, P_3, P_4\}$ is the set of light graphs in $\mathcal{P}_4$.

For the family of plane graphs of minimum degree $\delta = 5$, consider the three slices from Figure 2. The set $\mathcal{M}_5$ is contained in the set formed from all connected subgraphs which belong to pulp of each of these three slices. Observe that 5-wheel, $\mathcal{O}_3 - C_3$ (octahedron without 3-cycle) and "snake"

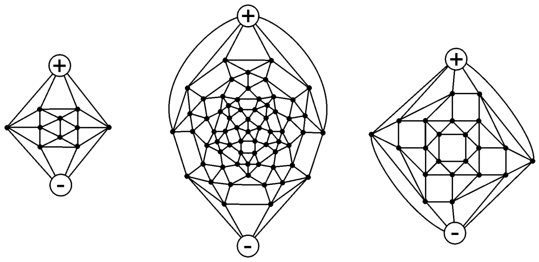

Figure 2: Some slices for case $\delta = 5$

of four triangles (i.e. triangulated C_6 which has two 2-, 3- and 4-vertices) are excluded.

In general, the list W of weight constraints could be short but the set of light graphs in $\mathcal{P}(W)$ could be arbitrary large. The next theorem gives a nice such example. In other words, it claims that if W is comprised of the pairs $(K_1, 4)$ and $(C_3, 6k+10)$ for $k \geq 8$, then all paths P_n with $n \leq k$ are light.

Theorem 2.3 *Let G be a planar graph of minimum degree $\delta \geq 4$ such that each triangle of G is of weight $\geq 6k+10$, for some fixed $k \geq 8$. Then, G contains P_k as a subgraph whose vertices are of degree $\leq 6k-1$.*

Proof. Suppose that the theorem is false and G is a counterexample. Every k-path of G contains a vertex of degree $\geq 6k$ (called *big* in the sequel). Initial charge is assigned to the vertices and faces as it is described in (1) with $\alpha = 1$. Now, redistribute the initial charge by the following rules:

Rule R1: Let u be a big vertex i-adjacent to a 4-vertex v. Then, u sends to v a charge $\frac{i}{2}$.

Rule R2: Each big vertex sends $\frac{1}{3}$ to each adjacent 5-vertex.

Rule R3: Each vertex x with $7 \leq d(x) \leq 6k-1$ sends $\frac{d(x)-6}{d(x)}$ to each adjacent 4- or 5-vertex.

Rule R4: Each face of size ≥ 4 sends $\frac{1}{2}$ to each incident 4- or 5-vertex.

Observe that if $[xyz]$ is a triangular face of G, x is a 4-vertex and none of y, z is big, then both y, z send a charge $2 - 6(\frac{1}{d(y)} + \frac{1}{d(z)})$ to x by R3. As $7 \leq d(y) \leq 6k-1,\ 7 \leq d(z) \leq 6k-1,\ d(y)+d(z) \geq 6k+6,\ k \geq 8$, it is easy to check that y, z send to x more than 1 in total. Similarly, if x is a 5-vertex, these two vertices send at least $\frac{1}{2}$ in total. In the sequel, such triangles are called *donators*.

We will prove that for every $x \in V(G) \cup F(G)$, $c^*(x) \geq 0$. To show this, several cases are considered.

Let f be a face of G of size $r = r(f)$. If $r = 3$, then $c^*(f) = 0$. Otherwise, $c^*(f) \geq 2 \cdot r - 6 - \frac{1}{2} \cdot r \geq 0$ for $r \geq 4$.

Let v be a 5-vertex of G. If v is incident with at least two faces of size ≥ 4, then $c^*(v) \geq -1 + 2 \cdot \frac{1}{2} = 0$. If v is incident with exactly one face of size ≥ 4, then it is adjacent either with at least two big vertices or with at least one donator sending $> \frac{1}{2}$ to v, so $c^*(v) > -1 + \frac{1}{2} + \frac{1}{2} = 0$. Otherwise, v is incident with at least two donators or with one such donator and two big vertices, or with at least three big vertices; in every case, $c^*(v) \geq 0$.

Let v be a 4-vertex of G. If v is not incident with triangular faces, then $c^*(v) \geq -2 + 4 \cdot \frac{1}{2} = 0$. If v is incident with one triangular face, then either it has a big neighbour, or this face is a donator, hence $c^*(v) \geq -2 + 3 \cdot \frac{1}{2} + \frac{1}{2} = 0$. If v is incident with two triangular faces, then it has a big neighbour sending 1, or a pair of big neighbours sending together ≥ 1, or it is incident with at least one donator, hence $c^*(v) \geq -2 + 2 \cdot \frac{1}{2} + 1 = 0$. If v is incident with three triangular faces, then it has a big neighbour sending 1 and another big neighbour sending $\geq \frac{1}{2}$, or one donator and one big neighbour sending $\geq \frac{1}{2}$, or it is incident with two non-adjacent donators, hence $c^*(v) \geq -2 + \frac{1}{2} + 1 + \frac{1}{2} = 0$. Finally, if v is incident only with triangular faces, then it has at least two big neighbours, or one big neighbour and one donator, or two non-adjacent donators, so $c^*(v) \geq -2 + 2 \cdot 1 = 0$.

If v is a d-vertex with $6 \leq d \leq 6k-1$, then $c^*(v) \geq d - 6 - d \cdot \frac{d-6}{d} = 0$.

Finally, let v be a big vertex of degree d. Note that v sends ≤ 1 to an adjacent 4- or 5-vertex. Then it is easy to see that v sends maximal charge when all neighbours of v which receive a charge from v are of degree 4 and form paths. Since these paths are on at most $k-1$ vertices and $d \geq 6k$, v has at least six big neighbours, thus $c^*(v) \geq d - 6 - 1 \cdot (d-6) = 0$. This completes the proof. $\square$

3 Edge weight and lightness

The class of 3-connected graphs as well as some of its subclasses (i.e. planar triangulations) have been mostly used as a realm for searching of light configurations. In this section, we will restrict to $\mathcal{P}_3$ and $\mathcal{T}_3$, the planar graphs with minimum degree 3 and its subclass of triangulations. As an additional condition for restriction in these classes of graphs will be used the edge weight. Note that the results of the previous section show that the only condition (based on bounded weight of a fixed subgraph H) which may force the existence of light graphs (different from K_1 and K_2 and bigger than H) in the family of plane graphs with $\delta \geq 3$ is the one concerning minimum vertex degree, and the one concerning the minimum edge weight. The first results concerning combination of these two conditions can be found in [14] – there was proved (in other terms) the lightness of $K_{1,6}$ in the class $\mathcal{T}(\{(K_1, 5), (K_2, 11)\})$. The class $\mathcal{P}(\{(K_1, 4), (K_2, 9)\})$ was studied in [16], where it was proved that $C_3, C_4, C_5, C_6, K_{1,3}, K_{1,4}, P_3, P_4$ are light; observe that it implies the lightness of these graphs in the class $\mathcal{P}_5$. Here, we will relax the condition of minimum vertex degree ≥ 4 and give the necessary and sufficient condition for lightness of some small graphs regarding the minimum edge weight; the results for C_4 and $K_{1,4}$ generalize the ones in [16].

For the sake of simplicity, we will denote by $\mathcal{P}(w)$ and $\mathcal{T}(w)$ the graphs with edge weight $\geq w$ from $\mathcal{P}_3$ and $\mathcal{T}_3$, respectively. In many cases, it holds that if H is light in $\mathcal{P}(w)$, then it is also light in $\mathcal{T}(w)$ and $\mathcal{P}(u)$ for $u \geq w$.

As we mentioned already, the only light graphs in the class of planar 3-connected graphs are the paths. In what follows, the concept of melons is used for proving the heaviness of paths which are not contained in their pulp:

Proposition 3.1 *The k-path P_k is not light in $\mathcal{P}(w_k)$, for k and w_k from Table* 1.

k	4	5	7	11	12	22
w_k	7	8	9	10	11	12–13

Table 1: Heavy paths

Proof. For $w_k = 7, 8, 9, 10$ and 11, consider slices at Figure 3 and construct corresponding melons. In melons so obtained, every 4- (5-, 7-, 11- and 12-) path contains a pole, thus, a vertex of arbitrarily high degree.

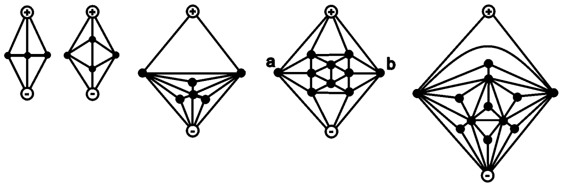

Figure 3: The slices for $w = 7, 8, 9, 10, 11$

For $w_k = 12$ or 13, take the slice used for $w = 10$ and into each triangular face insert a new vertex, then join it to vertices on the face boundary; also, add two new 3-vertices and join each of them to + and - pole and to the vertex a (b). In the melon constructed from these slices, each 22-path contains a pole. $\square$

Note that a slight modification – enlarging the slices before gluing them to melon – enables us to show that every path on $\geq k$ vertices is not light in the corresponding family $\mathcal{P}_3(w_k)$; details are left to the reader.

By a result of Borodin [4], the 3-path P_3 is light in the family $\mathcal{P}(7)$. One can easily see, using the constructions from the second section that neither C_3 nor a graph on ≥ 4 vertices is light in $\mathcal{P}(7)$. Thus, $\mathcal{L}(\mathcal{P}(7)) = \{P_1, P_2, P_3\}$.

For $\mathcal{P}(8)$, graphs $C_3, C_4, K_{1,3}^+, K_4^-$ (the graph K_4^- can be constructed from K_4 by deleting an edge) are not light - just consider the double pyramid. Hence, together with construction from the second section we obtain that the set $\mathcal{L}(\mathcal{P}(8))$ consists of P_1, P_2, P_3 and P_4 (the lightness of the last two graphs follows by Theorem 3.4).

For $\mathcal{P}(9)$, the similar construction based on third slice of Figure 3 shows that no graph on ≥ 7 vertices can be light. Also, first parts of Theorems 3.5, 3.7 show that also C_3, C_4, C_5, C_6 are not light. Thus, except the graphs mentioned above and $K_{1,4}$, the only possible light graphs are P_5, P_6, three trees with maximum degree 3 on 6 vertices and $K_{1,4}$ with one subdivided edge; the lightness of these graphs has not been considered so far.

Since any plane triangulation is 3-connected, every k-path P_k is light in $\mathcal{T}(w)$ for $6 \leq w \leq 13$. The full characterization of light graphs in the case $6 \leq w \leq 8$ gives:

Lemma 3.2 *$\mathcal{L}(\mathcal{T}(w)) = \{P_k \mid k \geq 1\}$ for $6 \leq w \leq 8$.*

Proof. Let G be a plane graph different from P_k and let m be an integer. Without loss of generality assume that $m \equiv 0 (\mathrm{mod}\ 3)$ and $m \gg |V(G)|$. Transform G into plane triangulation T by adding new edges in arbitrary way (this is always possible). Into each 3-face $[xyz]$ of T insert a $(K_2, u, v; m)$-tripod (u, v being endvertices of K_2) and identify its tripoles with vertices x, y, z. The resulting graph belongs to $\mathcal{T}(8)$ (hence also to $\mathcal{T}(7)$ and $\mathcal{T}(6)$) and each its subgraph isomorphic to G contains a vertex of degree $\geq m$. Thus, G is heavy in these classes. $\square$

For $w \geq 9$, the whole set $\mathcal{L}(\mathcal{T}(w))$ is not known. Fabrici [5] proved that every plane graph H with $\Delta(H) \geq 5$ is heavy in $\mathcal{T}_5$ (hence also in $\mathcal{T}(10)$). Inspired by his construction, we will give some necessary conditions for light graphs in the case $w \geq 9$:

Proposition 3.3 *Every light graph G in $\mathcal{T}(w)$, $w \in \{9, 10, 11\}$ is contained in pulps of $(Z^{(w)}, u, v; n)$-tripods, where*

- *$Z^{(9)}$ is triangle or the second graph of Figure 4;*
- *$Z^{(10)}$ is third or fourth graph of Figure 4; moreover, G is a subgraph of some k-antiprism;*
- *$Z^{(11)} = K_4^-$ and u, v are its 3-vertices.*

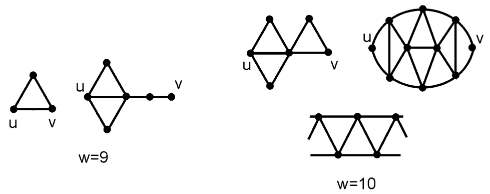

Figure 4: Basic parts of tripods for $w = 9, 10$

Proof. The proof goes in the same way as in Lemma 3.2, with tripods mentioned above. At the end we put some extra edges if necessary to obtain triangulations. For the second claim in case $w = 10$, a "tripod" of original construction of Fabrici [5] is used (its pulp is k-sided antiprism). $\square$

The following table shows the minimum values w for which are the particular graphs light in $\mathcal{P}(w)$ and $\mathcal{T}(w)$.

	C_3	C_4	$K_{1,3}$	$K_{1,4}$	P_3	P_4
$\mathcal{P}(w)$	10	10	9	9	7	8
$\mathcal{T}(w)$	9	10	9	9	6	6

Table 2: Edge–weight conditions for lightnes of particular graphs

In the rest of this section will be proved the bounds from the above table. Let G be light in $\mathcal{P}(w)$ and $w \leq u$. If G is a subgraph of at least one graph from $\mathcal{P}(u)$, then G is also light in $\mathcal{P}(u)$. The same holds for triangulations. Thus, in the following proofs, when proving that a graph is light in $\mathcal{P}(w)$ for $w \in [u, 13]$, we consider only the case $w = u$.

Theorem 3.4 *The 4-path P_4 is light in $\mathcal{P}(w)$ if and only if $w \in [8, 13]$.*

Proof. To prove that P_4 is not light in $\mathcal{P}(w)$ for $w \leq 7$, consider the $(S, u, v; m)$-melon, where S is the first slice from Figure 3 and u, v are its non-adjacent vertices. This graph belongs to the class $\mathcal{P}(7)$ and every its 4-path contains a vertex of degree $3m$.

Now, we will prove the other direction. Let G be a counterexample. Vertices of degree ≥ 192 and faces of size ≥ 4 are called *big*, vertices of degree 3, 4 or 5 are *small*. We will assume that every 4-path of G contains a big vertex.

If f_1 and $f_2 = [xyv]$ are two adjacent faces at the edge xy such that x or y is not big, and v is of degree 3, then we say that v is *$\triangle$-adjacent* to f_1. Alternatively, if both x, y are not big and v is of degree 4, we say that v is *$\triangle^*$-adjacent* to f_1. Moreover, if f_1 is a 3-face $[uxy]$, then we also say that u is $\triangle$- or $\triangle^*$-adjacent to v.

Initial charge is assigned to the vertices and faces as it is described in (1) with $\alpha = 1$. Now, redistribute the initial charge by the following rules:

Rule R1: Let u be a big vertex i-adjacent to a small vertex v. Then, u sends to v a charge $\frac{i+1}{2}$ or $\frac{i}{2}$ or $\frac{1}{2}$, if its degree is 3 or 4 or 5, respectively.

Rule R2: Each big face and big vertex sends $\frac{1}{2}$ to each $\triangle$-adjacent 3-vertex and each $\triangle^*$-adjacent 4-vertex. (In this case, we say that there is a *diagonal transfer* of $\frac{1}{2}$ through an edge.)

Rule R3: Let f be a big face and u be a small vertex incident with f. Then, f sends to u

(a) $\frac{1}{5}$ if u is a 5-vertex except of the case when $f = [xuyv]$ is of size 4 and x, y are big. In this exceptional case, f sends no charge to u;

(b) $\frac{1}{2}$ if u is 4-vertex;

(c) 1 if u is 3-vertex except of the case when $f = [xuyv]$ is of size 4, x is big, y is a 5-vertex and u, v are 3-vertices. In this exceptional case, f sends $\frac{9}{10}$ to u (and v).

In what follows, we will prove that for every $x \in V(G) \cup F(G)$, $c^*(x) \geq 0$.

Let f be a face of G of size r. Suppose first that $r \geq 6$. Denote by t_i ($i = 3, 4, 5$) the number of transfers of charge from f to incident vertices of degree 3, 4 and 5, respectively, and by t'_j ($j = 3, 4$) the number of transfers of $\frac{1}{2}$ to vertices which are $\triangle$- and $\triangle^*$-adjacent to f. Then, $t'_3 + t'_4 \leq r$ and $2t_3 + t_4 + t_5 \leq r$. After the initial charge of f is distributed, we have $c^*(f) \geq 2r - 6 - t_3 - \frac{t_4}{2} - \frac{t_5}{5} - \frac{t'_3}{2} - \frac{t'_4}{2} \geq 2r - 6 - (t_3 + \frac{t_4+t_5}{2}) - \frac{t'_3+t'_4}{2} \geq 2r - 6 - \frac{r}{2} - \frac{r}{2} = r - 6 \geq 0$.

Suppose now that $r = 5$. If f is incident with two 3-vertices, then $c^*(f) \geq 4 - 2 \cdot 1 - \frac{1}{2} - \frac{1}{5} > 0$. Similarly, if f is incident with one 3-vertex, then $c^*(v) \geq 4 - 1 - 3 \cdot \frac{1}{2} - 2 \cdot \frac{1}{5} > 0$. Thus, assume that all vertices incident with f are of degree at least 4. If at least one of them is of degree 4, then $c^*(f) \geq 4 - 4 \cdot \frac{1}{2} > 0$. In the remaining case all vertices incident with f are of degree at least 5, and so $c^*(f) \geq 4 - 4 \cdot \frac{1}{2} - 2 \cdot \frac{1}{5} > 0$.

Finally, suppose that $r = 4$. If f is incident with at least three big vertices, then $c^*(f) \geq 2 - 2 \cdot \frac{1}{2} > 0$. Suppose that f is incident with exactly two big vertices. If they are not diagonally opposite, then $c^*(f) \geq 2 - 1 - \frac{1}{2} - \frac{1}{5} > 0$. If they are diagonally opposite on the 4-face f, then consider the two remaining vertices of f. If at least one of them is of degree 5, then it receives no charge from f (by R3(a)), and so $c^*(f) \geq 2 - 4 \cdot \frac{1}{2} = 0$. Finally, if only one vertex incident with f is big, then either $c^*(f) \geq 2 - 2 \cdot \frac{9}{10} - \frac{1}{5} = 0$ (by R3(c)) or $c^*(f) \geq 2 - 2 \cdot 1 = 0$.

Now, we will consider the vertices of G. Let v be a d-vertex of G. Denote by $v_1, v_2, \ldots, v_d$ the neighbours of v around it, and by f_i the face which contains the subwalk $v_i v v_{i+1}$ (index modulo d).

Suppose first that v is a 5-vertex. If v is adjacent to at least two big vertices, then $c^*(v) \geq -1 + 2 \cdot \frac{1}{2} = 0$. If v is adjacent to exactly one big vertex, say v_1, then f_2, f_3, f_4 are big faces, and each of them sends $\frac{1}{5}$ to v. So, $c^*(v) \geq -1 + \frac{1}{2} + 3 \cdot \frac{1}{5} > 0$. Otherwise, v is not adjacent with a big vertex. Then, all faces around v are big, and hence $c^*(v) \geq -1 + 5 \cdot \frac{1}{5} = 0$.

Suppose that v is a 4-vertex. If all faces incident with v are triangular, then v is adjacent to at least two big vertices, and $c^*(v) \geq -2 + 2 \cdot 1 = 0$ by R1. Similarly, if all faces incident with v are big, then $c^*(v) \geq -2 + 4 \cdot \frac{1}{2} = 0$ by R3(b). Suppose that f_1 is the only big face incident with v. Then, v is adjacent to at least two big vertices. If at least one of v_3, v_4 is big, then $c^*(v) \geq -2 + \frac{1}{2} + 1 + \frac{1}{2} = 0$. Otherwise, both v_1 and v_2 are big, and there is a diagonal transfer of $\frac{1}{2}$ through $v_3 v_4$ from a big vertex or a big face by R2. Hence, $c^*(v) \geq -2 + \frac{1}{2} + 2 \cdot \frac{1}{2} + \frac{1}{2} = 0$.

Suppose now that v is incident with two big faces. If f_2, f_4 are big, then at least two of the neighbours of v are big, and $c^*(v) \geq -2 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 0$. Without lose of generality, we may assume that f_2, f_3 are big. If v_1 is big or v_2 and v_4 are big, then $c^*(v) \geq -2 + 2 \cdot \frac{1}{2} + 1 = 0$. So, consider the case when v_2 and v_3 are big (the case for v_3, v_4 being big is symmetrical). Then, we may assume that v_1 and v_4 are not big. So, there is a diagonal transfer of $\frac{1}{2}$ through $v_4 v_1$ from a big vertex or a big face by R2. Hence, $c^*(v) \geq -2 + 2 \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$.

Finally, suppose that v is incident with three big faces, say f_1, f_2, f_3. If one of v_1, v_4 is big, then $c^*(v) \geq -2 + 3 \cdot \frac{1}{2} + \frac{1}{2} = 0$. Otherwise, like above, there is a diagonal transfer of $\frac{1}{2}$ through $v_4 v_1$, and we infer that $c^*(v) \geq -2 + 4 \cdot \frac{1}{2} = 0$.

Now, we will consider the case when v is a 3-vertex. Suppose first that all faces incident with v are triangular. Note that v has at most two non-big neighbours. If v is adjacent to at least two big vertices, then $c^*(v) \geq -3 + 2 \cdot \frac{3}{2} = 0$. Let v_1 be the only big neighbour of v. Then, there are three diagonal transfers of $\frac{1}{2}$ through each of the edges $v_1 v_2, v_2 v_3, v_3 v_1$. Hence, $c^*(v) \geq -3 + \frac{3}{2} + 3 \cdot \frac{1}{2} = 0$.

Now, suppose that f_2 is the only big face incident with v. Without loss of generality, if v_2 or v_3 is big, then we will assume that v_3 is a big vertex. However, if v_1 and v_3 are big, then $c^*(v) \geq -3 + \frac{3}{2} + 1 + \frac{9}{10} > 0$. If both v_2 and v_3 are big, then $c^*(v) \geq -3 + 2 \cdot 1 + 1 = 0$. If v_1 is the only big neigbour of v, then there are two diagonal transfers through $v_1 v_2$ and $v_3 v_1$. Thus,

$c^*(v) \geq -3 + \frac{3}{2} + 1 + 2 \cdot \frac{1}{2} > 0$. Similarly, if v_3 is the only big neighbour of v, then there are two diagonal transfers through v_1v_2 and v_3v_1. Observe also that v receives 1 from f_2 (otherwise a light $P_4 = v_1vv_2x$ is found, where x is the neighbour of v_3 on f_2). Thus, $c^*(v) \geq -3 + 1 + 2 \cdot \frac{1}{2} + 1 = 0$.

Consider the case when v is incident with two big faces, say f_1 and f_2. If at least two of the neighbours of v are big, then $c^*(v) \geq -3+2\cdot 1+\frac{9}{10}+\frac{1}{2} > 0$. Suppose that v_2 is the only big neighbour. Then, there is a diagonal transfer of $\frac{1}{2}$ through v_3v_1 and, moreover, f_1 and f_2 send 1 to v (otherwise a light P_4 is found). Thus, $c^*(v) \geq -3 + \frac{1}{2} + \frac{1}{2} + 2 \cdot 1 = 0$. Finally, we may assume that v_1 is the only big neighbour. Then, $c^*(v) \geq -3 + 1 + 2 \cdot 1 = 0$.

Now, suppose that all three faces incident with v are big. If v is adjacent to at least one big vertex, then $c^*(v) \geq -3+\frac{1}{2}+3\cdot\frac{9}{10} > 0$. Otherwise, each of the neighbouring faces sends 1 to v, and we infer that $c^*(v) = -3+3\cdot 1 = 0$.

Suppose that v is a d-vertex, $6 \leq d \leq 191$. Since v neither receives nor sends charge, we obtain that $c^*(v) = c(v) = d - 6 \geq 0$.

Finally, let v be a big vertex of degree d. In order to calculate easier the final charge of v, we are doing an averaging, i.e. we estimate how much in average sends v to its neighbours. Consider the following averaging of the charge sent from v:

(A1) If a 3-vertex v_i receives $\frac{3}{2}$ from v, then it donates $\frac{3}{8}$ to each of v_{i-1}, v_{i+1}.

(A2) If a 3-vertex v_i receives 1 from v and f_{i-1} is a triangle, then v_i donates $\frac{1}{4}$ to v_{i-1}.

(A3) If a 4-vertex v_i receives 1 from v, then it donates $\frac{5}{16}$ to each of v_{i-1}, v_{i+1}. Moreover, if $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$ form a fan of 4 consecutive triangles (with common vertex v), v_{i-1}, v_{i+1} are 4-vertices and v_{i-2}, v_{i+2} are big, then v_i donates $\frac{1}{32}$ to each of v_{i-2}, v_{i+2}.

(A4) If a 4-vertex v_i receives $\frac{1}{2}$ from v and f_{i-1} is a triangle, then v_i donates $\frac{1}{8}$ to v_{i-1}.

(A5) Each 5-vertex v_i donates $\frac{3}{32}$ to each its neighbour $x \in \{v_{i-1}, v_{i+1}\}$ such that $[xvv_i]$ is a triangle. Moreover:

   (a) If $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$ form a fan of 4 consecutive triangles (with common vertex v), v_{i-1}, v_{i+1} are 3-vertices and v_{i-2}, v_{i+2} are big, then v_i donates $\frac{1}{16}$ to each of v_{i-2}, v_{i+2}.

   (b) if $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$ form a fan of 4 consecutive triangles (with common vertex v), v_{i-1} is a 3-vertex, v_{i+1} is a 4-vertex (or

v_{i+1} is a 3-vertex and v_{i-1} is a 4-vertex) and v_{i-2}, v_{i+2} are big, then v_i donates $\frac{1}{16}$ to v_{i-2} (or v_{i+2}).

(A6) Each vertex which receives $\frac{1}{2}$ from v through an edge xy by R2, donates $\frac{1}{4}$ to each of x, y.

After this averaging, any vertex which receives charge by a diagonal transfer from v fully redistributes this charge. Consider now, an arbitrary neighbour v_i of v.

Suppose that v_i is a 3-vertex. Then, v_i keeps $\leq \frac{3}{2} - 2 \cdot \frac{3}{8} + 2 \cdot \frac{3}{32} = \frac{15}{16}$.

Suppose now that v_i is a 4-vertex. Then, after the averaging, v_i preserves maximum charge in the case described by the second part of (A3), or in the case when v_{i+1} (or v_{i-1}) is a 4-vertex receiving 1 from v and there is diagonal transfer through $v_i v_{i+1}$ (or $v_{i-1} v_i$). In the former case, v_i preserves $1 - 2 \cdot \frac{5}{16} + 2 \cdot \frac{5}{16} - 2 \cdot \frac{1}{32} = \frac{15}{16}$. And, in the latter case v_i preserves $1 - 2 \cdot \frac{5}{16} + \frac{5}{16} + \frac{1}{4} = \frac{15}{16}$ if v_i is 2-adjacent to v or it preserves $\frac{1}{2} - \frac{1}{8} + \frac{5}{16} + \frac{1}{4} = \frac{15}{16}$ if v_i is 1-adjacent to v.

Suppose that v_i is a 5-vertex of v. Then, after the averaging, v_i preserves maximum charge in the case when v_{i+1} is a 4-vertex receiving 1 from v, there is a diagonal transfer through $v_i v_{i+1}$, and f_{i-1} is a big face. In this case, v_i keeps $\frac{1}{2} - \frac{3}{32} + \frac{1}{4} + \frac{5}{16} = \frac{31}{32}$.

If v_i is a k-vertex with $6 \leq k \leq 191$, then v_i preserves $\leq \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$. And, if v_i is a big neighbour of v, then v_i preserves $\leq 2 \cdot \frac{3}{8} + 2 \cdot \frac{1}{16} = \frac{7}{8}$.

Thus, we obtain that each neighbour of v receives in average $\leq \frac{31}{32}$ from v. Therefore, $c^*(v) \geq d - 6 - \frac{31}{32} d \geq 0$ for $d \geq 192$. This completes the proof. □

Theorem 3.5 *The 3-cycle C_3 is light in $\mathcal{P}(w)$ if and only if $w \in [10, 13]$.*

Proof. Let G^* be the graph comprised of $2n$-cycles $C_b = a_1 b_1 a_2 b_2 \cdots a_n b_n a_1$ and $C_c = a_1 c_1 a_2 c_2 \cdots a_n c_n a_1$ and vertices b and c such that $N(b) = V(C_b)$ and $N(c) = V(C_c)$. Note that G^* is a planar graph with edge weight at least 9 whose every 3-cycle has weight $2n+9$. This proves that the 3-cycle is not light in $\mathcal{P}_3(w)$ for $w < 10$.

Now, we will prove the other direction. Suppose that the theorem is false and G is a counterexample. Every edge of G has weight ≥ 10. Suppose also that every triangle has weight $\geq k$ where k is sufficiently large, say $k = 52$. Initial charge is assigned to the vertices and faces as it is described in (1) with $\alpha = 2$. Now, redistribute the initial charge by the following rule:

Rule R: Let v be a vertex incident with a face f. Then, v sends $\frac{c(v)}{d(v)}$ to f.

We will prove that for every $x \in V(G) \cup F(G)$, $c^*(x) \geq 0$. By rule R, it is clear that every vertex of G has a final charge 0. Let f be an arbitrary face of G and let $r = r(f)$. If $r \geq 6$, obviously $c^*(v) \geq c(v) \geq 0$. If $r = 5$ then f has at least three incident vertices of degree ≥ 5 and each of these vertices sends at least $\frac{2 \cdot 5 - 6}{5}$ to f. So, $c^*(v) \geq -1 + 3 \cdot \frac{4}{5} > 0$. Suppose now that $r = 4$. If f is incident with at least three vertices of degree ≥ 5, then similarly as before, $c^*(v) \geq -2 + 3 \cdot \frac{4}{5} > 0$. Otherwise, f is incident with two vertices of degree ≥ 6 and so $c^*(v) \geq -2 + 2 \cdot 1 = 0$. Finally, suppose that $r = 3$. Let $f = [x_1 x_2 x_3]$ and $d_1 = d(x_1)$, $d_2 = d(x_2)$, and $d_3 = d(x_3)$. We may assume that $d_1 \leq d_2 \leq d_3$. By rule R, $c^*(f) = -3 + \frac{2d_1 - 6}{d_1} + \frac{2d_2 - 6}{d_2} + \frac{2d_3 - 6}{d_3}$. By the assumptions that $3 \leq d_1 \leq d_2 \leq d_3$, $10 \leq d_1 + d_2$ and $52 \leq d_1 + d_2 + d_3$, it is easy to see that the maximum value of $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3}$ is $\frac{1}{2}$ for $d_1 = 3,\ d_2 = 7,\ d_3 = 42$. Hence,

$$c^*(f) = 3 - 6\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3}\right) \geq 3 - 6\left(\frac{1}{3} + \frac{1}{7} + \frac{1}{42}\right) = 0.$$

$\square$

Theorem 3.6 *The 3-cycle C_3 is light in $\mathcal{T}(w)$ if and only if $w \in [9, 13]$.*

Proof. Let G^* be a graph of n-sided double pyramid. We see that G^* has edge weight 8 and every 3-cycle has weight $n + 8$.

Now, we will prove the other direction. Suppose that the theorem is false and G be a counterexample. Every edge of G has weight ≥ 9. Suppose also that every triangle has weight $\geq k$ where k is sufficiently large, say, $k = 52$. Initial charge is assigned to the vertices and faces as it is described in (1) with $\alpha = 1$. Now, redistribute the initial charge by the following rule:

Rule R: A vertex v of degree ≥ 24 sends $\frac{1}{3}$ to each 5-vertex, 1 to each 4-vertex, and $\frac{3}{2}$ to each 3-vertex.

We will prove that for every $x \in V(G) \cup F(G)$, $c^*(x) \geq 0$. It is clear that every face of G has a final charge 0. Let v be an arbitrary vertex of G and let $d = d(v)$. If $6 \leq d \leq 23$ then $c^*(v) = c(v) \geq 0$. If $d = 5$ then v has at least three neighbours of degree ≥ 24. So $c^*(v) \geq -1 + 3 \cdot \frac{1}{3} \geq 0$. If $d = 4$ then v

has at least two neighbours of degree ≥ 24, and so $c^*(v) \geq -2 + 2 \cdot 1 = 0$. Similarly, for $d = 3$ it follows that $c^*(v) \geq -3 + 2 \cdot \frac{3}{2} = 0$.

Finally, suppose that $d \geq 24$. Denote by $v_1, v_2, \ldots v_d$ the neighbours of v as they appear around v. In order to calculate easier the final charge of v, we are doing an averaging. If $d(v_i) = 3$ then let v_i donate $\frac{3}{8}$ (of the charge that it receives from v) to each of v_{i-1} and v_{i+1}. And, if $d(v_i) = 4$ then let v_i donate $\frac{1}{6}$ (of the charge received from v) to each of v_{i-1} and v_{i+1}. Note that after the averaging takes part, each neighbour of v receives from v at most $\frac{3}{4}$. So, $c^*(v) \geq d - 6 - \frac{3}{4}d \geq 0$. This completes the proof. $\square$

Theorem 3.7 *The* 4*-cycle* C_4 *is light in* $\mathcal{P}(w)$ *if and only if* $w \in [10, 13]$.

Proof. Let G^* be the graph comprised of $2n$ cycle $C_a = a_1 b_1 a_2 b_2 \cdots a_n b_n a_1$, n-cycle $C_b = b_1 b_2 \cdots b_n b_1$, and vertices a and b with $N(a) = V(C_a)$ and $N(b) = V(C_b)$. G^* is a planar graph with edge weight 9 and every 4-cycle has weight $\geq n + 15$.

Now, we will prove the other direction. Suppose that the theorem is false and G is a counterexample. Every edge of G has weight ≥ 10. Suppose also that every 4-cycle has weight $\geq k$ where k is sufficiently large, say $k = 381$. Vertices of degree ≥ 96 are called *big* and those of degree $\in [6, 95]$ are called *intermediate*; other vertices are called *small.* Thus, every 4-cycle of G contains a big vertex and if two small vertices are adjacent, then they are of degree 5.

Denote by $f_\triangle(v)$ the number of 3-faces incident with v. Let $f = [x_1 x_2 \cdots x_n]$ be a face. Suppose that vertex u is not incident with f. If $[u x_1 x_2]$ is a 3-face and x_1 and x_2 are intermediate vertices then we say that u is $\triangle$*-incident* with f. Moreover, if $d(x_n) = d(x_3) = 3$ then u is $\triangle^*$*-incident* with f. Thus, if u and f are $\triangle^*$-incident then they are also $\triangle$-incident. Denote by $\triangle(f)$ the number of 3- and 4-vertices $\triangle$-incident with f. If x, y are 2-adjacent and u, v are remaining two vertices of their triangles, we say that u, v are $\triangle$*-adjacent.*

Initial charge is assigned to the vertices and faces as it is described in (1) with $\alpha = 1$. Now, redistribute the initial charge by the following rules:

Rule R1: Let v be an intermediate vertex. Then, v equally distribute the charge $d(v) - 6$ to its neighbours of degree 3.

Rule R2: Let v be a big vertex adjacent to a vertex u of degree ≤ 4.

(a) If u is 2-adjacent to v, then v sends $\frac{7}{4}$ to u.

(b) If u is 1-adjacent to v, then v sends $\frac{5}{4}$ to u.

(c) If u is 0-adjacent to v, then v sends $\frac{7}{8}$ to u.

Rule R3: Let v be a big vertex adjacent to a 5-vertex u. Then, v sends $\frac{7}{8}$ to u.

Rule R4: Let v be a big vertex and let v be $\triangle$-adjacent to a 3- or 4-vertex u. Then, v sends $\frac{7}{8}$ to u.

Rule R5: Let f be a face of length ≥ 4 incident with a vertex u of degree ≤ 5.

(a) Let $d(u) = 3$ or 4. If $r(f) = 4$ then f sends 1 to u and otherwise it sends $\frac{3}{2}$ to u.

(b) If $d(u) = 5$ then f sends $\frac{1}{2}$ to u.

Rule R6: Let f be a face of length ≥ 4 and let f be $\triangle$-incident with a 3- or 4-vertex u.

(a) If $r(f) = 4$ then f sends 1 to u.

(b) Let $r(f) \geq 5$. If f and u are $\triangle^*$-incident then f sends $\frac{17}{20}$ to u, and otherwise f sends $\frac{4}{5}$ to u.

Rule R7: Let f be a face of length ≥ 4. After rules R5 and R6 are used, the rest of positive charge of f is equally distributed to its incident 3-vertices.

Now, we will prove that for every $x \in V(G) \cup F(G)$, $c^*(x) \geq 0$. Let v be a vertex of G and let $d = d(v)$. Denote by $v_1, v_2, \ldots, v_d$ the neighbours of v as they appear around v. Denote by f_i the face which contains the subwalk $v_i v v_{i+1}$. If v is a intermediate vertex, then by rule R1, we have $c^*(v) \geq 0$. Suppose that v is a big vertex. We do the following averaging: If a vertex v_i receives charge c from v then v_i donates $\frac{c}{4}$ to each of v_j ($j \in \{i-1, i+1\}$) such that f_j is a triangle. If u is $\triangle$-adjacent to v where $[uv_iv_{i+1}]$ and $[vv_iv_{i+1}]$ are 3-faces, then u donates $\frac{7}{16}$ to each of v_i and v_{i+1}. Now, by rules R2-R4, it is easy to see that each neighbour of v receives in average at most $\frac{15}{16}$ from v. So, $c^*(v) \geq d - 6 - \frac{15d}{16} \geq 0$.

Suppose now that $d = 5$. If $f_\triangle(v) \leq 3$, then by rule R5(b), we have $c^*(v) \geq -1 + 2 \cdot \frac{1}{2} = 0$. If $f_\triangle(v) = 5$ then it is adjacent to two big vertices. By rule R3 each of these two neighbours sends $\frac{7}{8}$ to v and so $c^*(v) > 0$.

Finally, if $f_\triangle(v) = 4$ then one face incident with v is of size ≥ 4. By R5(b), this face sends $\frac{1}{2}$ to v. And, there is also a big vertex which sends $\frac{7}{8}$ to v. Hence, we infer $c^*(v) > 0$.

Suppose that $d = 4$. We argue similarly as in the previous case (i.e. $d = 5$). If $f_\triangle(v) \leq 2$ then by R5(a) we obtain that $c^*(v) \geq -2+2\cdot 1 = 0$. If $f_\triangle(v) = 4$ then there are two big neighbours of v and each one sends $\frac{7}{4}$ to v by R2(a). Finally, if $f_\triangle(v) = 3$ then one face sends 1 to v by R5(a), and there is a big vertex which sends $\geq \frac{5}{4}$ to v by rule R2. Hence, $c^*(v) > 0$.

Finally, let us suppose that v is a 3-vertex. We will consider the following cases:

$f_\triangle(v) = 0$: By R5(a), it follows that $c^*(v) \geq -3 + 3 \cdot 1 = 0$.

$f_\triangle(v) = 1$: We may assume that f_3 is a triangle. Note that each of f_1 and f_2 sends ≥ 1 to v. If some of v_1, v_2, v_3 is a big vertex, then it sends $\geq \frac{7}{8}$ to v by R2 and each of other two vertices sends $\geq \frac{1}{7}$ to v by R1. Hence, $c^*(v) \geq -3 + 2 \cdot 1 + 2 \cdot \frac{1}{7} + \frac{7}{8} > 0$. So, assume that none of v_1, v_2, v_3 is a big vertex. If f_1 is a 4-cycle, say $f_1 = [v_1 v v_2 x]$, then x is a big vertex, and hence f_1 sends 2 to v by R7. And, if $r(f_1) \geq 5$ then f_1 sends $\frac{3}{2}$. Similar arguing holds for f_2. Thus, $c^*(v) \geq -3+2\cdot\frac{3}{2} \geq 0$.

$f_\triangle(v) = 2$: We may assume that f_1 and f_2 are triangles. Then, f_3 sends at least 1 to v. Note that one of v_1, v_2, v_3 is a big vertex. Suppose first that v_2 is a big vertex. Then, each of v_1 and v_3 sends $\geq \frac{1}{6}$ to v and v_2 sends $\frac{7}{4}$ to v. Thus, $c^*(v) \geq -3+2\cdot\frac{1}{6}+\frac{7}{4}+1 > 0$. Suppose now that v_2 is intermediate. In this case, v_1 or v_3 is a big vertex, say v_1. Note that v_1 sends $\frac{5}{4}$ to v. If v_3 is also big vertex then $c^*(v) \geq -3+2\cdot\frac{5}{4}+1 > 0$. So assume that v_3 is intermediate. Let f' be a face incident with the face f_2 at the edge v_2v_3. If $r(f') \neq 3$ then f' sends $\geq \frac{4}{5}$ to v by R6. Otherwise, let x be a vertex such that $f' = [v_2 v_3 x]$. Then, x is a big vertex and it sends $\frac{7}{8}$ to v by rule R4. However, in both cases, we have $c^*(v) \geq -3 + 1 + \frac{4}{5} + \frac{5}{4} > 0$.

$f_\triangle(v) = 3$: Without lose of generality, we may assume that v_1 is a big vertex. Then, it sends $\frac{7}{4}$ to v. If v_2 or v_3 is a big vertex then $c^*(v) \geq -3 + 2 \cdot \frac{7}{4} > 0$. So, assume that v_2 and v_3 are intermediate vertices. Each of v_2 and v_3 sends $\geq \frac{1}{5}$ to v by R1. Let f' be the face incident with f_2 at edge v_2v_3. Suppose first that f' is a triangle, say $f' = [v_3 v_2 x]$. In this case, x is a big vertex and it sends $\frac{7}{8}$ to v by R4. So, $c^*(v) \geq -3 + 2 \cdot \frac{1}{5} + \frac{7}{4} + \frac{7}{8} > 0$. If f' is a 4-face, then

f' sends 1 to v. Hence, $c^*(v) \geq -3 + \frac{7}{4} + 2 \cdot \frac{1}{5} + 1 > 0$. Finally, assume that $r(f') \geq 5$. Then, v is $\triangle^*$- or $\triangle$-incident with f'. In the first case, we have $c^*(v) \geq -3 + \frac{7}{4} + 2 \cdot \frac{1}{5} + \frac{17}{20} = 0$. And, in the second case observe that either v_2 or v_3 sends $\geq \frac{1}{4}$ to v by R1. So, $c^*(v) \geq -3 + \frac{7}{4} + \frac{1}{4} + \frac{1}{5} + \frac{4}{5} = 0$.

Let $f = [x_1 x_2 \cdots x_r]$ be an arbitrary face of G. Suppose first that $r(f) \geq 5$. Consider the following averaging of charge sent from f by rules R5 and R6: if x_i is a 3- or 4-vertex, then it sends $\frac{3}{8}$ to each of x_{i-1} and x_{i+1}. If v is $\triangle^*$-incident (resp. $\triangle$-incident) with f at edge $x_i x_{i+1}$ then v splits $\frac{17}{40}$ (or $\frac{2}{5}$, respectively) to each x_i and x_{i+1}. Note that after this every vertex x_i receives at most $\frac{4}{5}$ from f. Thus, charge $\bar{c}(f)$ of f after rules R5 and R6 being applied satisfies $\bar{c}(f) \geq 2r - 6 - \frac{4}{5}r \geq 0$.

Suppose now that f is a 4-face. We may assume that x_1 is a big vertex. Let $r_{3,4}$ be the number of 3- and 4-vertices incident with f, i.e. $r_{3,4} = r_3(f) + r_4(f)$. Obviously, $r_{3,4} \leq 2$. To conclude the proof, consider the following three cases:

$r_{3,4} = 2$: Then x_2 and x_4 are 3- or 4-vertices and $c^*(f) \geq 2 - 2 \cdot 1 = 0$.

$r_{3,4} = 1$: If x_3 is 3- or 4-vertex, then $c(f) \geq 2 - 1 = 1$. Otherwise assume that x_2 is a 3- or 4-vertex. Note that $r_5(f) + \triangle(f) \leq 1$ and hence $c^*(f) \geq 2 - 1 - \frac{4}{5} > 0$.

$r_{3,4} = 0$: Then $r_5(f) + \triangle(f) \leq 2$ with equality only if $\triangle(f) = 2$. In all cases, it is easy to see that $c^*(f) \geq 0$.

□

Theorem 3.8 *The star $K_{1,4}$ is light in $\mathcal{P}(w)$ if and only if $w \in [9, 13]$.*

Proof. To show that $K_{1,4}$ is heavy in $\mathcal{P}(w)$ for $w \leq 8$, consider the double n-pyramid. In this graph, every $K_{1,4}$ contains a vertex of degree n.

Now, we will prove the other direction. Suppose that theorem is false and G is a counterexample. For the purpose of the proof, a vertex of degree at least 432 is called *big*, a vertex of degree between 6 and 431 is *intermediate*; all other vertices are called *small*. A face of size at least 4 is called *big*.

Let f be a big face and let x_1, x_2, x_3, x_4 be four consecutive vertices on f. Let $[x_2 x_3 v]$ be a 3-face and v be a 3-vertex. If x_1 (or x_4) is big and x_2, x_3 are intermediate, we say that f is $\triangle$*-adjacent* to v. If x_1, x_3 are big and x_2 is intermediate (or x_2, x_4 are big and x_3 is intermediate), we say that f is $\triangle^\circ$-*adjacent* to v. Finally, if $[x_2x_3v]$ is a 3-face, v is a 4-vertex, x_2 is a 5-vertex and x_3 is a 5- or intermediate vertex (or vice versa), we say that v is $\triangle^*$-adjacent to f. In all these cases, the edge x_2x_3 is a *diagonal* for v and f.

The initial assignment of charge is as in (1) with $\alpha = 1$. The local redistribution of charge preserving its total sum is performed by the following rules:

Rule R1: Each big face sends

(a) 1 to each $\triangle$-adjacent 3-vertex;

(b) $\frac{1}{2}$ to each $\triangle^\circ$-adjacent 3-vertex;

(c) $\frac{1}{2}$ to each $\triangle^*$-adjacent 4-vertex.

The remaining charge of this face is then equally distributed to all incident vertices of degree 3 and 4.

Rule R2: Let u be a big vertex i-adjacent to a small vertex v. Then, u sends to v charge $\frac{i+1}{2}$ or $\frac{i}{2}$ or $\frac{1}{2}$, if its degree is 3 or 4 or 5, respectively.

Rule R3: Each big vertex u sends $\frac{1}{3}$ to each adjacent 6-vertex v except in the case when v is adjacent to less than 3 small vertices. In this exceptional case, u sends $\frac{2}{9}$ to v.

Rule R4: Each intermediate vertex sends $\frac{1}{3}$ to each adjacent small vertex.

Rule R5: Let $[uxy]$, $[xyv]$ be adjacent triangular faces, x, y be intermediate, u be big, and v be a 3-vertex. Then, u sends 1 to v (through the edge xy).

Rule R6: Let $[uxy]$, $[xyv]$ be adjacent triangular faces, x be intermediate, u and y be big, and v be a 3-vertex. Then, u sends $\frac{1}{2}$ to v (through the edge xy).

Rule R7: Let $[uxy]$, $[xyv]$ be adjacent triangular faces, x be a 5-vertex, y be a 5- or intermediate vertex, u be big, and v be a 4-vertex. Then, u sends $\frac{1}{2}$ to v (through edge xy).

Now, we will prove that for every $x \in V(G) \cup F(G)$, $c^*(x) \geq 0$. To this end, several cases have to be considered.

Let f be a face. If it is a triangle, then $c^*(f) = 0$. Suppose that $r = r(f) \geq 4$. Let t_1, t_2, t_3 be the number of transfers of charge from f to $\triangle$-, $\triangle^\circ$-, $\triangle^*$-adjacent small vertices, respectively. Let s be the number of 3- and 4-vertices incident with f. Then, $2s + t_1 + t_2 + t_3 \leq r$ and the remaining charge of f after application of R1(a)-(c) is $2r - 6 - t_1 - \frac{1}{2}(t_2 + t_3)$. After R1(a)-(c) are applied, each 3- or 4-vertex incident with f receives at least

$$\frac{2r - 6 - t_1 - \frac{1}{2}(t_2 + t_3)}{(r - t_1 - t_2 - t_3)/2} = 2\left(1 + \frac{\frac{1}{2}(t_2 + t_3) + r - 6}{r - t_1 - t_2 - t_3}\right).$$

Hence, a face of size $r \geq 6$ can send at least 2 to each incident 3- or 4-vertex. Consider the case $r = 5$. Observe that at most two vertices incident with f are of degree 3 or 4, and if there are two of them, only one transfer by R1(c) can be applied. In this case, each of these two vertices receives at least $\frac{7}{4}$. If R1(a) or R1(b) is applied, then f is incident with at most one 3- or 4-vertex, which subsequently receives ≥ 3 from f. Finally, consider the $r = 4$. If one of rules R1(a)-(c) is applied, then at most one 3- or 4-vertex is incident with f and it receives ≥ 1 from f. Otherwise, at most two such vertices are incident with f and each of them receives $\geq \frac{2 \cdot 4 - 6}{2} = 1$. We can conclude that each big face sends at least 1 to each incident 3- or 4-vertex. Moreover, if it is of size ≥ 5 then it sends at least $\frac{7}{4}$ to each incident 3- or 4-vertex.

Let v be a d-vertex. Denote by $v_1, v_2, \ldots, v_d$ the neighbours of v around it, and denote by f_i the face which contains the subwalk $v_i v v_{i+1}$ (index modulo d). If f_i is a triangle, then the face which is different from f_i and incident with $v_i v_{i+1}$ is denoted by f_i'.

Suppose first that v is a 3-vertex. Suppose also that v is adjacent to at least two big vertices. If v is not incident with a big face, then, by R2, we obtain that $c^*(v) \geq -3 + 2 \cdot \frac{3}{2} = 0$. If v is incident with only one big face, then each of the big neighbours sends ≥ 1 to v. Thus, by R1 and R2, $c^*(v) \geq -3 + 1 + 2 \cdot 1 = 0$. If v is incident with at least two big faces, then each big neighbour of v sends $\geq \frac{1}{2}$ to it, so we have $c^*(v) \geq -3 + 2 \cdot 1 + 2 \cdot \frac{1}{2} = 0$.

Now, let v be adjacent to exactly one big vertex, say v_1; then the remaining neighbours v_2, v_3 are intermediate (it follows from the fact that $w \geq 9$). If v is incident with at least two big faces, then, by rules R1, R2, and R4 it follows that $c^*(v) \geq -3 + 2 \cdot 1 + \frac{1}{2} + 2 \cdot \frac{1}{3} > 0$. Suppose that v is incident with only one big face. If this face is f_2, then $c^*(v) \geq -3 + \frac{3}{2} + 1 + 2 \cdot \frac{1}{3} > 0$. Suppose now that this face is f_1 or f_3, say f_3. If $r(f_3) \geq 5$, then $c^*(v) \geq -3 + 1 + \frac{7}{4} + 2 \cdot \frac{1}{3} > 0$. Otherwise, let $f_3 = [v_3 v v_1 x]$ be a 4-face. If x is a big vertex, then f_3 sends at least 2 to v, and we infer easily that $c^*(v) > 0$. So, x is intermediate or small. Then, each neighbour of v_3 different from x, v, v_2 is big. Hence, the face f'_2 is either big or a triangle $[v_2 v_3 y]$ where y is a big vertex. In both cases, by an additional transfer through the edge $v_2 v_3$ by rule R5 or R1, vertex v receives 1. Finally, we obtain $c^*(v) \geq -3 + 1 + 1 + 1 + 2 \cdot \frac{1}{3} > 0$.

Suppose now that v is incident with three triangles. Let $f'_2 = [v_2 v_3 x]$ be a triangle. If x is big, then by rule R5, we infer $c^*(v) \geq -3 + \frac{3}{2} + 2 \cdot \frac{1}{3} + 1 > 0$. Otherwise, x is not a big vertex. Then, all neighbours of v_{2+j}, $j = 0, 1$, except v, x and v_{3-j} are big and v receives $\frac{1}{2} + \frac{1}{2}$ due to two additional transfers by R6 or R1 through the edges $v_1 v_2$ and $v_1 v_3$. Thus, $c^*(v) \geq -3 + \frac{3}{2} + 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} > 0$. Let $r(f'_2) \geq 4$ and $x_1 v_3 v_2 x_0$ be a part of its boundary walk. If x_0 or x_1 is big, then by R1(a) we have $c^*(v) \geq -3 + \frac{3}{2} + 2 \cdot \frac{1}{3} + 1 > 0$. Otherwise, x_0 and x_1 are not big. Then, all neighbours of v_{2+j}, $(j = 0, 1)$, except v, v_{3-j} and x_j are big. As before, v receives $\frac{1}{2} + \frac{1}{2}$ from two additional transfers by R6 or R1, which gives $c^*(v) > 0$.

Finally, suppose that all neighbours of v are not big. Then they are intermediate. If v is incident with at least two big faces, then $c^*(v) \geq -3 + 2 \cdot 1 + 3 \cdot \frac{1}{3} = 0$. Suppose that v is incident with exactly one big face, say f_3. Then, all neighbours of v_2 except v, v_1 and v_3 are big and v receives $1 + 1$ from two additional transfers by R5 or R1 (through the edges $v_1 v_2$ and $v_2 v_3$). Thus, $c^*(v) \geq -3 + 3 \cdot \frac{1}{3} + 1 + 2 \cdot 1 > 0$. If all faces incident with v are triangles, then all neighbours of v_j $(j = 1, 2, 3)$ except v and themselves are big and v receives $1 + 1 + 1$ from three additional transfers by R5 or R1.

Let v be a 4-vertex. Then, v is adjacent to at least one big vertex, say v_1 (otherwise, a light $K_{1,4}$ is found). If v is incident with at least two big faces, then $c^*(v) \geq -2 + 2 \cdot 1 = 0$. Similarly, if v is incident with exactly one big face different from f_1 or f_4, then $c^*(v) \geq -2 + 1 + 1 = 0$. So, suppose first that f_4 is the only big face incident with v. If some of v_2, v_3, v_4 is big then $c^*(v) \geq -2 + 1 + \frac{1}{2} + \frac{1}{2} = 0$. So, assume that v_1 is the only big neighbour of v. If two of v_2, v_3, v_4 are intermediate, then $c^*(v) \geq -2 + 1 + \frac{1}{2} + 2 \cdot \frac{1}{3} > 0$. Otherwise, at least two of them are 5-vertices. Consider the faces f'_2 and f'_3. If some of them is triangular, it has to contain a big vertex (to avoid light $K_{1,4}$) sending $\frac{1}{2}$ to v by R7; otherwise v is $\triangle^*$-adjacent to this face and receives $\frac{1}{2}$ by R1(c). Thus, $c^*(v) \geq -2 + 1 + \frac{1}{2} + 2 \cdot \frac{1}{2} > 0$.

Finally, suppose now that v is incident with four triangles. If each of v_2, v_3, v_4 is intermediate, then $c^*(v) \geq -2 + 1 + 3 \cdot \frac{1}{3} = 0$. If precisely two of them are intermediate, then the third one is a 5-vertex. In this case by rule R1(c) or R7, vertex v receives additional $\frac{1}{2}$, and so $c^*(v) =$

$-2+2\cdot\frac{1}{3}+1+\frac{1}{2}>0$. Finally, we may assume that at least two of v_2, v_3, v_4 are 5-vertices. Then, by R1(c) or R7, vertex v receievs $\frac{1}{2}+\frac{1}{2}$, and we infer that $c^*(v)\geq -2+2\cdot\frac{1}{2}+1=0$.

If v is a 5-vertex, then v is adjacent to at least two big vertices, and hence by R2, $c^*(v)\geq -1+2\cdot\frac{1}{2}=0$. If v is a 6-vertex, then v is adjacent to at least three big vertices and to at most three small vertices. So, by rules R3 and R4, $c^*(v)\geq 0+3\cdot\frac{1}{3}-3\cdot\frac{1}{3}=0$ or $c^*(v)\geq 0+3\cdot\frac{2}{9}-2\cdot\frac{1}{3}=0$. Let v be an intermediate vertex of degree >6. Then v is adjacent to at most three small vertices. Thus, by rule R4, we infer that $c^*(v)\geq 1-3\cdot\frac{1}{3}=0$.

Finally, let v be a big vertex of degree d. Consider the following averaging of the charges sent from v:

(A1) Each 3-vertex donates $\frac{3}{8}$ from its received charge to each its neighbour lying on triangular face with v.

(A2) Each 4-vertex donates $\frac{1}{8}$ from its received charge to each its neighbour lying on a triangular face with v.

(A3) Each 6-vertex donates $\frac{c}{4}$ from its received charge c (by rule R3, $c=\frac{1}{3}$ or $\frac{2}{9}$) to each its neighbour lying on triangular face with v; moreover, if such a neighbour is a big, then 6-vertex donates additional $\frac{c}{4}$ to it.

(A4) Let v_{i-1} and v_i be 6-vertices, v_{i+1} be a 3-vertex adjacent to v_i, and let v sends a charge through $v_{i-1}v_i$ by R5. Then, v_i sends (additional) $\frac{1}{18}$ to v_{i+1}.

(A5) If a 3-vertex receives by the rule R5 a charge from v through the edge v_iv_{i-1}, then it donates $\frac{1}{2}$ from its charge to each of v_{i-1}, v_i.

(A6) If a 3-vertex receives by the rule R6 a charge from v through the edge v_iv_{i-1}, then it donates $\frac{1}{4}$ from its charge to each of v_{i-1}, v_i.

(A7) If a 4-vertex receives by the rule R7 a charge from v through the edge v_iv_{i-1}, then it donates $\frac{1}{4}$ from its charge to each of v_{i-1}, v_i.

After this averaging, each 3- or 4-vertex which receives some charge by rules R5-R7 from v fully redistributes this charge by (A5)-(A7). It is not hard to observe that each 3-neighbour of v keeps $\leq\frac{3}{2}-2\cdot\frac{3}{8}+4\cdot\frac{1}{18}=\frac{35}{36}$ of charge received from v. Similarly, each 4-neighbour preserves at most $1-2\cdot\frac{1}{8}+2\cdot\frac{1}{12}=\frac{11}{12}$. Each intermediate neighbour of v with degree at least 7 keeps $\leq\frac{1}{2}+\frac{3}{8}+\frac{1}{18}=\frac{67}{72}$ and each big neighbour of v keeps at most $2\cdot(\frac{1}{4}+2\cdot\frac{1}{12})+\frac{1}{12}=\frac{5}{6}$.

Consider now a 5-neighbour v_i of v. If there is no transfer from v by R5-R7 through the edges v_iv_{i+1} and v_iv_{i-1}, then v_i keeps at most $\frac{1}{2}+2\cdot\frac{1}{8}=\frac{3}{4}$. If there are transfers from v by R5-R7 through both of the edges v_iv_{i+1} and v_iv_{i-1}, then a copy of light $K_{1,4}$ is obtained. Finally, assume that there is a transfer only through the edge v_iv_{i-1}. In this case, v_i keeps maximum charge if v_{i-1} is a 6-vertex and v_{i+1} is a 4-vertex. Then, the charge that v_i preserves is $\leq \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{12}=\frac{23}{24}$.

Finally, consider a 6-neighbour v_i of v. If there is no transfer from v by R5-R7 through the edges v_iv_{i+1} and v_iv_{i-1}, then v_i keeps $\leq \frac{1}{3}-2\cdot\frac{1}{12}+2\cdot\frac{3}{8}=\frac{11}{12}$. Let there be transfers through both of the edges $v_{i-1}v_i, v_iv_{i+1}$. Then, at least one of v_{i-1}, v_{i+1} is big (otherwise a light $K_{1,4}$ is found), say v_{i+1}. If v_{i-1} is not a 6-vertex, then v_i keeps maximum charge when v_{i-1} is an intermediate vertex of degree ≥ 7. In this case, the charge is $\leq \frac{2}{9}-3\cdot\frac{1}{18}+\frac{1}{4}+\frac{1}{2}=\frac{29}{36}$. Otherwise, v_{i-1} is a 6-vertex, and then similarly, v_i keeps $\leq \frac{2}{9}-2\cdot\frac{1}{18}+\frac{1}{4}+\frac{1}{2}=\frac{31}{36}$. Now, suppose that there is only one such transfer, say from v through the edge $v_{i-1}v_i$. If v_{i+1} is not a 3-vertex, then v_i keeps $\leq \frac{2}{9}-2\cdot\frac{1}{18}+\frac{1}{18}+\frac{1}{2}+\frac{1}{8}=\frac{19}{24}$. So, assume that v_{i+1} is a 3-vertex. If v_{i-1} is big, then v_i keeps $\leq \frac{1}{3}-3\cdot\frac{1}{12}+\frac{1}{4}+\frac{3}{8}=\frac{17}{24}$. If v_{i-1} is a 5-vertex, then v_i keeps $\leq \frac{1}{3}-2\cdot\frac{1}{12}+\frac{1}{4}+\frac{3}{8}=\frac{19}{24}$. Assume now that v_{i-1} is an intermediate vertex. If v_{i-1} is of degree ≥ 7, then v_i is adjacent with exactly two small vertices. Thus, v_i keeps at most $\frac{2}{9}-2\cdot\frac{1}{18}+\frac{1}{2}+\frac{3}{8}=\frac{71}{72}$. Assume now that v_i is a 6-vertex. In this case v_{i-1} has at most two small neighbours as well. Then, both v_{i-1}, v_i receive $\frac{2}{9}$ from v by rule R3 and (A4) is applied. Thus, v_i preserves again at most $\frac{3}{8}+\frac{1}{2}+\frac{2}{9}-\frac{1}{18}-\frac{1}{18}=\frac{71}{72}$.

We obtain that every neighbour of v receives average charge $\leq \frac{71}{72}$ from v. Hence, $c^*(v)\geq d-6-d\cdot\frac{71}{72}\geq 0$ for $d\geq 432$. This completes the proof. $\square$

The following table gives the sum of results on light structures in families $\mathcal{P}_3(w)$ and $\mathcal{T}_3(w)$. Denote by n the number of vertices of G.

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	$\mathcal{P}_3$	$\mathcal{P}_3(7)$	$\mathcal{P}_3(8)$	$\mathcal{P}_3(9)$	$\mathcal{P}_3(10)$
light structures	P_1, P_2	P_1, P_2 P_3	P_1, P_2 P_3, P_4	P_1, P_2 $P3, P_4$ $K_{1,3}, K_{1,4}$	P_1, P_2, P_3 P_4, C_3, C_4 $K_{1,3}, K_{1,4}$
heavy structures	G with $n \geq 3$	G with $n \geq 4$	G with $n \geq 4$	G with $n \geq 7$	G with $n \geq 11$
	$\mathcal{T}_3$	$\mathcal{T}_3(7)$	$\mathcal{T}_3(8)$	$\mathcal{T}_3(9)$	$\mathcal{T}_3(10)$
light structures	all paths	all paths	all paths	all paths, C_3 $K_{1,3}, K_{1,4}$	all paths, C_3 $C_4, K_{1,3}, K_{1,4}$
heavy structures	non-paths	non-paths	non-paths	see Prop. 3.3	G with $\Delta(G) \geq 5$

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