A stronger form of the theorem constructing a rigid binary relation on any set

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Summary. On every set A there is a rigid binary relation i.e. such a relation $\mathcal{R} \subseteq A \times A$ that there is no homomorphism $\langle A, \mathcal{R} \rangle \to \langle A, \mathcal{R} \rangle$ except the identity (Vopěnka et al. [1965]). We prove that for each infinite cardinal number κ if card $A \leq 2^{\kappa}$, then there exists a relation $\mathcal{R} \subseteq A \times A$ with the following property:

$$\forall x \in A \; \exists_{\substack{\{x\} \subseteq A(x) \subseteq A \\ \text{card } A(x) \leq \kappa}} \; \forall_{\substack{f : A(x) \to A \\ f \neq \text{id}_{A(x)}}} \; f \text{ is not a homomorphism of } \mathcal{R}$$

which implies that $\mathcal{R}$ is rigid. If a relation $\mathcal{R} \subseteq A \times A$ has the above property, then card $A \leq 2^{\kappa}$.

On every set A there is a rigid binary relation, i.e. such a relation $\mathcal{R} \subseteq A \times A$ that there is no homomorphism $\langle A, \mathcal{R} \rangle \to \langle A, \mathcal{R} \rangle$ except the identity ([8] and also [2],[3] [5]). Conjectures 1 and 2 below strengthen this theorem.

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Conjecture 1 ([6],[7]). If κ is an infinite cardinal number and card $A \leq 2^{2^{\kappa}}$, then there exists a relation $\mathcal{R} \subseteq A \times A$ which satisfies the following condition:

$$(\kappa^{*}) \qquad \forall_{\substack{x,y \in A \\ x \neq y}} \exists_{\substack{\{x\} \subseteq A(x,y) \subseteq A \\ \text{card } A(x,y) \leq \kappa}} \forall_{\substack{f : A(x,y) \to A \\ f(x) = y}}$$

f is not a homomorphism of $\mathcal{R}$.

Proposition 1a ([7]). If κ is an infinite cardinal number, $\mathcal{R} \subseteq A \times A$ satisfies condition (κ^{*}) and card $\widetilde{A} \leq$ card A, then there exists a relation $\widetilde{\mathcal{R}} \subseteq \widetilde{A} \times \widetilde{A}$ which satisfies condition (κ^{*}).

Proposition 1b ([6]). If $\mathcal{R} \subseteq A \times A$ satisfies condition (κ^{*}), then $\mathcal{R}$ is rigid. If κ is an infinite cardinal number and a relation $\mathcal{R} \subseteq A \times A$ satisfies condition (κ^{*}), then card $A \leq 2^{2^{\kappa}}$.

Theorem 1 ([7]). Conjecture 1 is valid for $\kappa = \omega$.

Conjecture 2 ([6],[7]). If $\kappa \neq 0$ is a limit cardinal number and card $A \leq 2^{\sup\{2^{\alpha} : \alpha \in \text{Card}, \alpha < \kappa\}}$, then there exists a relation $\mathcal{R} \subseteq A \times A$ which satisfies the following condition:

$$(\kappa^{**}) \qquad \forall_{\substack{x,y \in A \\ x \neq y}} \exists_{\substack{\{x\} \subseteq A(x,y) \subseteq A \\ \text{card } A(x,y) < \kappa}} \forall_{\substack{f : A(x,y) \to A \\ f(x) = y}}$$

f is not a homomorphism of $\mathcal{R}$.

Proposition 2a ([7]). If $\kappa \neq 0$ is a limit cardinal number, $\mathcal{R} \subseteq A \times A$ satisfies condition (κ^{**}) and card $\widetilde{A} \leq$ card A, then there exists a relation $\widetilde{\mathcal{R}} \subseteq \widetilde{A} \times \widetilde{A}$ which satisfies condition (κ^{**}).

Proposition 2b ([6]). If $\mathcal{R} \subseteq A \times A$ satisfies condition (κ^{**}), then $\mathcal{R}$ is rigid. If $\kappa \neq 0$ is a limit cardinal number and a relation $\mathcal{R} \subseteq A \times A$ satisfies condition (κ^{**}), then card $A \leq 2^{\sup\{2^{\alpha} : \alpha \in \text{Card}, \alpha < \kappa\}}$.

Theorem 2 ([6],[7]). Conjecture 2 is valid for $\kappa = \omega$.

In this article we prove a changed form of Conjecture 1 which holds for all infinite cardinal numbers κ, see Theorem 3, Corollary and Theorem 5.

Theorem 3. If κ is an infinite cardinal number and card $A \leq 2^{\kappa}$, then there exists a relation $\mathcal{R} \subseteq A \times A$ which satisfies the following condition:

$$(\kappa^{\diamond}) \qquad \forall x \in A \ \exists_{\substack{\{x\} \subseteq A(x) \subseteq A \\ \text{card } A(x) \leq \kappa}} \forall_{\substack{f : A(x) \to A \\ f \neq \text{id}_{A(x)}}}$$

f is not a homomorphism of $\mathcal{R}$.

Proof. It is known ([1] and also [2],[5]) that for each infinite cardinal number κ there exists a rigid symmetric relation $R \subseteq \kappa \times \kappa$. Let Φ denote the family of all relations $S \subseteq \kappa \times \kappa$ which satisfy:

(1) $R \subseteq S$,

(2) for each $\alpha, \beta \in \kappa$ if $\alpha \neq \beta$, then $\alpha S \beta$ or $\beta S \alpha$,

(3) for each $\alpha, \beta \in \kappa$ if $\alpha S \beta$ and $\beta S \alpha$, then $\alpha R \beta$ and $\beta R \alpha$.

Since R is rigid

(4) $R \subseteq \{(\alpha, \beta) : \alpha, \beta \in \kappa, \alpha \neq \beta\}$.

By (1) and (3) the following Lemma 1 holds true.

Lemma 1. If $S_1, S_2 \in \Phi$ and $f : \langle \kappa, S_1 \rangle \to \langle \kappa, S_2 \rangle$ is a homomorphism, then $f : \langle \kappa, R \rangle \to \langle \kappa, R \rangle$ is a homomorphism.

Lemma 2. For every $S_1, S_2 \in \Phi$ if $S_1 \neq S_2$, then $\mathrm{id}_\kappa : \langle \kappa, S_1 \rangle \to \langle \kappa, S_2 \rangle$ is not a homomorphism.

Proof. Applying (3) and (4) we obtain two cases. First case: there exist $\alpha, \beta \in \kappa$, $\alpha \neq \beta$ such that $(\alpha, \beta) \in S_1$ and $(\alpha, \beta) \notin S_2$, so id_κ is not a homomorphism. Second case: there exist $\alpha, \beta \in \kappa$, $\alpha \neq \beta$ such that $(\alpha, \beta) \in S_2$ and $(\alpha, \beta) \notin S_1$. By (2) $(\beta, \alpha) \in S_1$. It suffices to prove that $(\beta, \alpha) \notin S_2$. Suppose, on the contrary, that $(\beta, \alpha) \in S_2$. By (3) $(\alpha, \beta) \in R$, so by (1) $(\alpha, \beta) \in S_1$, a contradiction.

Lemma 3. card $\Phi = 2^\kappa$.

Proof. Let $T := \{\{\alpha, \beta\} : \alpha, \beta \in \kappa, \alpha \neq \beta, (\alpha, \beta) \notin R\}$. It suffices to prove that card $T = \kappa$. Suppose, on the contrary, that card $T < \kappa$. Hence card $\bigcup T < \kappa$ and consequently card $(\kappa \setminus \bigcup T) = \kappa$. For each $\alpha, \beta \in \kappa \setminus \bigcup T$ if $\alpha \neq \beta$, then $(\alpha, \beta) \in R$. From this and (4) any non-identical injection from κ into $\kappa \setminus \bigcup T$ is a homomorphism of R. This contradiction completes the proof of Lemma 3.

Now we turn to the main part of the proof. For each $\emptyset \neq \Psi \subseteq \Phi$ we define the relation $R_\Psi \subseteq (\kappa \times \Psi) \times (\kappa \times \Psi)$ by the following formula:

$$\forall \alpha, \beta \in \kappa \forall S_1, S_2 \in \Psi \Big(((\alpha, S_1), (\beta, S_2)) \in R_\Psi \iff (\alpha, \beta) \in S_1 = S_2 \Big).$$

In other words, the graph corresponding to the relation R_Ψ is a disjoint union of graphs belonging to Ψ. By Lemma 3 it suffices to prove that R_Ψ satisfies condition $(\kappa^\diamond)$. Let $(\lambda, S_1) \in \kappa \times \Psi$. We prove that $(\kappa \times \Psi)((\lambda, S_1)) := \kappa \times \{S_1\}$ satisfies condition $(\kappa^\diamond)$.

Suppose, on the contrary, that $f : \kappa \times \{S_1\} \to \kappa \times \Psi$ is a homomorphism of R_Ψ and $f \neq \mathrm{id}_{\kappa \times \{S_1\}}$. Then there exist $\alpha, \beta \in \kappa$ and

$S_2 \in \Psi$ such that $f((\alpha, S_1)) = (\beta, S_2)$ and $(\alpha, S_1) \neq (\beta, S_2)$. By (2) for each $\gamma \in \kappa \setminus \{\alpha\}$ $\alpha S_1 \gamma$ or $\gamma S_1 \alpha$. From this for each $\gamma \in \kappa \setminus \{\alpha\}$ $(\alpha, S_1) R_\Psi (\gamma, S_1)$ or $(\gamma, S_1) R_\Psi (\alpha, S_1)$. Therefore $f((\alpha, S_1)) R_\Psi f((\gamma, S_1))$ or $f((\gamma, S_1)) R_\Psi f((\alpha, S_1))$ and consequently $(\beta, S_2) R_\Psi f((\gamma, S_1))$ or $f((\gamma, S_1)) R_\Psi (\beta, S_2)$. In both cases there exists a $\delta \in \kappa$ such that $f((\gamma, S_1)) = (\delta, S_2)$. It implies that f maps $\kappa \times \{S_1\}$ into $\kappa \times \{S_2\}$. Let $\pi : \{S_1\} \to \{S_2\}$. There is a uniquely determined transformation $\widetilde{f} : \kappa \to \kappa$ such that $f = \langle \widetilde{f}, \pi \rangle$. Obviously, $\widetilde{f}(\alpha) = \beta$ and $\widetilde{f} : \langle \kappa, S_1 \rangle \to \langle \kappa, S_2 \rangle$ is a homomorphism. By Lemma 1 $\widetilde{f} : \langle \kappa, R \rangle \to \langle \kappa, R \rangle$ is a homomorphism. Since R is rigid $\widetilde{f} = \mathrm{id}_\kappa$. Therefore $\alpha = \widetilde{f}(\alpha) = \beta$ and $\mathrm{id}_\kappa : \langle \kappa, S_1 \rangle \to \langle \kappa, S_2 \rangle$ is a homomorphism. On the other hand, $\alpha = \beta$ and $(\alpha, S_1) \neq (\beta, S_2)$ implies $S_1 \neq S_2$. It is impossible by Lemma 2. This contradiction completes the proof of Theorem 3.

Remark 1. It is easy to observe that condition $(\kappa^\diamond)$ implies condition (κ^*). Obviously, if $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^\diamond)$, then $\mathcal{R}$ is rigid.

Theorem 3 follows easily from Perminov's ([4]) Theorem 4 on rigid tournaments. By a tournament on A we will understand a relation $Z \subseteq A \times A$ satisfying:

$$\forall_{\substack{x, y \in A \\ x \neq y}} \quad (x, y) \in Z \vee (y, x) \in Z,$$

$$\forall x, y \in A \quad (x, y) \in Z \Rightarrow (y, x) \notin Z.$$

Tournaments in [4] are defined differently, but Theorem 4 holds true for our definition. Unfortunately, the proof in [4] is only sketched.

Theorem 4 ([4]). For each infinite cardinal number κ there exists a family Γ of 2^κ rigid tournaments on κ such that there is no homomorphism between any two distinct tournaments belonging to Γ.

Corollary. For every $\Delta \subseteq \Gamma$ a disjoint union of tournaments belonging to Δ satisfies condition $(\kappa^\diamond)$. It gives a new proof of Theorem 3.

We will show an alternative, algebraic method for proving Theorem 3. This method described in the proof of Theorem 5 gives more, namely a symmetric relation satisfying condition $(\kappa^\diamond)$. Unfortunately, in contradistinction to the relation constructed in the proof of Theorem 3, a direct description of such a relation is very complicated.

Theorem 5. If κ is an infinite cardinal number and $\mathrm{card}\, A \leq 2^\kappa$, then there exists a symmetric relation $\mathcal{R} \subseteq A \times A$ which satisfies condition $(\kappa^\diamond)$.

Proof. Following [5] let **Graph** denote the category of graphs and their homomorphisms. The objects of **Graph** are couples (X, R) with $R \subseteq X \times X$, the morphisms from (X, R) to (X', R') are triples $((X', R'), f, (X, R))$ with $f : X \to X'$ such that $(f(x), f(y)) \in R'$ whenever $(x, y) \in R$, and it is viewed as a concrete category endowed with the natural forgetful functor. Let $\mathcal{C}_n$ $(n \geq 3)$ denote the category of connected n-chromatic undirected graphs and their homomorphisms. It is known (see [5], Theorem 4.12 on page 113), that for every $n \geq 3$ there is a strong embedding $F : \mathbf{Graph} \to \mathcal{C}_n$ that transforms objects of the cardinality κ into objects of the cardinality κ.

Let $R \subseteq \kappa \times \kappa$ be a rigid symmetric relation (undirected graph). Considering all possible orientations of R we obtain 2^κ rigid graphs with the property that there is no homomorphism between any two distinct graphs. Using the strong embedding $F : \mathbf{Graph} \to \mathcal{C}_n$ $(n \geq 3)$ constructed in [5] we obtain a family Γ of 2^κ rigid undirected graphs on κ with the property that there is no homomorphism between any two distinct graphs. Since Γ consists of connected graphs, for every $\Delta \subseteq \Gamma$ a disjoint union of graphs belonging to Δ satisfies condition $(\kappa^\diamond)$, the proof is similar to the proof that R_Ψ satisfies condition $(\kappa^\diamond)$. This completes the proof of Theorem 5.

Theorem 6. If κ is an infinite cardinal number and a relation $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^\diamond)$, then card $A \leq 2^\kappa$.

Proof. Suppose, on the contrary, that $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^\diamond)$ and card $A > 2^\kappa$. For each $x \in A$ we choose the set $A(x)$ from condition $(\kappa^\diamond)$. Let $\mathcal{B} := \{A(x) : x \in A\}$. Since $\bigcup \mathcal{B} = A$ we conclude that card $\mathcal{B} =$ card A. For each $B \in \mathcal{B}$ we choose a bijective $f_B : \text{card } B \to B$ and define the relation $R_B \subseteq \text{card } B \times \text{card } B$ by the following formula:

$$\forall \alpha, \beta \in \text{card } B \ \Big((\alpha, \beta) \in R_B \Longleftrightarrow (f_B(\alpha), f_B(\beta)) \in \mathcal{R} \Big).$$

Let

$$\mathcal{B} \ni B \xrightarrow{h} (\text{card } B, R_B) \in \bigcup_{\substack{\alpha \in \text{Card} \\ \alpha \leq \kappa}} \{\alpha\} \times \mathcal{P}(\alpha \times \alpha).$$

Since

$$\text{card } \mathcal{B} = \text{card } A > 2^\kappa = \text{card} \bigcup_{\substack{\alpha \in \text{Card} \\ \alpha \leq \kappa}} \{\alpha\} \times \mathcal{P}(\alpha \times \alpha)$$

we conclude that there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \neq B_2$ and $h(B_1) = h(B_2)$. Hence $\langle B_1, \mathcal{R}\rangle \overset{f_{B_2} \circ (f_{B_1})^{-1}}{\longrightarrow} \langle B_2, \mathcal{R}\rangle$ is a non-identical isomorphism. This contradiction completes the proof.

Conjecture 3. If $\kappa \neq 0$ is a limit cardinal number and card $A \leq \sup\{2^{\alpha} : \alpha \in \mathrm{Card}, \alpha < \kappa\}$, then there exists a relation $\mathcal{R} \subseteq A \times A$ which satisfies the following condition:

$$(\kappa^{\diamond\diamond}) \qquad \forall x \in A \ \exists_{\substack{\{x\} \subseteq A(x) \subseteq A \\ \mathrm{card}\, A(x) < \kappa}} \ \forall_{\substack{f : A(x) \to A \\ f \neq \mathrm{id}_{A(x)}}}$$

f is not a homomorphism of $\mathcal{R}$.

Remark 2. Obviously, if $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^{\diamond\diamond})$, then $\mathcal{R}$ is rigid. By Theorem 3 Conjecture 3 holds true if card $A < \sup\{2^{\alpha} : \alpha \in \mathrm{Card}, \alpha < \kappa\}$.

Theorem 7. If $\kappa \neq 0$ is a limit cardinal number and a relation $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^{\diamond\diamond})$, then card $A \leq \sup\{2^{\alpha} : \alpha \in \mathrm{Card}, \alpha < \kappa\}$.

Proof. Suppose, on the contrary, that $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^{\diamond\diamond})$ and card $A > \sup\{2^{\alpha} : \alpha \in \mathrm{Card}, \alpha < \kappa\}$. For each $x \in A$ we choose the set $A(x)$ from condition $(\kappa^{\diamond\diamond})$. Let $\mathcal{B} := \{A(x) : x \in A\}$. Since $\bigcup \mathcal{B} = A$ we conclude that card $\mathcal{B} =$ card A. For each $B \in \mathcal{B}$ we choose a bijective $f_B : \mathrm{card}\, B \to B$ and define the relation $R_B \subseteq \mathrm{card}\, B \times \mathrm{card}\, B$ by the following formula:

$$\forall \alpha, \beta \in \mathrm{card}\, B \ \Big((\alpha, \beta) \in R_B \Longleftrightarrow (f_B(\alpha), f_B(\beta)) \in \mathcal{R}\Big).$$

Let

$$\mathcal{B} \ni B \xrightarrow{h} (\mathrm{card}\, B, R_B) \in \bigcup_{\substack{\alpha \in \mathrm{Card} \\ \alpha < \kappa}} \{\alpha\} \times \mathcal{P}(\alpha \times \alpha).$$

Since

$$\mathrm{card}\, \mathcal{B} = \mathrm{card}\, A > \sup\{2^{\alpha} : \alpha \in \mathrm{Card}, \alpha < \kappa\} = \mathrm{card} \bigcup_{\substack{\alpha \in \mathrm{Card} \\ \alpha < \kappa}} \{\alpha\} \times \mathcal{P}(\alpha \times \alpha)$$

we conclude that there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \neq B_2$ and $h(B_1) = h(B_2)$. Hence $\langle B_1, \mathcal{R}\rangle \overset{f_{B_2} \circ (f_{B_1})^{-1}}{\longrightarrow} \langle B_2, \mathcal{R}\rangle$ is a non-identical isomorphism. This contradiction completes the proof.

Proposition 3. By Theorems 3 and 7, if $\kappa \neq 0$ is a limit cardinal number, $\mathcal{R} \subseteq A \times A$ satisfies condition $(\kappa^{\diamond\diamond})$ and card $\widetilde{A} \leq$ card A, then there exists a relation $\widetilde{\mathcal{R}} \subseteq \widetilde{A} \times \widetilde{A}$ which satisfies condition $(\kappa^{\diamond\diamond})$.

Theorem 8. Conjecture 3 is valid for $\kappa = \omega$ i.e. there exists a relation $\mathcal{R} \subseteq \omega \times \omega$ satisfying condition $(\omega^{\diamond\diamond})$.

Proof. The relation $\mathcal{R} := \{(i, i+1) : i \in \omega\} \cup \{(0,2)\} \subseteq \omega \times \omega$ satisfies condition $(\omega^{\diamond\diamond})$. Indeed, for each $i \in \omega$ the set $A(i) := \{j \in \omega : j \leq i+2\}$ is adequate for property $(\omega^{\diamond\diamond})$.

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