List distance labelings of graphs

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Abstract

In this paper, we study the distance choosability - the list counterpart of the distance labelings. In particular, we show that the Alon-Tarsi theorem for choosability in graphs has an analogous version for the choosability of distance labelings, a notion stemming from channel assignment. We apply this result on paths and cycles for labeling with a condition at distance two.

1 Introduction

One of the main issues concerning the efficient use of radio spectra in telecommunication is the design and analysis of efficient algorithms for frequency assignment. The main task is to assign transmitters frequencies from the shortest possible range while maintaining an adequate quality of

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the signal. The signal clarity depends on the possible interference of simultaneous transmitting from close sources. The level of interference depends mainly on distance of transmitters, but also on other factors, i.e. on the hilliness of the terrain (see [5]).

The graph theoretic model of the frequency assignment transforms the geometric instance into a discrete structure. In the most simplest case we draw a graph whose vertices represent transmitters and adjacent transmitters should use different frequencies. Clearly, it is a graph coloring but in this simple version the coloring approach for channel assignment provides only rough level on channel separation. In practice there are used several levels on channel separation so we adopt a notion of distance constrained labeling [11, 12], where close vertices should have labels separated by a specified parameter that depends on the graph distance. These labelings were investigated in [10, 14, 16, 19] also for their interesting graph-theoretic properties.

Moreover we consider a situation, where transmitters my support only a limited list of frequencies (due to physical or organizational reasons). Then our aim is to select labels for vertices from the given lists such that all distance constraints are met. We call this the *distance constrained list labeling* problem (we omit the term distance constrained if no confusion may appear).

We note here that list version of graph coloring was introduced by Vizing [18] and independently by Erdős, Rubin, and Taylor [6] and extensively studied by many authors, for a surveys see [1, 13, 17, 20].

Graph labeling generalizes graph coloring, so we expect that its computational complexity should not be simpler in general. The graph coloring problem belongs among the hardest combinatorial optimization problems, it is strongly *NP*-hard and also inapproximable within the factor $n^{1/7-\varepsilon}$, where n denotes the number of vertices of the graph [4].

On the other hand, the First-fit coloring heuristic might bring satisfactory results for restricted classes of graph that might appear in practice [8], and is also easily extendible for the list coloring problem. The First-fit method, selects colors for vertices of a graph G as follows: Order vertices arbitrarily, process them one by one and assign to a vertex the smallest available color, that means the least number from the assigned list that has not been yet used on an adjacent vertex.

It is well known, that if the vertices of the graph can be ordered such that each vertex is adjacent to at most d predecessors, then the lists of size $d+1$ assure that the First-fit algorithm finds a suitable coloring. This ordering can be transformed to an acyclic orientation with maximum indegree at most d, where edges are oriented towards succcessors. Observe that this approach on d-regular graphs gives relatively weak bound of $d+1$ colors on each vertex, which is very close to the maximum degree.

Alon and Tarsi [2] extended this result to special orientations (see Theorem 1). In specific cases a suitable orientation might bring twice better upper bound than the First-fit method, e.g. in the case of bipartite k-regular graphs. However, the proof is not constructive, that means, that we only know that such list coloring with bounded number of colors exists, and there is not known a tool which finds it in polynomial time yet.

We reproduce the statement of Theorem 1 in Section 3. The hard part of its practical application is the proof of the inequality between the number of even and odd Eulerian subgraphs for a particular orientation of the underlying graph. Even from graphs derived from cycles it might not be an easy task. On the other hand, this method is up to now the only way to prove a statement that every 4-regular graph composed of a Hamiltonian cycle and a set of disjoint triangles is 3-choosable [9].

The main contribution of this paper is the proof of an analogon of the Alon-Tarsi theorem for distance constrained list labelings. A similar approach for so called T-colorings was shown in [3]. Furthermore, as an example for the application for the extended theorem we provide sharp upper bound on the sizes of lists for the list labelings of cycles and paths with constraints $(2,1)$, as an extension of the labeling results of [10], also with a possible application in liear and circular transmitter networks.

2 Preliminaries

We assume that the set of natural numbers does not contain zero, i.e. $\mathbb{N} = \{1, 2, \dots\}$.

Let $G = (V, E)$ be a finite undirected loopless multigraph with vertices $V = \{v_1, \dots, v_n\}$ and the multiset of edges E over $\binom{V}{2}$. We say that a multigraph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

The symbols C_n and P_n denote the cycle and the path on n vertices.

An orientation of multigraph G provides a directed multigraph $\vec{D}$, on the same vertex set V. For each edge $(u, v) \in E(G)$, the graph $\vec{D}$ contains one of ordered pairs $[u, v]$ or $[v, u]$, where the first vertex in the pair is called *tail* and the second is called *head*. We define *indegree* and *outdegree* of an directed multigraph as follows:

$$\mathrm{indeg}(u) = |\{[v,u] \in E(\vec{D})\}| \qquad \mathrm{outdeg}(u) = |\{[u,v] \in E(\vec{D})\}|.$$

A directed multigraph is called *Eulerian* if for every vertex u : $\mathrm{indeg}(u) = \mathrm{outdeg}(u)$. The set of all Eulerian subgraphs of a directed multigraph $\vec{D}$ is denoted by $\mathcal{E}(\vec{D})$. We also define sets $\mathcal{E}_o(\vec{D})$ and $\mathcal{E}_e(\vec{D})$ as sets of Eulerian subgraphs with odd and with even number of edges, respectively. Thus, $\mathcal{E}(\vec{D}) = \mathcal{E}_o(\vec{D}) \cup \mathcal{E}_e(\vec{D})$.

A *list assignment* of G is a function L which assigns to each vertex $u \in V(G)$ a *list* $L(u) \subseteq \mathbb{N}$ of *admissible* colors for u. The number $|L(u)|$ we call the *size* of list $L(u)$. An *L-coloring* is a function $c : V(G) \to \mathbb{N}$ such that $c(u) \in L(u)$ for each $u \in V(G)$ and such that $c(u) \neq c(v)$ whenever u and v are adjacent. If G admits a L-coloring, it is *L-colorable*. We say that G is *k-choosable*, if admits an L-coloring for every list assignment L such that $|L(u)| \geq k$ for each $u \in V(G)$. The smallest k for which G is k-choosable is called the *choice number* of G and it is denoted by $\chi^\ell(G)$.

In this paper we address a generalization of graph coloring called distance labelings. For a sequence $P = (p_1, \dots, p_k)$ of positive integers called *distance constraints* and list-assignment L of a graph G we define an L_P-*labeling* as a mapping $c : V(G) \to \mathbb{N}$ such that

(1) $\forall u \in V(G) : c(u) \in L(u)$, and

(2) $|c(u) - c(v)| \geq p_i$ for every $u, v \in V(G)$ at distance at most $i \leq k$.

It follows from the definition that without loss of generality we may assume that numbers in P appear in the decreasing order. To avoid a possible misunderstanding we note here that P stands for distance constraints only in Section 3, while in Section 4 the symbol P_n means the path on n vertices.

The non-list version of the above concept can be obtained if we select all lists of available labels as the integer interval $[1, n]$ for some fixed n. Then the smallest n for which G has a labeling satisfying distance constraints P is denoted by $\chi_P(G)$. In the literature is also used the graph invariant $\lambda_P(G) = \chi_P(G) - 1$, but since we consider list labelings we found more convenient the invariant χ_P than λ_P.

Similarly, as above, the minimum number k for which G has an L_P-labeling, for every list assignment L with lists of size k, is denoted by $\chi^\ell_P(G)$. Observe that in general $\chi_P(G) \leq \chi^\ell_P(G)$.

For only one distance constraint $P = (1)$ the $L_{(1)}$-labeling becomes equal to the list coloring, i.e. $\chi^\ell_{(1)}(G) = \chi^\ell(G)$. Similarly for $p_1 = p_2 = \cdots =$

$p_k = 1$, we get $\chi^{\ell}_{(1,\ldots,1)}(G) = \chi^{\ell}(G^k)$, where G^k is the k-th power of G, i.e. a graph which arise from G by adding edges connecting vertices at distance at most k. Therefore, the notion of distance-constrained labeling generalizes the concept of coloring of powers of graphs as well.

3 Alon-Tarsi theorem for distance labeling

One of the most interesting results in the theory of list-colorings is the following result [2].

Theorem 1 (Alon and Tarsi) *Let G be a graph and $\vec{D}$ be its orientation satisfying $|\mathcal{E}_e(\vec{D})| \neq |\mathcal{E}_o(\vec{D})|$. If L is a list assignment of G such that $|L(u)| \geq \operatorname{indeg}(u) + 1$ for all vertices $u \in V(G)$, then the graph G is L-colorable.*

Through this section we assume that $P = (p_1, \ldots, p_k)$ is a fixed k-tuple of distance constraints. Denote by G^P the multigraph with the same vertex set as G such that every two vertices of distance $i \in \{1, \ldots, k\}$ are connected with a cluster of $2p_i - 1$ multiple edges.

We extend Theorem 1 for the concept of distance constrained list labelings as follows:

Theorem 2 *Let L be a list assignment of a graph G and P be a tuple of distance constraints. Suppose that for an orientation $\vec{D}$ of G^P it holds that $|\mathcal{E}_e(\vec{D})| \neq |\mathcal{E}_o(\vec{D})|$ and $\operatorname{indeg}(u) + 1 \leq |L(u)|$ for every vertex $u \in V(\vec{D})$. Then, G admits an L_P-labeling.*

Proof: We follow the ideas from [2, 3]. Assign to every vertex u a variable x_u and consider a polynomial in $\mathbb{N}^{|V(G)|}$:

$$f^P_G = \prod_{i=1}^{k} \prod_{j=1-p_i}^{p_i-1} \prod_{\substack{u,v \in V(G) \\ dist(u,v)=i}} (x_u - x_v - j).$$

In the definition of polynomial f^P_G as well as in the forthcoming definition of polynomial g^P_G, we assume that vertices of G are linearly ordered, and that u is a predecessor of v. If c is an L_P-labeling of G then due to the statement (2) of the definition of L_P-labeling , the polynomial f^P_G evaluated at $x_u = c(u)$ has a nonzero value, since every term in the product has a nonzero value as well. Moreover, from (1) follows that for each vertex $u \in V(G)$ there exist a polynomial $r_u(x_u)$ in x_u of degree at most $|L(u)|-1$ such that

$$\prod_{s \in L(u)} (x_u - s) = 0 = x_u^{|L(u)|} - r_u(x_u).$$

We expand f_G^P into a linear combination of monomials and recursively replace every occurrence of $x_u^{|L(u)|}$ by $r_u(x_u)$, until the degree of every variable x_u in every monomial of the modified polynomial h_G^P is at most $|L(u)|-1$. Observe that the new polynomial h_G^P retains its value of f_G^P for all selections $x_u \in L(u)$.

Now consider the multigraph G^P, its orientation $\vec{D}$ and a polynomial

$$g_G^P = \prod_{(u,v) \in E(G^P)} (x_u - x_v),$$

where the term $(x_u - x_v)$ appears in g_G^P as many times as the edge (u, v) in $E(G)$. It follows from the construction of G^P that every monomial with a nonzero coefficient in g_G^P appears also with the same coefficient in f_G^P.

Corollary 2.3. of [2] shows, that in the polynomial g_G^P, the coefficient by the monomial $M = \prod_{u \in V} x_u^{\mathrm{indeg}(u)}$ is equal to $|\mathcal{E}_o(\vec{D})| - |\mathcal{E}_e(\vec{D})|$, perhaps with the multiplicative (-1) factor. By our assumptions, M has nonzero coefficients in all polynomials g_G^P, f_G^P, and h_G^P, since:

(1) M was not reduced, because it contains no $x_u^{|L(u)|}$, and

(2) monomial M cannot be obtained by such reduction from other monomial $\prod x_u^{d_u}$, because the reduction decreases the value $\sum d_u \leq |E(G^P)|$ and the sum of degrees in M attains the upper bound $|E(G^P)|$.

We may summarize that in this moment we know that h_G^P contains a nonzero coefficient by monomial M, and the degree of every variable in every monomial is at most $\mathrm{indeg}(u) = |L(u)| - 1$.

If we assume that the value of h_G^P is zero for all possible selections $x_u \in L(u)$, we get a contradiction with Lemma 2.1. in [2], since such polynomial $h_G^P \equiv 0$.

We conclude the proof by the argument that any selection of variables $x_u \in L(u)$ having a non-zero value of h_G^P is in one-to-one correspondence with a feasible L_P-labeling of the graph G. □

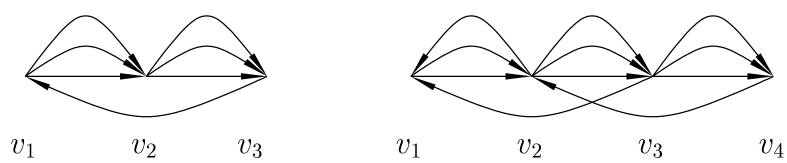

Figure 1: Orientations of graphs $P_3^{(2,1)}$ and $P_4^{(2,1)}$.

4 Distance choosability for paths and cycles

It is not difficult to show that for k distance constraints $(1, \dots, 1)$ and the path P_n on n vertices $\chi_{(1,\dots,1)}(P_n) = \chi^{\ell}_{(1,\dots,1)}(P_n) = \min(n, k)$. Recently, Prowse and Woodall [15] proved that the equality $\chi_{(1,\dots,1)}(C_n) = \chi^{\ell}_{(1,\dots,1)}(C_n)$ holds also for cycles.

In this section we apply Theorem 2 for $L_{(2,1)}$-labelings and prove that the equality $\chi_{(2,1)}(G) = \chi^{l}_{(2,1)}(G)$ holds for both paths and cycles.

Proposition 3 *Let P_n be a path on $n \geq 2$ vertices. Then,*

$$\chi^{\ell}_{(2,1)}(P_n) = \begin{cases} 3, & n = 2 \\ 4, & n = 3, 4 \\ 5, & n \geq 5. \end{cases}$$

Proof: Since $\chi^{\ell}_{(2,1)}(P_n) \geq \chi_{(2,1)}(P_n)$ and the above equality is satisfied for $\chi^{\ell}_{(2,1)}(P_n)$ replaced by $\chi_{(2,1)}(P_n)$ (see [10]) it will be enough if we prove $\chi^{\ell}_{(2,1)}(P_n)$ is smaller than or equal to the right side of the above expression. Assume the the vertices of the path P_n appear in order $v_1, v_2, \dots, v_n$. When $n = 2$ the statement is straightforward. In the case $n \geq 5$, the First-fit algorithm works assuming that every vertex has 5 available colors and vertices are ordered as they appear on the path.

For $n = 3$ or 4, let $\vec{D_3}, \vec{D_4}$ be the oriented graphs obtained from $P_3^{(2,1)}$ and $P_3^{(2,1)}$ as depicted in Fig. 1.

Both graphs $\vec{D_3}$ and $\vec{D_4}$ have maximum indegree 3. It is easy to calculate that $|\mathcal{E}_o(\vec{D_3})| = 9$ and $|\mathcal{E}_e(\vec{D_3})| = 1$. Graphs from $\mathcal{E}_o(\vec{D_4})$ may have only 3 or 5 edges. Their number is 15 and 21, respectively. Similarly, graphs from $\mathcal{E}_e(\vec{D_4})$ may have 0,2,6, or 8 edges. Their number is 1,2,18, and 9, respectively. Thus, $|\mathcal{E}_o(\vec{D_4})| = 36$ and $|\mathcal{E}_e(\vec{D_4})| = 30$. Now, the statement follows by Theorem 2. $\square$

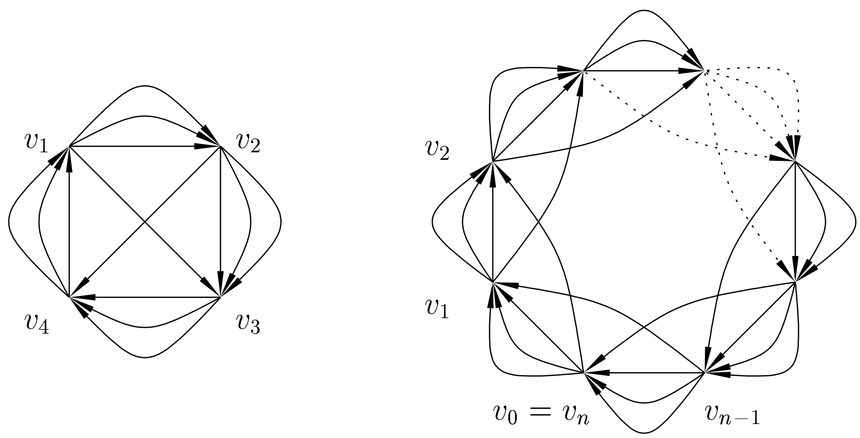

Figure 2: Orientations of graphs $C_4^{(2,1)}$ and $C_n^{(2,1)}$.

In [10] is proven that $\chi_{(2,1)}(C_n) = 5$ for cycles. We will extend this result for distance constrained list labelings.

Theorem 4 *If C_n is a cycle on $n \geq 3$ vertices, then $\chi^{\ell}_{(2,1)}(C_n) = 5$.*

Proof: Since $\chi^{\ell}_{(2,1)}(C_n) \geq \chi_{(2,1)}(C_n) = 5$ it will be enough if we prove $\chi^{\ell}_{(2,1)}(C_n) \leq 5$. Assume that the vertices in the graph C_n appear in the order $v_1, v_2, \ldots, v_n = v_0, v_1$.

Suppose first that $n = 3$. Let L be a list assignment of C_3 with lists of size 5. Then assume that a_1 is the smallest color in $L(v_1) \cup L(v_2) \cup L(v_3)$ and $a_1 \in L(v_1)$. Now let a_2 be the smallest color from $L(v_2) \cup L(v_3)$ greater than $a_1 + 1$. We may assume that $a_2 \in L(v_2)$. Finally, let a_3 be the greatest color from $L(v_3)$. Now it is easy to see that labeling c with $c(v_i) = a_i$ for each $i = 1, 2, 3$ is an $L_{(2,1)}$-labeling.

Consider the case $n \geq 4$. Let $\vec{C}$ be the digraph constructed from $C_n^{(2,1)}$ so that every edge is oriented towards the vertex with greater index by one or two. All computations on indices i of v_i are done modulo n with the only exception that for diagonals of C_4 we use the classical addition. These orientations are depicted in Fig. 2. We call the edges of form $[v_i, v_{i+1}]$ *short*, while those of type $[v_i, v_{i+2}]$ are *long*. We use the following simpler notation without the argument $\vec{C}$, i.e. $\mathcal{E}_e = \mathcal{E}_e(\vec{C})$ and $\mathcal{E}_o = \mathcal{E}_o(\vec{C})$.

Suppose now that $n = 4$. Then $\mathrm{indeg}(v_1) = \mathrm{indeg}(v_2) = 3$ and $\mathrm{indeg}(v_3) = \mathrm{indeg}(v_4) = 4$. Observe that each Eulerian subgraph with odd number of edges contains at least one of the arcs $[v_1, v_3]$ and $[v_2, v_4]$. Now it is easy to evaluate that the Eulerian subgraphs from $\mathcal{E}_o$ may have 3, 7, or 11 edges. The number of such subgraphs are 18, 162, and 18, respectively. The number of Eulerian subgraphs with 0, 4, 6, 8, 10, or 12 edges is 1, 81, 27, 81, 27, 1, respectively. Thus, $|\mathcal{E}_e| = 218$ and $|\mathcal{E}_o| = 198$, and the equality $\chi^{\ell}_{(2,1)}(C_4) = 5$ follows directly from Theorem 2.

We now prove the case of cycles of length at least 5. The basic idea is as follows: We derive recursive formulas for the numbers of Eulerian subgraphs with specified degrees. Then we provide an explicit formula for the difference of number of even and odd Eulerian subgraphs and show that it never attains zero. Because the indegree of every vertex in the proposed orientation $\vec{C}$ is 4, we conclude, that for lists of size 5 a feasible $L_{(2,1)}$-labeling of C_n always exists.

Suppose that $n \geq 5$. Let $S \subseteq \{0, 1, 2, 3, 4\}$. We denote by $\mathcal{E}_e^S$ and $\mathcal{E}_o^S$ the Eulerian subgraphs of $\vec{C}$ with indegrees in S and with even and odd number of edges, respectively. Moreover, we assume that for every number from S there exists at least one vertex in $\vec{C}$ with indegree equal to this number. For simplicity, in the expression $\mathcal{E}^S$ we write the set S without braces, thus $\mathcal{E}^{0,1}$ means the same as $\mathcal{E}^{\{0,1\}}$.

Observe that whenever S contains i, j with $|i - j| \geq 2$, then no such Eulerian subgraph exists, hence $\mathcal{E}_e^S = \mathcal{E}_o^S = \emptyset$. The total number of edges in $\vec{C}$ is even. The complement operation $\phi : \mathcal{E} \to \mathcal{E}$ with $\phi(H) = \vec{C} - E(H)$ is one-to one mapping between sets $\mathcal{E}_e^i$ and $\mathcal{E}_e^{4-i}$, or $\mathcal{E}_e^{i,i+1}$ and $\mathcal{E}_e^{4-i,3-i}$. Due to this complementarity we get that $|\mathcal{E}_e^0| = |\mathcal{E}_e^4|$, $|\mathcal{E}_e^{0,1}| = |\mathcal{E}_e^{3,4}|$, etc. The same equalities hold for odd subgraphs as well. Thus, it is sufficient to consider only graph classes $\mathcal{E}_e^0, \mathcal{E}_e^{0,1}, \mathcal{E}_e^1, \mathcal{E}_e^{1,2}, \mathcal{E}_e^2$, and their counterparts for odd Eulerian subgraphs.

The subgraph with no edges is even Eulerian, hence $|\mathcal{E}_e^0| = 1$ and $|\mathcal{E}_o^0| = 0$. The number of edges of a 1-regular digraph is equal to the number of its vertices, hence $|\mathcal{E}_o^1| = 0$ for n even and $|\mathcal{E}_e^1| = 0$ for n odd. In the other case there are 3^n cycles composed of short edges and one Eulerian subgraph composed by all long edges. We have $|\mathcal{E}_o^1| = 3^n + 1$ for odd n and $|\mathcal{E}_e^1| = 3^n + 1$ for n even.

Now, we calculate the order of $\mathcal{E}_e^{0,1}$ and $\mathcal{E}_o^{0,1}$. In $\vec{C}$, we remove the edge $[v_{n-1}, v_1]$, and cut the vertex $v_n = v_0$ into two new vertices v_0, v_n such that the new v_0 is incident with all outgoing edges and the new v_n with all incoming. The remaining graph $\vec{P}$ is isomorphic to the uniform orientation of the graph $P_{n+1}^{(2,1)}$. Denote by x_n^e (and by x_n^o) the number of directed paths of even (and of odd) length from the vertex v_0 to v_n in $\vec{P}$. The numbers x_n^e and x_n^o are defined by the recursive relations:

$$\begin{array}{lll} x_1^e = 0, & x_2^e = 9, & x_n^e = 3x_{n-1}^o + x_{n-2}^o \\ x_1^o = 3, & x_2^o = 1, & x_n^o = 3x_{n-1}^e + x_{n-2}^e. \end{array}$$

Observe that the number of graphs from $\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1$ (resp. $\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1$) which contain an edge incident with the vertex v_0 and do not contain $[v_{n-1}, v_1]$ is x_n^e (resp. x_n^o). Similarly, there are x_{n-2}^o (and x_{n-2}^e) graphs from $\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1$ (and $\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1$) which contain the long edge $[v_{n-1}, v_1]$ and no edge incident with v_0. Finally, only the subgraph induced by all n long edges has the property that contains both the edge $[v_{n-1}, v_1]$ and an edge incident with the vertex v_0. In total we got that

$$\begin{array}{rcl} |\mathcal{E}_o^{0,1} \cup \mathcal{E}_o^1| & = & x_{n-1}^o + x_{n-2}^e + (n \bmod 2) \\ |\mathcal{E}_e^{0,1} \cup \mathcal{E}_e^1| & = & x_{n-1}^e + x_{n-2}^o + 1 - (n \bmod 2), \end{array}$$

where $(n \bmod 2)$ is 1 for n odd and 0 for n even.

Since every 2-regular digraph has even number of edges, it follows that $\mathcal{E}_o^2 = \emptyset$. Now we show that $\mathcal{E}_e^2$ contains precisely $2 \cdot 3^n$ graphs. Since, we can connect v_i and v_{i+1} with two edges in three different ways, it follows that the number of graphs from $\mathcal{E}_e^2$ which contains only short edges is 3^n. If a graph from $\mathcal{E}_e^2$ contains a long edge then it contains all long edges of $\vec{C}$. Similarly as above, we obtain that $\mathcal{E}_o^2$ contains exactly 3^n such graphs. Thus, $|\mathcal{E}_e^2| = 2 \cdot 3^n$.

We proceed by calculating $|\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2|$. Denote by $\mathcal{P}_n$ the set of subgraphs in $\vec{P}$, that define an 2-flow from v_0 to v_n with even number of edges. More formally such graphs satisfy $\text{outdeg}(v_i) = \text{indeg}(v_i)$ for all $i = 1, \ldots, n-1$ and $\text{indeg}(v_n) = \text{outdeg}(v_0) = 2$. We call such graphs *even 2-paths*.

Note that every even 2-path starts/ends with either two short edges or one short and one long edge. In what follows $p_n(a, b)$ (where $a, b \in \{l, s, *\}$) denotes the numbers of graphs from $\mathcal{P}_n$ with the following property: if $a = l$ then this 2-path starts with one long and one short edge, if $a = s$ then it starts with two short edges and if $a = *$ then the type of the edges by which this 2-path starts is not important. Similarly b denotes the way this graph ends. For example, $p_n(*, *) = |\mathcal{P}_n|$ and $p_n(s, l)$ is the number of graphs from $\mathcal{P}_n$ which start with two short edges and ending with one long and one small edge. For simplicity, we write $p_n = p_n(*, *)$. Observe that for every $x \in \{l, s, *\}$ and $n \geq 2$:

$$\begin{aligned} p_n(x,l) &= p_n(l,x) = p_n(*,x) - 3p_{n-1}(*,x) \\ p_n(x,s) &= p_n(s,x) = 3p_{n-1}(*,x). \end{aligned}$$

From these relations we derive that for $n \geq 3$:

$$\begin{aligned} p_n(l,l) &= p_n - 6p_{n-1} + 9p_{n-2} \\ p_n(s,l) &= 3p_{n-1} - 9p_{n-2} \\ p_n(s,s) &= 9p_{n-2}. \end{aligned}$$

In a similar way as above define an *odd 2-path* in $\vec{P}$. Denote by $\mathcal{Q}_n$ the set of odd 2-paths in $\vec{P}$. We define a similar parameter $q_n(a,b)$ for graphs from $\mathcal{Q}_n$ and obtain the similar equations as above for $p_n(a,b)$. Similarly we write $q_n = q_n(*, *)$.

Now consider an even 2-path in $\vec{P}$. The three possible ways of building of such path as an extension of shorter one are depicted in Fig. 3. This directly implies two recursive relations between p_n and q_n:

$$p_n = 3p_{n-1} + q_{n-1} + 6q_{n-2} \quad \text{and} \quad q_n = 3q_{n-1} + p_{n-1} + 6p_{n-2}.$$

Note that $p_1 = 3$, $p_2 = 9$, $q_1 = 0$, and $q_2 = 9$. Thus, the above two relations provide the recursive definition of sequences (p_n) and (q_n).

We now focus our attention back to graphs from $\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2$ and $\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2$. In exactly p_n graphs from $\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2$ the vertex v_0 has outdegree two. Now we count the number of graphs from $\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2$, in which v_0 has outdegree one. Such graphs necessarily contain the long edge $[v_{n-1}, v_1]$. Denote by e^+ (resp. e^-) the edge which goes out (resp. in) from v_0. The number of graphs in which both e^+ and e^- are short is exactly $q_n(s,s)$. Similarly the number of graphs for which e^+ and e^- are long edges is $q_n(l,l)/9$. Finally the number of graphs in which e^+ is long and e^- is short or vice-versa is $q_n(l,s)/3 + q_n(s,l)/3$. Thus, we obtain that

$$\begin{aligned} |\mathcal{E}_e^{1,2} \cup \mathcal{E}_e^2| &= p_n + q_n(s,s) + \frac{q_n(l,s) + q_n(s,l)}{3} + \frac{q_n(l,l)}{9} \\ &= p_n + \frac{1}{9}q_n + \frac{4}{3}q_{n-1} + 4q_{n-2}. \end{aligned}$$

In a similar way one can show that

$$|\mathcal{E}_o^{1,2} \cup \mathcal{E}_o^2| = q_n + \frac{1}{9}p_n + \frac{4}{3}p_{n-1} + 4p_{n-2}.$$

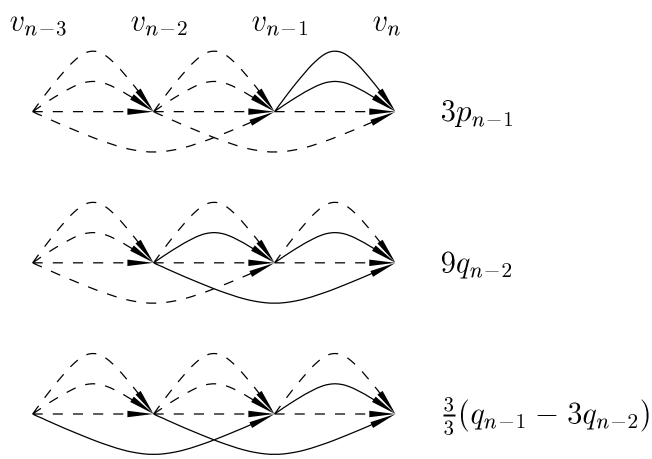

Figure 3: Composition of a 2-path from shorter 2-paths.

Now, the term $|\mathcal{E}_e| - |\mathcal{E}_o|$ can be expressed as

$$\begin{aligned}
|\mathcal{E}_e| - |\mathcal{E}_o| &= 2 + 2(x^e_{n-1} + x^o_{n-2} + 1 - (n \bmod 2)) + \\
&+ 2(p_n + \frac{1}{9}q_n + \frac{4}{3}q_{n-1} + 4q_{n-2}) - 2 \cdot 3^n - \\
&- \Big[2(x^o_{n-1} + x^e_{n-2} + (n \bmod 2)) + 2(q_n + \frac{1}{9}p_n + \frac{4}{3}p_{n-1} + 4p_{n-2})\Big] \\
&= 2\Big[2 - 3^n - 2(n \bmod 2) + x^e_{n-1} - x^e_{n-2} - x^o_{n-1} + x^o_{n-2} + \\
&+ \frac{8}{9}(p_n - q_n) + \frac{4}{3}(q_{n-1} - p_{n-1}) + 4(q_{n-2} - p_{n-2})\Big]
\end{aligned}$$

To simplify above expression define $x_n = x^o_n - x^e_n$ for every n. Then, we infer that $x_1 = 3$, $x_2 = -8$, and $x_n = -3x_{n-1} - x_{n-2}$. Hence, one can calculate

$$x_n = \left(-\frac{1}{2} - \frac{3\sqrt{5}}{10}\right)\left(\frac{-3-\sqrt{5}}{2}\right)^n + \left(-\frac{1}{2} + \frac{3\sqrt{5}}{10}\right)\left(\frac{-3+\sqrt{5}}{2}\right)^n.$$

Similarly define, $r_n = q_n - p_n$ for every n. Then, we obtain that $r_1 = -3$,

n	3	4	5	6	7	8	9	10
$\|\|\mathcal{E}_e\| - \|\mathcal{E}_o\|\|/3^n$	3.25	2.86	0.12	1.57	1.71	3.18	9.02	2.06

Table 1: Relative difference for small n

$r_2 = 0$, and $r_n = 2r_{n-1} - 6r_{n-2}$. Hence

$$r_n = \left(-\frac{1}{2} - \frac{i}{\sqrt{5}}\right)(1 - i\sqrt{5})^n + \left(-\frac{1}{2} + \frac{i}{\sqrt{5}}\right)(1 + i\sqrt{5})^n.$$

Assume that for φ and ψ we have

$$\sqrt{6}e^{\pm i\varphi} = 1 \mp i\sqrt{5} \qquad \text{and} \qquad \frac{3}{2\sqrt{5}}e^{\pm i\psi} = -\frac{1}{2} \mp \frac{i}{\sqrt{5}}.$$

Now, in turn we have

$$r_n = \frac{3}{2\sqrt{5}}\cos(n\varphi - \psi)(\sqrt{6})^n.$$

For $n < 11$ we have evaluated the difference of the number of even and odd Eulerian subgraphs as shown in Table 4.

Now, assume that $n \geq 11$. For $i \geq 10$, observe that

$$\frac{|x_i|}{3^i} < 0.15 \quad \text{and} \quad \frac{|r_i|}{3^i} < 0.15$$

By using the following identities

$$x^e_{i-1} - x^e_{i-2} - x^o_{i-1} + x^o_{i-2} = -4x_{i-1} - x_n \quad \text{and}$$

$$\frac{8}{9}(p_i - q_i) + \frac{4}{3}(q_{i-1} - p_{i-1}) + 4(q_{i-2} - p_{i-2}) = -\frac{14}{9}r_i + \frac{8}{3}r_{i-1},$$

we obtain

$$\begin{aligned}\frac{||\mathcal{E}_e| - |\mathcal{E}_o||}{3^n} &\geq 2 - \frac{1}{3^n} - \frac{4|x_{n-1}|}{3 \cdot 3^{n-1}} - \frac{|x_n|}{3^n} - \frac{4|x_{n-1}|}{3 \cdot 3^{n-1}} - \frac{14|r_n|}{9 \cdot 3^n} - \frac{8|r_{n-1}|}{9 \cdot 3^{n-1}} \\ &> 2 - \frac{1}{3^{11}} - 0.15\left(\frac{4}{3} + 1 + \frac{4}{3} + \frac{14}{9} + \frac{8}{9}\right) \\ &> 0.\end{aligned}$$

This argument concludes the proof of Theorem 4. □

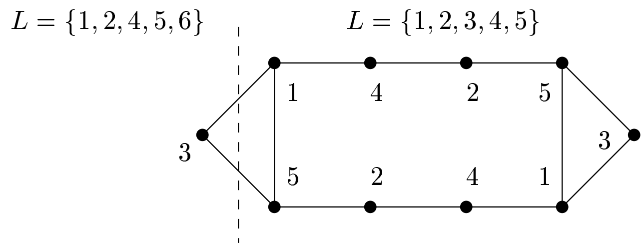

Figure 4: A graph satisfying the strict inequality $\chi_{(2,1)}(G) < \chi^{\ell}_{(2,1)}(G)$

5 Conclusion and open problems

In this study we have proved and analogon of the Alon-Tarsi theorem for distance labelings and have applied this result on paths and cycles and distance constraints $(2,1)$. Since this method brings a non constructive improvement up to the factor at most two the list sizes against the classical First-fit algorithm, we believe that the methods and calculations involved for computing odd and even Eulerian subgraphs are interesting in the theoretical sense rather than applicable in practice.

On the other hand, by use of this method we proved that the equality $\chi_{(2,1)}(G) = \chi^{\ell}_{(2,1)}(G)$ holds for all cycles and paths. This is a special case of the general inequality

$$\chi_P(G) \leq \chi^{\ell}_P(G).$$

It would be interesting to classify all graphs for which the equality holds with respect to given distance constraints P. For the standard graph coloring and choosability, i.e. $P = (1)$, there are known constructions of graphs satisfying the strict inequality [6]. The easiest example of such graph is the complete bipartite graph K_{k,k^k} satisfying $\chi(K_{k,k^k}) = 2 < k < \chi^{\ell}(K_{k,k^k})$.

We expect that in the case of distance constrained labeling it would be easier to construct such examples of graphs as there are more constraints on the possible labeling.

As an example we would like to present a graph depicted in Fig. 4. A feasible labeling showing $\chi_{(2,1)}(G) = 5$ is indicated by the numbers by the vertices, while the for the indicated lists it is impossible to find a feasible $L_{(2,1)}$-labeling. Therefore, $\chi^{\ell}_{(2,1)}(G) > 5 = \chi_{(2,1)}(G)$.

Another possible direction of the further research would consider the relation of distance constrained labeling with similar distance constraints.

Observe that in the non-list version we may assume that distance constraints have no common divisor [7, 10]:

Let $P = (p_1, p_2, \dots, p_s)$ and we denote by kP the tuple $(kp_1, kp_2, \dots, kp_s)$. Then for every graph G it holds that

$$k(\chi_P(G) - 1) = \chi_{kP}(G) - 1.$$

It could be an interesting adventure to search for some necessary as well as sufficient conditions under which the application of *list labeling* the equality is still maintained:

$$k(\chi^{\ell}_P(G) - 1) = \chi^{\ell}_{kP}(G) - 1.$$

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