Homomorphism Bounded Classes of Graphs

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Abstract

A class $\mathcal{C}$ of graphs is said to be *H-bounded* if each graph in the class $\mathcal{C}$ admits a homomorphism to H. We give a general necessary and sufficient condition for the existence of bounds with special local properties. This gives a new proof of Häggkvist-Hell theorem [5] and implies several cases of the existence of triangle free bounds for planar graphs.

1 Introduction

In this paper we study mainly coloring of graphs in the setting of graph homomorphism. Recall that a homomorphism from G to H is any edge-preserving mapping $f : V(G) \longrightarrow V(H)$, (i.e. $xy \in E(G) \Longrightarrow f(x)f(y) \in E(H)$. The existence of a homomorphism from G to H is denoted by $G \longrightarrow H$, this sometime is called H-coloring. This notion captures the coloring problems by means of the following observation:

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$$\chi(G) \leq k \quad \text{iff} \quad G \longrightarrow K_k.$$

It also allows us to treat many combinatorial problems in a more general setting, for example in the context of partial order, as the existence of a homomorphism defines a quasiorder $\leq$ which is called *coloring order*

$$G \leq H \text{ if and only if G} \to H.$$

The following is then the main concept of this paper: We say that a class $\mathcal{C}$ of graphs is *bounded by* H if $G \leq H$ (or, equivalently, $G \longrightarrow H$) for any $G \in \mathcal{C}$. This simple order-theoretic concept may take the form of a profound problem when applied to a concrete class of graphs.

Remark on terminology: In several earlier papers any graph H which bounds a class $\mathcal{C}$ is called universal. We believe that this is a bit confusing as the notion of universal graph (for a class $\mathcal{C}$) is usually reserved for those graphs H which belong to the class $\mathcal{C}$ (i.e. which is a greatest elements of the class $\mathcal{C}$). We believe that the present notations are more fitting in our context.

The characterization of boundedness (and the estimation of the size of H) is the basic problem of chromatic (extremal) theory. In particular, the 4CT asserts that the class $\mathcal{P}$ of all planar graphs is bounded by K_4. Another related classical result is the Grötzsch theorem [4]: The class of all triangle free planar graphs is K_3-bounded, see any graph-theory textbook, or [13] for a short proof. In our setting of the Grötzsch theorem a certain asymmetry of the statement becomes apparent: "triangle free bounded by triangle". This led to the following problems posed in [8], [9]:

Problem 1 *Is the class of K_4-free planar graphs bounded by a K_4-free graph?*

Problem 2 *Is the class of K_3-free planar graphs bounded by a K_3-free graph?*

One can ask whether the class of all planar graph is bounded by a K_5-free graph. Note that the answer to this last problem is positive by the virtue of 4CT [2], [11], but in view of the difficulty of the proofs we should perhaps ask for an independent proof (not relying on a computer).

The purpose of this paper is to give a necessary and sufficient condition for the existence of bounds of this type. This is related to the following concept which is of independent interest:
Given a graph property P we say that a proper coloring of a graph G is (m, P)-coloring if the subgraph of G induced by any m-color classes has property P.
Roughly speaking, we prove that for certain properties P a class $\mathcal{C}$ of graphs is H-bounded with H having a (m, P)-coloring iff all the graphs $G \in \mathcal{C}$ have a (m, P)-coloring by a fixed number of colors. This holds for properties P like $P(k)$, k-colorablity, or more generally, *locally F-colorable* graphs, see Theorem 7 below. Notice that $(k, P(k-1))$-colorability for a graph H exists iff H is a K_k-free and that $(m, P(2))$-colorability implies high odd girth.

To formulate the next result we introduce the following:
For a finite set $\mathcal{F}$ of graphs we denote by $Forb_h(\mathcal{F})$ the class of all graphs G satisfying $F \not\longrightarrow G$ for any $F \in \mathcal{F}$.
The main result of [5] and [3] can be reformulated as follows:

Theorem 3 *For any $d \geq 1$ and any finite class of graphs $\mathcal{F}$, the class of connected graphs $Forb_h(\mathcal{F})$ with maximum degree d and chromatic number k is bounded by a k-chromatic graph in $Forb_h(F)$.*

Below (in Section 3) we give a proof of this result as a consequence of our main result (see Theorem 7 and Proposition 3). One can see that most of our results are partial results toward solving the following:

Problem 4 *Is it true that for any finite set of connected graphs $\mathcal{F}$ any set of planar graphs in $Forb_h(\mathcal{F})$ is bounded in $Forb_h(\mathcal{F})$?*

The paper is organized as follows: In section 2 we reduce the boundedness of classes of graphs to (m, P)-colorings while in Section 3 we prove the existence of (m, P)-colorings for bounded degree graphs. In Section 4 we present results on triangle free and large odd girth planar graphs. Section 5 contains some further remarks and open problems.

2 Construction of bounds for bounded classes

For a graph property P we introduced the notion of an (m, P)-coloring of a graph G (recall: the subgraph induced by the union of any m color classes should have property P). We also denote by $\chi_{m,P}(G)$ the minimal number of classes in a (m, P)-coloring of G (providing that it exists).

The following definition is a key construction, compare [1, 10, 12] for constructions of similar flavour:

Definition Let m and n be positive integers and U a graph, then let $\Pi = \Pi(n, m, U)$ be the graph whose vertex set is the set of ordered pairs (i, ϕ), where $1 \leq i \leq n$ and ϕ is a function from the m-sets of $\{1, 2, \dots n\}$ which contain i, to $V(U)$ and whose edge set is the set $((i, \phi), (j, \psi))$ for which $i \neq j$ and $\phi(S)$ is adjacent to $\psi(S)$ for all m-sets of $\{1, 2, \dots n\}$ which contain both i and j.
The graph $\Pi(n, m, U)$ is n-partite and has order $n \times |V(U)|^{\binom{n-1}{m-1}}$.

Proposition 5 *Let P_U be the property of U-colorability. If $\chi_{m,P_U}(G) \leq n$ then there is a homomorphism from G to $\Pi(n, m, U)$.*

Proof. Let $c : V(G) \to [n]$ be an (m, P)-coloring of G, then for each $S \subseteq [n]$ of cardinality m the vertices colored from S induce a subgraph G_S of G with property P and so there is a homomorphism $\rho_S : G_S \to U$. Now define $f : V(G) \to V(\Pi(n, m, U))$ by $f(v) = (c(v), \phi)$, where ϕ is defined by $\phi(S) = \rho_S(v)$. We must show that f is a homomorphism. If u and v are adjacent vertices in G then set $f(u) = (i, \phi)$ and $f(v) = (j, \psi)$. Since c is a proper coloring $i \neq j$ and, if $\{i, j\} \subseteq S$ then $\phi(S) = \rho_S(u)$ which is adjacent to $\rho_S(v) = \psi(S)$ so f is a homomorphism. $\square$

For special type of properties we prove the converse of this result.
Definition Let G and U be graphs. The graph G is said to be *m-locally U-colorable* if every subgraph of G induced by m or fewer vertices admits a homomorphism into U.
Examples
i. m-locally K_2-colorable graphs are graphs with odd girth at least m.
ii. k-locally K_{k-1}-colorable graphs are just graphs not containing K_k.

Proposition 6 *The graph $\Pi(n, m, U)$ is m-locally U-colorable.*

Proof. Let Π_1 be the subgraph of $\Pi(n, m, U)$ induced by vertices $\{(i_k, \phi_k)\}_{k=1}^m$. Let S=$\{i_1, i_2, \ldots i_m\}$ and let S be any k-subset containing S'. Then the mapping $(i_k, \phi_k) \to \phi_k(S)$ is a homomorphism. $\square$

Together these propositions give

Theorem 7 *A class $\mathcal{C}$ of graphs is bounded by a m-locally U-colorable graph if and only if $\{\chi_{m,P_U}(G)|\ G \in \mathcal{C}\}$ is bounded where P_U stands for U-colorability.*

Proof. Propositions 5 and 6 give the "if" part of the theorem. Conversely, if a class $\mathcal{C}$ of graphs has a m-locally U-colorable universal graph H, then H itself gives an (m, P_U)-coloring of all the graphs in $\mathcal{C}$. $\square$

We will be mainly concerned with the following two instances of this theorem (propositions 8 and 9).

Proposition 8 *For each n and each class $\mathcal{C}$ of graphs the following statements are equivalent, where $P(n-1)$ is the property of $(n-1)$-colorability.*

(a) $\mathcal{C}$ is bounded by a K_n-free graph.

(b) $\{\chi_{n,P(n-1)}(G)|\ G \in \mathcal{C}\}$ is bounded.

(c) $\{\chi_{n+1,P(n-1)}(G)|\ G \in \mathcal{C}\}$ is bounded.

Proof. Observe that K_n-free is equivalent to both n-locally K_{n-1}-colorable and $(n+1)$-locally K_{n-1}-colorable (since the smallest n-chromatic graph other than K_n has $n+2$ vertices) and apply theorem 7. $\square$

Our second particular choice is the property $B = P(2)$ of being bipartite.

Proposition 9 *For each odd n and each class $\mathcal{C}$ of graphs the following statements are equivalent where B is the property of the class of bipartite graphs.*

(a) $\mathcal{C}$ is bounded by a graph of odd girth n.

(b) $\{\chi_{n-2,B}(G)|\ G \in \mathcal{C}\}$ is bounded.

(c) $\{\chi_{n-1,B}(G)|\ G \in \mathcal{C}\}$ is bounded.

Proof. Having odd girth n is equivalent to both $(n-1)$-locally K_2-colorability and $(n-2)$-locally K_2-colorability. Now apply theorem 7. $\square$

3 Proof of Theorem 3

Let $\mathcal{F}$ be a finite set of connected graphs. Put $m = \max\{|V(F)|; F \in \mathcal{F}\}$ and let U is the disjoint union of all (non-isomorphic) graphs in $Forb_h(\mathcal{F})$ which have at most m vertices. Then $Forb_h(\mathcal{F})$ is precisely the class of all m-locally U-colorable graphs. (This is easy to see: any $G \in Forb_h(\mathcal{F})$ is clearly m-locally U-colorable. If G is m-locally U-colorable then the existence of a homomorphism $F \to G$ for an $F \in \mathcal{F}$ would violate the U-colorability of the image of F in G.)
Let $\mathcal{C}_d$ be the subclass of $Forb_h(\mathcal{F})$ of all graphs G with the maximal degree $\Delta(G) \leq d$. We prove that for any $G \in \mathcal{C}_d$ holds $\chi_{m,P_U}(G) \leq d^{2m+1}$. This will finish the proof as, by Theorem 7, the class $\mathcal{C}_d$ is then bounded by a graph H in $Forb_h(\mathcal{F})$. Moreover if all graphs in $\mathcal{C}_d$ are k-colorable then $\mathcal{C}_d$ is bounded by the k-colorable graph $H \times K_k \in Forb_h(\mathcal{F})$.
Put $t = 2m$. Given $G \in \mathcal{C}_d$ consider the graph $G^{(t)}$ on the same set of vertices where two distinct vertices are joined by an edge iff they are joined in G by a path of length $\leq t$. As $\Delta(G^{(t)}) < d^{t+1}$ there exists a d^{t+1}-coloring of G so that any two distinct vertices of G with their distance $\leq t$ are colored differently. We prove that this is an (m, P_U)-coloring: Let G' be a subgraph of G induced by any m-classes. As any two vertices of G' colored by the same color have the distance $\geq 2m$, any path of length $< 2m$ will take distinct colors. Thus any connected component of G' has at most m vertices and thus, by the definition of U, the graph G' is U-colorable.

4 Bounds with given odd girth

The problem of bounds for set of planar graphs with a given odd girth has been studied in various papers. The Grötszch theorem is of this type. We use $\mathcal{P}_k$ to denote the set of all planar graphs with odd girth at least k (note that $\mathcal{P}_{2k} = \mathcal{P}_{2k+1}$). Grötzsch's theorem then simply states that C_3 is a bound for $\mathcal{P}_4$. Another result of this type is Zhu's recent proof that C_{2k+1} is a bound for $\mathcal{P}_{8k-3}$, and it is conjectured that C_{2k+1} even bounds $\mathcal{P}_{4k+1}$ [15].
In this section we will study the following problem:

Problem 10 *For any given $k \geq 1$ does there exists a bound H for the subclass $\mathcal{P}_{2k+1}$ of planar graphs with $oddgirth(H) = 2k+1$?*

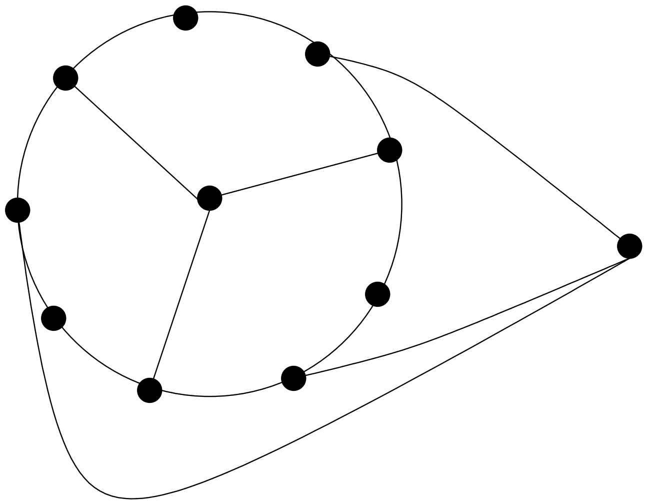

Figure 1: H

By Proposition 9 this is equivalent to asking whether $\{\chi_{2k-1,B}(G)|\ G \in \mathcal{P}_{2k+1}\}$ (or, equivalently, $\{\chi_{2k,B}(G)|\ G \in \mathcal{P}_{2k+1}\}$) is bounded.
In this setting the conjecture of [8] reads as follows:

Conjecture 11 *$\mathcal{P}_4$ is bounded by a triangle free graph.*

By applying Proposition 9 this is equivalent to:

Conjecture 12 *There exists a number l for which every $G \in \mathcal{P}_4$ has an l-coloring such that every odd cycle takes at least 4 different colors.*

Examples like the graph of Fig. 1 shows that a triangle-free bound for $\mathcal{P}_4$, if it exists, can not be very small. In fact any such a graph has to contain graph H of Fig. 1 as subgraph (as any 2 vertices of this graph are joined by a path of length 3). It is also not hard to see that any triangle-free bound for $\mathcal{P}_4$ is nonplanar:

Theorem 13 *$\mathcal{P}_4$ is not bounded by a triangle-free planar graph H.*

Proof. For a contradiction, suppose that H is a planar triangle-free bound and assume H has minimum number of vertices. It follows that H is a core (i.e. it does not have a proper retract). We show that H has minimum degree at least four. Since H is also triangle-free, this contradicts Euler's formula.

Let B be the graph obtained by joining two 5-cycles with an edge. This graph contains a set of four vertices $\{a, b, c, d\}$, each of which is at distance three from the other three. Form a new graph H' by taking a copy H_1 of H and joining each vertex v of H_1 to the vertices $\{a, b, c, d\}$ of a copy B_v of B. Since H' is still in P_4, there is homomorphism from H' to H. As H is a core, the restriction of this homomorphism to H_1 is an automorphism of H. Since also the images of the vertices corresponding to $\{a, b, c, d\}$ in each B_v are disjoint, each vertex of H has degree at least four, as required. $\square$

The following related result puts our results in yet another context.

Proposition 14 *The class of planar graphs with girth at least* $4k$ *is bounded by a graph* H *with* $oddgrith(H) = 2k + 1$.

Proof. Let P have girth at least $4k$, then the dual $P^\star$ of P has edge connectivity at least $4k$ and so, by the theorem of Tutte and Nash-Williams, [7, 14], has $2k$ edge disjoint spanning trees $T_1, T_2, \ldots, T_{2k}$. Let D_i be the union of T_i and T_{i+1} (taking subscripts modulo $2k$). Each D_i is 2-edge connected and so admits a nonzero Z_4-flow, by the 4CT, which may be extended to a flow ϕ_i on all of $P^\star$ by defining it to be zero on edges not in D_i. Define T_0 to be the set of edges that do not lie in any T_i $(i \geq 1)$. For each edge e in T_0 define a Z_2 flow by choosing any cycle containing the edge e with every other edge contained in some T_i and letting the value of the flow be 1 for edges in this cycle, 0 otherwise. Let ϕ_0 be the sum of these flows over all $e \in T_0$.

The cartesian product of the ϕ_i $(0 \leq i \leq 2k)$ is thus a nonzero $Z_2 \times Z_4^{2k}$-flow ϕ on $P^\star$. If e is an oriented edge in T_i $(i \geq 1)$,then $\phi(e)$ vanishes on all of the last $2k$ coordinates except the ith the $(i-1)$th. This flow induces a corresponding $Z_2 \times Z_4^{2k}$-coloring $\tilde{\phi}$ on P. In view of Proposition 9, it suffices to prove that $\tilde{\phi}$ gives each odd cycle in P at least $2k$ colors. In fact we prove this for every cycle.

Let C be a cycle in P which takes m vertex colors under $\tilde{\phi}$. Define an edge-colored graph H whose vertex set is the set of colors taken by vertices in C and join vertices c and c' by an edge colored i if there are adjacent vertices v and v' in C, colored c and c' respectively, which are joined by an

edge whose dual is in T_i (this edge coloring is well defined because, if the dual of vv' is in T_i then $c - c'$ is nonzero on exactly the ith and the $(i-1)$th of the last $2k$ coordinates for $i > 0$ and zero on all of the last $2k$ coordinates for $i = 0$). Since each of the trees T_i $(i \geq 1)$ meets each cutset in $P^\star$, H has at least one edge of each colour $i \geq 1$.
Choose one edge of each color $i \geq 1$ in H, and suppose $|V(H)| \leq 2k$. In this case a subset of these edges forms a cycle in H with vertices, in order, $c_1, c_2, \dots c_s$. Since

$$\sum (c_{j+1} - c_j) = 0,$$

(taking subscripts modulo s) and each difference $c_{j+1} - c_j$ is nonzero on a different adjacent pair of the last $2k$ coordinates, we must have $s \geq 2k$. Thus, in any case, $|V(H)| \geq 2k$ and, since $m = |V(H)|$, the proof is completed. □

5 Remarks and open problems

1. If m is odd and B is the property of being bipartite, then the (m, B)-colorings, considered in the previous section, are only possible on graphs with odd girth at least $m + 2$. On the other hand nothing is said about the number of colors taken by an even cycle. We now consider colorings which are (m, B)-colorings when the odd girth is at least $m + 2$ but are defined for all graphs and for which there is a specified minimum number of colors on even as well as odd cycles.
In view of Proposition 9, a proof of the following conjecture would give an affirmative answer to problem 10.

Conjecture 15 *For each n there exists a constant C_n for which every planar graph admits an coloring by at most C_n colors, for which every cycle of length $\geq k$ takes at least* $\min\{\lceil (k/2) \rceil + 1, n\}$ *distinct colors.*

The proof of Proposition 14 shows that this result holds for planar graphs of girth at least $2n$.
Remark For each fixed k, there is no constant C for which every planar graph admits a C-coloring in which every cycle of length k takes more than $\lceil (k/2) \rceil + 1$ distinct colors. For even k, a graph with many independent paths of length $k/2$ with common end points shows this (by the pigeonhole principle, if there are enough paths, some two must take the same colors

and hence the circuit they form takes at most $k/2+1$ colors. If we join two adjacent interior points of each path by a path of length two, a similar argument gives the bound for odd cycles. Hence the numbers in the conjecture above cannot be increased).

These ideas can be taken further. Roughly speaking, we try to use a fixed number of colors to color each planar graph in such a way that each subgraph takes as many different colors as possible.

For a planar graph G, we define $v(G)$ to be the largest number n for which there is some constant C such that every planar graph has a C coloring in which every subgraph isomorphic to G takes at least n colors. Clearly $v(G)$ is at most the order of G, but is generally less.
Example Let S be a star with $k+1$ vertices. We show that $v(S)=2$. Clearly $v(S)\geq 2$. On the other hand, for every n, there is a planar graph P such that, for every n-coloring of P, there is a two colored subgraph of P isomorphic to S. We need only take P to be the star with $1+kn$ vertices. By the pigeonhole principle some k vertices other than the centre must take the same color. These vertices together with the star centre induce a two-colored S in P.

As another example, we have also seen that $v(G)\leq \lceil (k/2)\rceil+1$ for a k-cycle.

The following is a strengthening of Conjecture 15.

Conjecture 16 *For each n there exists a constant A_n for which every planar graph admits an coloring by at most A_n colors for which every subgraph G takes at least* $\min\{v(G),n\}$ *distinct colors.*

2. For a graph G, we denote by G^k the graph with the same set of vertices where the edges correspond to pairs of distinct vertices joined by a trail of length k. We confine our attention to odd k

It is an easy observation that if k is odd and a class $\mathcal{C}$ is bounded by a graph in H with odd girth at least $\geq k+2$ then the graphs in $\mathcal{C}$ also have odd girth at least $\geq k+2$ and H^k bounds the graphs G^k whenever $G\in\mathcal{C}$. Thus there is a common bound (namely $|V(H)|$) for $\chi(G^k)$ when $G\in\mathcal{C}$.

We do not know whether the converse to this statement is true. This can be formulated as follows:

Conjecture 17 *If k is odd the graphs in $\mathcal{C}$ also have odd girth at least $\geq k+2$ and $\{\chi(G^k); G \in \mathcal{C}\}$ is bounded then $\mathcal{C}$ is bounded by a graph with odd girth at least $k+2$.*

In particular the following is a weakening of Conjecture 11

Conjecture 18 $\{\chi(G^3); G \in \mathcal{P}_4\}$ *is bounded*

We do not know any example of a graph $G \in P_4$ for which $\chi(G^3) > 11$. The graph of the Fig. 1 shows that 11 can be attained.

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