A Note on Gauss codes.

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Abstract

This paper is concerned with closed curves on a sphere and properties of their Gauss codes. We enlighten here one implication of a Rosenstiehl's theorem and moreover, we show another property of a Gauss code in theorem 1.10. As far as I know this property has not been noticed before. Recently, P.O. de Mendez has found an easy proof of this property which we include.

0. Introduction First, let's make some definitions.

0.1. Definition: A **graph** G is an ordered set $G = (V, E)$ where V is a finite set of vertices, $E \subseteq \binom{V}{2}$ are edges.

0.2. Definition: A **closed cross curve on the sphere** S **in general position** is an image of a continuous function $f : I \to S, I =< a, b >\subset \mathbf{R}$ closed interval bounded by $a, b \in \mathbf{R}$ such that if x, y, z are mutually different numbers then $f(x) \neq f(y)$ or $f(x) \neq f(z)$. For $x \neq y$ and $f(x) = f(y)$, $f(x)$ is a crossing point. Also we demand that $f(a) = f(b)$ and that we have an orientation of the curve. Further we shall talk just about a **curve**.

0.3. Definition: A set of curves on a sphere is in **general position** if no sphere's point is a triple image and wherever two of curves touch they cross each other. Moreover we demand for the curves to have a finite number of common points-double images.

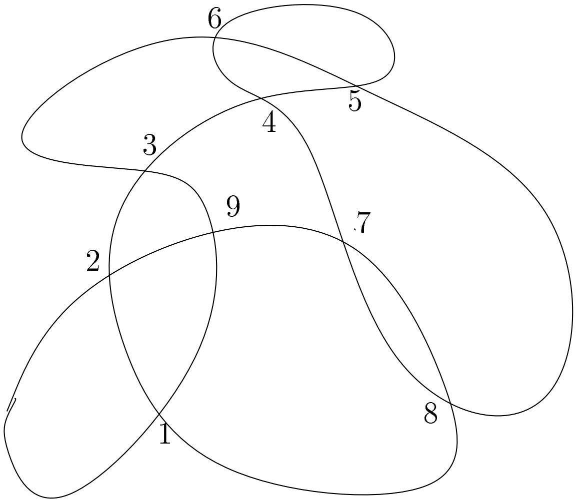
0.4. Definition: A **drawing of a graph** $G = (V, E)$ onto a sphere S is a set of functions $f_V : V \to S, f_e(e \in E) :< 0, 1 >\to S$, such that f_V is a one to one mapping and each $f_e(e = \{v_1, v_2\})$ is a curve with endpoints v_1, v_2. We also assume that image of f_V is disjoint with each $f_e[(0, 1)]$. The image of f_e we shall often call simply edge e.

1. Gauss codes

First, let's make some definitions.

1.1. Definition: Having a curve Ω, let's denote its intersections with itself by some mutually different symbols. A **Gauss code** of Ω we shall call a word come up by passing Ω around through from a chosen starting point in direction of orientation with synchronously writing down the symbols belonging to intersections of the curve with itself passed through.

1.2.0. Note: If we denote n_Ω the number of Ω's crossing points, i.e. number of symbols, a Gauss-code of the oriented curve Ω has length $2n_\Omega$ as each symbol occurs twice exactly. We have exactly $2n_\Omega$ Gauss-codes of Ω, one cycled from another.



Gauss code=123456478563912978
Figure 1

1.2.1. Example: The curve from Figure 1 has these Gauss-codes: 123456478563912978, 234564785639129781, 345647856391297812, 456478563912978123, 564785639129781234, 647856391297812345, 478563912978123456, 785639129781234564, 856391297812345647, 563912978123456478, 639129781234564785, 391297812345647856, 912978123456478563, 129781234564785639, 297812345647856391, 978123456478563912, 781234564785639129, ...

1.3. Definition: Let C_Ω denote the set of (mutually different) symbols representing crossing-points of a curve Ω with itself, ω be a Gauss code of Ω. Between two occurences of the same symbol $c \in C_\Omega$ there is an intermediate word $\delta_\omega(c)$. There are two possible subwords $\delta_\omega(c)$ in all of Ω's Gauss-codes. Now **interlaced with** c we will call the symbols with exactly single occurence in $\delta_\omega(c)$.

1.4. Note: If ω is a Gauss code of a curve, c symbol of ω then each of two possible subwords $\delta_\omega(c)$ is corresponding to one of two **loop**s of Ω in between two passes of Ω through the crossing point named c. Then symbols interlaced with c in ω are mutual intersection points among these two loops.

1.5.1. Definition(s): We will also call a curve with no crossing point a **contour**. A **contours system**, shortly **contouring** will mean a finite

set of contours, not mutually intersecting each other. Having a contouring, a **continent** we will call a region boundered by contouring, i.e. a set of sphere's points with these properties:
(i) None of them lays on a contour,
(ii) each two of them may be connected by a path on the sphere not intersecting any of the contours and
(iii) it is maximal set as for inclusion with properties (i) and (ii).
Let's choose any of these cOntinents and call it O. Then **level** (in relation to O) of any point p of the sphere S not laying on any of the contours we will call the lowest possible number of contours crossed during a journey from any O's point to p. Level of continent we will define inductively as a level of any of its points. A sphere's point not laying on any of contours will be called **inner** (in relation to O) if its level is even, and **outer** (in relation to O) if its level is odd. Note that O's points are inner. Inductively, continents will be called inner or outer.

1.5.2. Definition: The curve Ω divides a sphere onto several parts-**discs** locally homeomorph to $\mathbf{R}^2$. More exactly, a disc is a region bordered by a curve, i.e. a set of sphere's points with these properties:
(i) None of them lays on the curve Ω,
(ii) each two of them may be connected by a path on the sphere not intersecting Ω and
(iii) it is maximal set as for inclusion with properties (i) and (ii)
(you can compare it with the definition of a continent).

1.6. Definition: Let's overpaint (for example by different colour) onto a curve Ω, starting anywhere on Ω but not in a crossing point, following the Ω's orientation.
Always, when being just a little piece in front of Ω's crossing point we avoid it and move into (still painting, holding a pen on a paper) another part of Ω incident to this crossing point, so that we don't continue overpainting Ω in a "natural" way, we continue by the only crossing point's incident flowline leaving the crossing point bordering the same disc as the old one.
This way we continue till we get to the point where we started. We have painted a contour - by getting to the starting point our overpainting is closed curve and as we haven't painted a crossing as we always avoid Ω's crossing points. Then we repeat this all - painting contours, starting on any flowline between some neighboring crossing points, that is not (almost) overpainted - while there is such one. In the end we get a contours system (Figure 2)), we shall talk about the Ω's **contouring**.
We consider contours to be oriented in direction induced by direction of Ω.

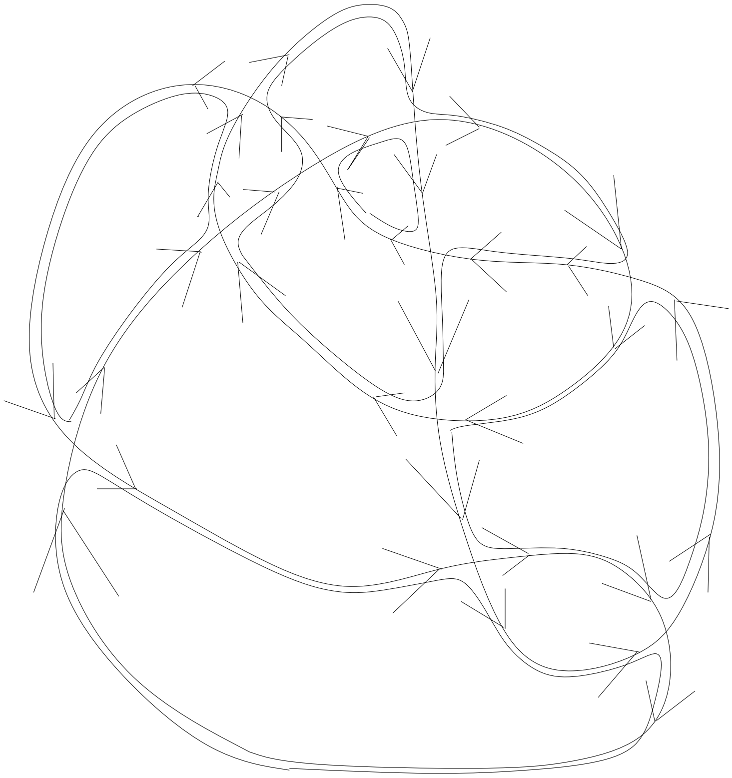


Figure 2

1.6.2.1. Note: For a contour of Ω, we can define its code as a sequence of Ω's crossing points passed by while drawing a contour. It is easy to construct it from Ω's Gauss code: We start at some occurence of some symbol passed by, we write it down. Then we take the symbol following it in the Gauss code, write it down and find its next occurence in the Gauss code. And we continue doing the same with temporarilly following symbol, until we get to the place in the Gauss code where we have started.

1.7. Having a contouring of Ω, each Ω-bordered disc naturally responds/belongs to some continent (the one that "almost contains" it), as contours are almost following the borders of the discs. For example, on the Figure 3 there is a continent "almost containing" discs B,F,L,J,H, another one "almost containing" discs D,E,K while disc A is the only one responding to "its" continent, like also C,I,G are.
Also each of Ω's crossing points, as they are not laying on any of the contours, is contained in some continent. For us, Ω-contouring's continent is

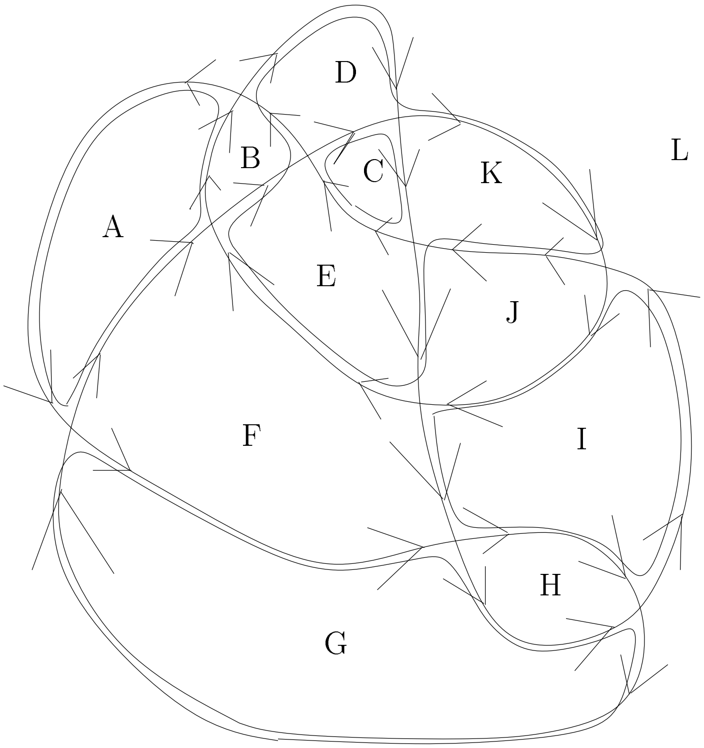


Figure 3: Discs responding to a continent

significantly characterised by a set of incident discs and Ω's crossing points. We can also talk about inside/outside discs and crossing points, as a property induced by responding contouring's continent. Even by choosing a crossing point or a disc (as inner) we are also choosing a continent (as inner).

1.8.1. Theorem (Rosenstiehl): A double-occurence word ω from alphabet C_ω as a set of its symbols is a Gauss code of a curve Ω if and only if it satisfies these three "parity" conditions:
(i) The subset of symbols interlaced with each symbol $c \in C_\omega$ of ω is even.
(ii) Whenever two symbols c and d are not interlaced they have an even number of common interlaced symbols.
(iii) There is a subset A of C_ω such that for $B := C_\omega \setminus A$ any two interlaced symbols c, d belong to the same set (A or B) if and only if they have an odd number of common interlaced symbols.

1.8.2. In [1] you can find the first **proof** of this theorem, with only a mistake in formulation of condition (iii). But there is another proof by H. de Fraysseix and P.O. de Mendez in [2], that seems to be easier to understand. Finally, another proof by P. Rosenstiehl [3] is to appear.

1.9.0. Remark: Set A from 1.8.1. is not uniquelly determined, not even up to complement. For a word ω representing a curve Ω there may exist more sets satisfying (iii) since the graph $G = (C_\omega, S =$set of pairs of interlaced symbols) need not be connected. But, given a curve Ω, Mr. Rosenstiehl's proof [1] leads to a constructive way to find certain such A: We make Ω's contours. Now, having chosen any continent O as inner, A we get right as the crossing points that are without loss of generality (instead of A we may always take its complement) inner in Ω-contouring.

1.9.1. Hypothesis: For any set A satisfying (iii) we can find a curve such that its contouring determines set A.

1.9.2. To persuade reader that this way constructed set A fulfills condition (iii) in 1.8.1., I reproduce the relevant thoughts, i.e. one implication of Mr. Rosenstiehl's proof [1]. So, let's have a curve Ω on a sphere, with some disc chosen as inner. We define a graph $G = G(\Omega) = (V =$inner discs, $E = \Omega$'s crossing points, each of them is realised by the pair of inner discs it touches). Let us point out that the graph $G^* := (V^* =$outer discs, $E^* = \Omega$'s crossing points, each of them being realised by the pair of outer discs it touches) is the geometric dual of G. We consider embeddings of G and G^* such that their edges named by the same Ω's crossing point cross each other in this Ω's crossing point as in Figure 4

We identify as far as the notation is concerned every subset of E with the corresponding vector of 2^E, where $2 = GF[2]$ is a Galois field with two elements. A scalar product $< X, Y >$ of two vectors X, Y is defined, and is equal to parity of the corresponding sets' X, Y common elements.

A **cycle** $\rho \in \mathcal{C}$ is the characteristic vector of the set of edges with an even incidence degree at each vertex, we denote the cycle space of G

$$\mathcal{C} \subseteq 2^E.$$

As G is planar, the cycle space $\mathcal{C}^*$ of G^* is exactly $\mathcal{C}^\perp$, taken as a subspaces of 2^E.

We can imagine our curve Ω to go along each edge e of G on one of its sides, cross e and then continue on the other side. (see Figure 5).

Ω traverses each edge twice in the same direction or once in each direction. Let A be the set of edges of G traversed twice in the same direction.

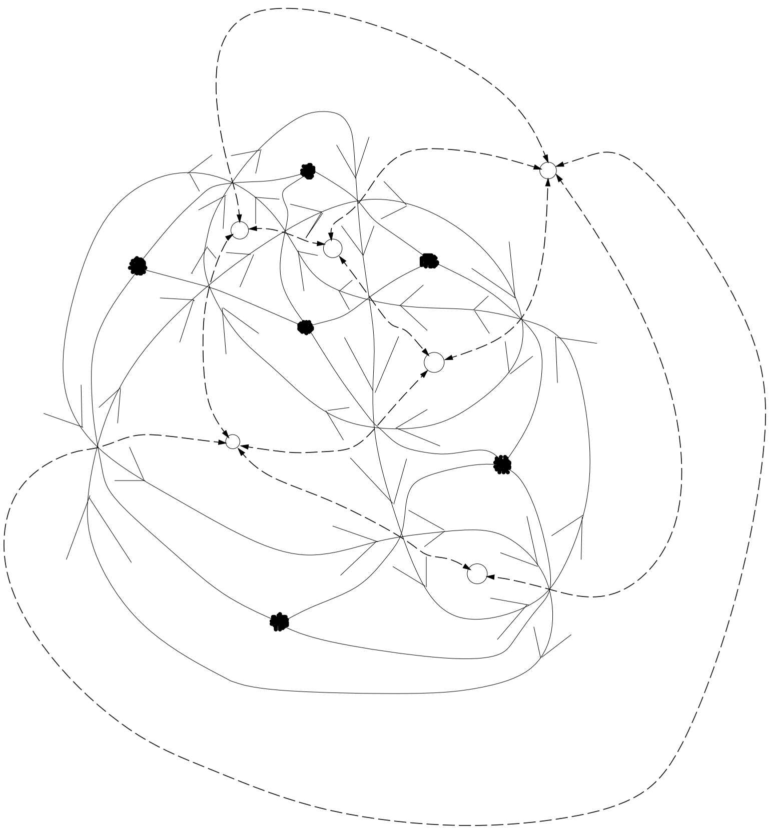


Figure 4: A-embeddings of $G(\Omega)$ (full lines) and G^*

1.9.2.1. Key lemma: A is exactly a set of inner crossing points of Ω-contouring.
Proof: Ω traverses edge e of G twice in the same direction if and only if the contour is near the edge-responding crossing point outside an inner disc (see Figure 6).
It's when the point's incident inner discs are in the same continent as the crossing point. So the crossing point must also be inner. Q.E.D.

Now, let ω be the Gauss code of Ω and let ϵe be the characteristic vector of the set of letters interlaced with e. Note that $< e, \epsilon e >= 0$. Let us define the following vectors relative to ω and A only:

$\alpha e = \epsilon e + e$ if $e \in A$ and $\alpha e = \epsilon e$ otherwise,

$\beta e = \epsilon e$ if $e \in A$ and $\beta e = \epsilon e + e$ otherwise.

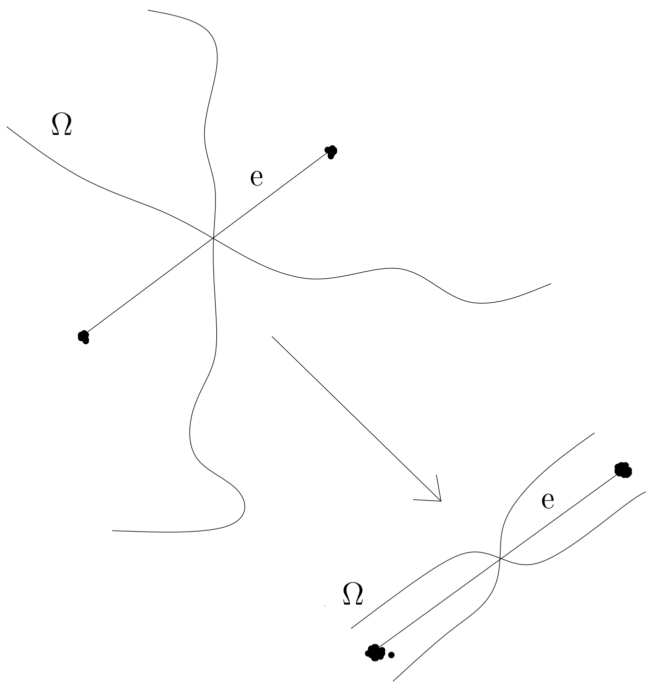


Figure 5

As we imagine Ω to go along each edge e of G, we can consider Ω to define a path in G. Now considering even a loop in Ω between two occurences of a symbol e as a path in G, we see that ϵe is a characteristic vector of crossing points met during going through the loop. By Figure 7 we see that in any event αe is a cycle of G. By duality βe is a cycle of G^*. Extending α and β to linear maps of E into 2^E we have

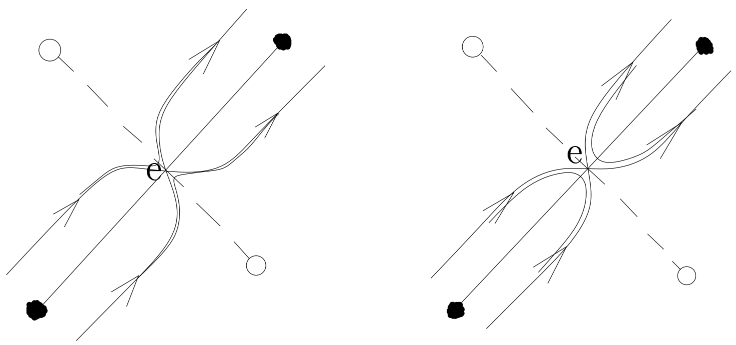
$$Im\alpha \subseteq \mathcal{C}, Im\beta \subseteq \mathcal{C}^* = \mathcal{C}^{\perp}.$$

Hence,

$$Im\alpha \perp Im\beta.$$

Now let us write orthogonality of αp and βq for any $p, q \in E$ in terms of scalar product, using the fact that $\alpha p = \epsilon p + Ap$ and $\beta q = \epsilon q + Bq$ for $B = E \setminus A$, where A, B are taken here as the characteristic mappings of the sets A, B ($Ap = p$ if $p \in A$, $Ap = 0$ otherwise). We get

- i if $p = q$ then $< \epsilon p, \epsilon p >= 0$ for $p \in E$
- ii if $p \neq q \& < p, \epsilon q >= 0$ then $< \epsilon p, \epsilon q >= 0$ for $p, q \in E$



$e \in A$ $e \notin A$

Figure 6

iii if $p \neq q \& < p, \epsilon q >= 1 then$
...........$< \epsilon p, \epsilon q > + < Ap, \epsilon q > + < \epsilon p, Bq >= 0$
.... or $< \epsilon p, \epsilon q > + < Ap, Ap > + < Bq, Bq >= 0$ for $p, q \in E$.

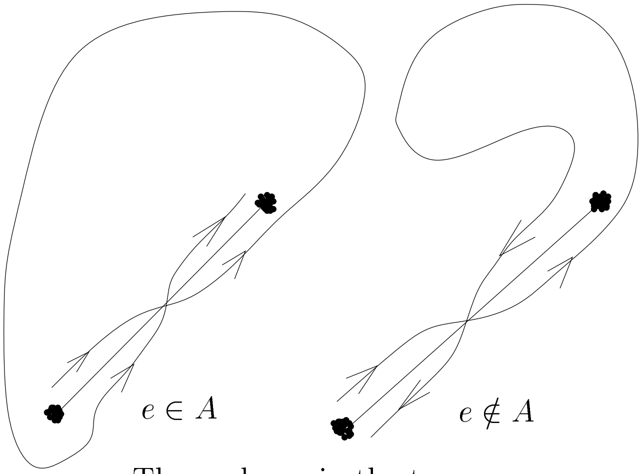
We recognize three parity conditions, QED.

1.9.3. Remark: As Ω-contouring's continents are significantly characterised by their discs and crossing points, we can even define a graph of a continent, being a subgraph (not necessarilly induced) of G from 1.9.2. for an inner continent or of G^* from 1.9.2. for an outer continent. Exactly, vertices of the continent's graph are continent's discs, its edges are continent's crossing points connecting discs. Then set A consists exactly of all edges of all continental subgraphs of G.

1.10. Theorem (Popule, 2000/'01): For each closed curve Ω on a sphere with the Gauss code ω from the symbols-set $C_\omega = C_\Omega$ and with $A \subseteq C_\omega$ constructed like in 1.9. each symbol $c \in C_\omega$ is interlaced with an even number of symbols -elements of A (and, according to 1.8.1.(i) of non-elements of A as well).

To prove this, we begin with some definitions and three lemmas .

1.10.1. First, we make extensions of some definitions from before. The point is that in definitions of a **disc** and a **contouring** we never used that we have just one curve. Hence, both definitions remain valid also for a set of curves in general position. See Figures 8 and 9.



The cycle αe in the two cases.
Figure 7

1.10.2. Again, having a contouring of a set of curves in general position, each disc naturally responds/belongs to some continent (the one that "almost contains" it). Also each of curves' crossing points, as they are not laying on any of contours, is contained in some continent. Again, for us, a curves-contouring's continent is significantly characterised by a set of belonging discs and incident curves' crossing points. Of course, again, choosing some continent as inner we can also talk about inside/outside discs and points, as a property induced by responding continent one. Finally, by choosing a point not laying on the contours we are also choosing incident continent as level 0.

1.10.3. Definition: Let Θ^O, Ξ^O be two curves-topological equivalents of a circle (i.e.contours) in general position. Let D be a disc of $\{\Theta^O, \Xi^O\}$. A **ray** $< D, p, \Phi^O >$ where p is some $\Theta^O \times \Xi^O$ crossing point on D's border and $\Phi^O \in \{\Theta^O, \Xi^O\}$ we will call a small part of Φ^O (not containing any other crossing point) with the same local orientation as Φ^O which begins or ends in the crossing point and borders the disc D. We shall talk about a ray **entering** (or **leaving** respectively) **the crossing point** if the the ray ends (or begins respectively) in the crossing point. (see Figure 10.) If the disc's crossing point's two rays both enter it, or both leave it, we shall talk about **disc's crossing point of type S(ame)**, otherwise of **type (differen)T**.

1.10.4. Lemma: Again, let Θ^O, Ξ^O be two oriented contours in general position, D be a disc of $\{\Theta^O, \Xi^O\}$. Then type (S/T) is the same for all the

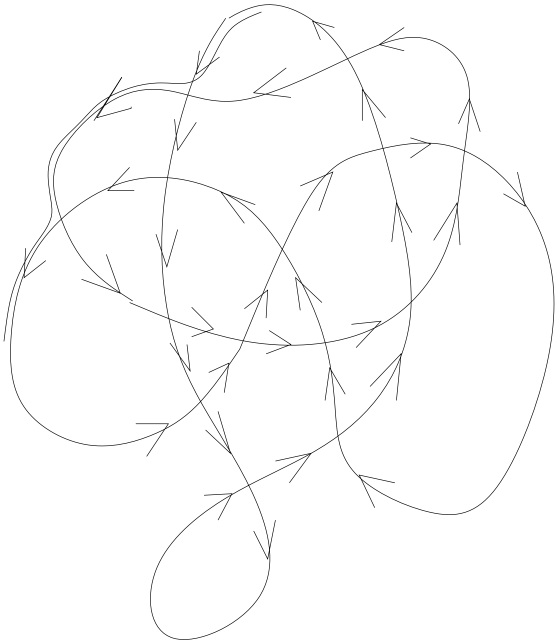


Figure 8

D's bordering crossing points.
Proof: I will prove that any two neighboring D-bordering crossing points always have the same type. This would obviously suffice.
For a contradiction suppose not. Then these two neighboring D-bordering crossing points correspond to the situation (see Figure 11.)
when a circle crosses another one immediately twice in a row in the same direction.
It's easy to see that this is not possible. Q.E.D.

1.10.5. Lemma: Let Θ^O, Ξ^O be two oriented contours in general position. Let's make (straightforward-) contouring on $\{\Theta^O, \Xi^O\}$ and choose one of continents as inner. Then the $\Theta^O \times \Xi^O$ mutual crossing points are of the same "colour" (inner/outer).
Proof: As discs make a "pseudo-chess-board" on the sphere, it's easy to see that it suffices to prove that all the crossing points neighboring any disc D are of the same "colour". If D's crossing point x is of type S then the contour doesn't separate x from the D and so x is inner if and only if the disc D is inner, while otherwise (type T) the contour separates x from D so the point x is inner if and only if the disc D is outer. Now we use previous lemma 1.10.4. QED

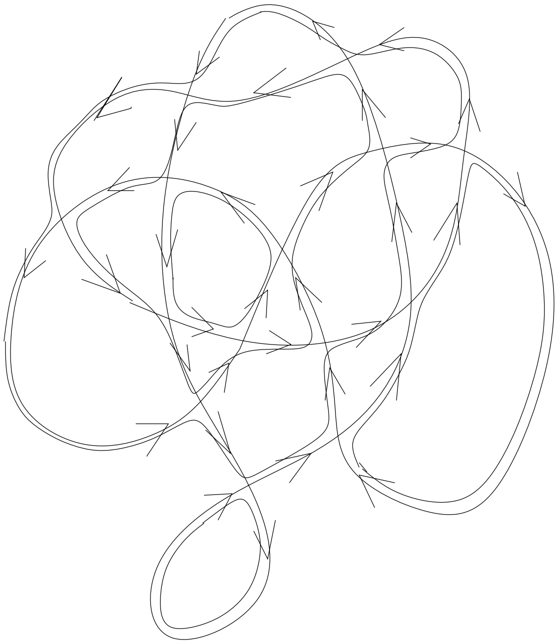


Figure 9

1.10.6. Lemma: Let P, Q be two contour-systems on a sphere. Then each two contours $\Theta^O \in P$, $\Xi^O \in Q$ have all their mutual crossing points of the same "colour" (inner/outer) in $P \cup Q$-contouring.

Proof: Let Θ^O, Ξ^O have some mutual crossing point p (otherwise the assertion is trivial), let then q be another $\Theta^O \times \Xi^O$-mutual crossing point. Discs bordered by $\{\Theta^O, \Xi^O\}$ are "overdiscs" of discs bordered by $P \cup Q$. Let's choose "colourings" of $\{\Theta^O, \Xi^O\}$- and $P \cup Q$- contourings such that each p-incident $P \cup Q$-disc has the same "colour" as its $\{\Theta^O, \Xi^O\}$-overdisc. Then also any $q-$ incident $P \cup Q$-disc's "colour" is the same as its "overdisc"'s (bordered by $\{\Theta^O, \Xi^O\}$) "colour" because from one any p-incident disc to this disc we can get by journey along Θ^O or Ξ^O with the same parity of crossings (i.e. meetings with) $P \cup Q$ as (with) $\Theta^O \cup \Xi^O$. Moreover, $\{\Theta^O, \Xi^O\}$-contouring is in any $\Theta^O \times \Xi^O$-mutual crossing point's neighborhood like $P \cup Q$-contouring, as the algorithm is locally the same. So the $P \cup Q$-contouring "colours" of p and q are the same as their $\{\Theta^O, \Xi^O\}$-contouring "colours". So it suffices to apply the previous lemma 1.10.5.: $\Theta^O \times \Xi^O$-mutual crossing points are of the same "colour". Q.E.D.

1.10.7. Definition: The **union-contouring** of a set S of curves in general position we will call a contouring of union of the sets of the contours of

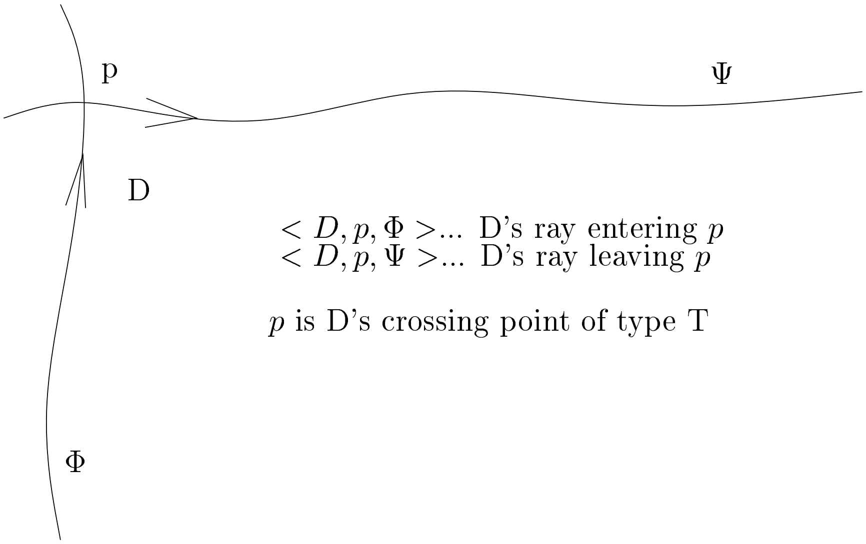


Figure 10: Ray entering/leaving a crossing point

curves-members of S. Again, each continent come up is significantly characterised by a set of its discs and incident crossing points.

1.10.8. Observation: Let Θ, Ξ be two curves on a sphere in general position. Then each two contours, one called Θ^O from Θ-contouring, the other called Ξ^O from Ξ-contouring have all their mutual crossing points of the same "colour" (inner/outer) in $\{\Theta, \Xi\}$-union-contouring by 1.10.6.

1.10.9. Lemma: Union-contouring of a set of curves is the same up to significant (i.e. as for discs and crossing points) characterisation as its contouring.
Proof: In each crossing point p they are locally the same, in the sense that the border of the same pair of four p-incident discs is "almost followed". QED

1.10.10. For Θ, Ξ -curves in general position number of $\Theta \times \Xi$-crossing points being outer in $\{\Theta, \Xi\}$-union-contouring is even because number of crossings of any two contours Θ^O, Ξ^O from Θ- and Ξ- contourings is even and they all have the same "colour" in the union-contouring by 1.10.8.

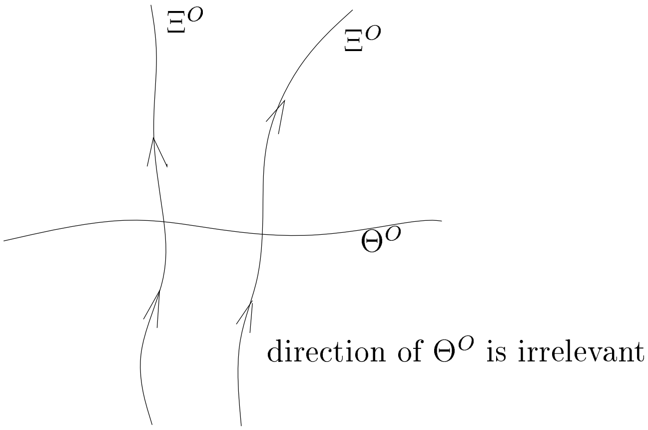


Figure 11

Proof of 1.10.: Any curve's crossing point c divides the curve Ω into two parts each of which may be individually considered a curve, let's call these curves Θ, Ξ (Θ, Ξ are passing, not crossing in c) - if we up to the Gauss codes' cyclic equivalence consider the Gauss code of Ω to be $\omega = \alpha c \beta c$ then Θ's and Ξ's Gauss codes are $\tilde{\alpha}$ and $\tilde{\beta}$ respectively, where $\tilde{\phi}$ rises from ϕ by removing all symbols with only single occurence in ϕ. They are exactly the symbols interlaced with c, and they are representing the mutual crossing points between Θ and Ξ, i.e. mutual crossing points between Θ- and Ξ-contourings.
We set a continent containing c as inner for all contourings we consider. Then also all other crossing points have the same "colour" in Ω-contouring as in $\{\Theta, \Xi\}$-contouring. By 1.10.9. the $\{\Theta, \Xi\}$-contouring is the same up to significant characterisation as $\{\Theta, \Xi\}$-union-contouring which implies that any crossing point's "colour" is the same in both of them. From 1.10.10. we know that number of outer crossing points in $\{\Theta, \Xi\}$-union-contouring is even. So the number of outer crossing points in Ω-contouring is even, that are exactly those in A (see also 1.9.). Q.E.D.

1.11. We can even write this theorem 1.10. in a stronger shape.
Theorem: For each closed curve Ω on a sphere with the Gauss code ω from the symbols-set C_ω and with any $A \subseteq C_\omega$ satisfying 1.8.1.(iii) each symbol $c \in C_\omega$ is interlaced with an even number of symbols-elements of A (and, according to 1.8.1.(i) of non-elements of A as well).
1.11.1. Recently, Mr. P.O. de Mendez has found an easy proof of this theorem based on the properties of interlacement graphs only.
Proof (P.O. de Mendez): We consider an interlacement graph. We denote the set of neighbors of any vertex p as $N(p)$. Let A be arbitrary

set satisfying (iii). Then $S(c) := \sum_{x \in N(c)} |N(c) \cap N(x)| = 2.$number of triangles in an interlacement graph with c as one of vertices, as each such triangle is taken twice exactly ($y \in N(x) \cap N(c)$ iff $x \in N(y) \cap N(c)$). Hence, $S(c)$ is even. But the sum of $S(c)$ may as well be divided into

$$S(c) = \sum_{x \in N(c) \cap A} |N(c) \cap N(x)| + \sum_{x \in N(c) \setminus A} |N(c) \cap N(x)|.$$

First, for $c \in A$, each $|N(r) \cap N(x)|$ in the second sum, i.e. for $x \in N(r) \setminus A$, is even, so the whole sum is even, so the first sum must be even as well. But its each member $|N(r) \cap N(x)|$ is odd according to condition (iii). Hence, the number of the first sum's members, i.e. cardinality of the set $N(r) \cap A$, is even.
Similarly, for $c \notin A$, each $|N(r) \cap N(x)|$ in the first sum, i.e. for $x \in N(r) \cap A$, is even, so the whole sum is even, so the second sum must be even as well. But its each member $|N(r) \cap N(x)|$ is odd. Hence, the number of the the second sum's members, i.e. cardinality of the set $N(r) \setminus A$, is even. So also, according to condition (i), $|N(r) \cap A|$ is even. Q.E.D.

1.12. Corollary: So we can instead of condition (iii) in 1.8.1.(Rosenstiehl) write condition (iii)'=condition come up by union of 1.8(iii) and 1.10.: There exist disjoint sets $A, B \subseteq C_\omega, A \cup B = C_\omega$ such that each letter of C_Ω is interlaced with an even number of elements of A and any pair of interlaced letters has an even number of common interlaced letters if and only if one of them is element of A, the other element of B. Or, we could in 1.8 even skip the condition (i) if in (iii)' we demand also (moreover) for each letter of C_ω to be interlaced with an even number of elements of B (like of elements of A).

1.13. The motivation to explore another properties of the set A could be beside others the Conway's thrackle conjecture, for that it suffices to prove that no C_{2m}^{2n} is a thrackle: Any drawing of C_{2m}^{2n} is obviously a closed curve. To prove impossibility of such a drawing it would suffice to prove non-existence of corresponding curve, i.e. of a Gauss code satisfying particular conditions. The more properties of the set A we know, i.e. the more conditions we can demand, the easier the proof could be.

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