Foreword

It has already become a tradition to organize one day meetings for Czech and Slovak graph theorists. DIMATIA Graph Thoery Days, as the title says, are organized by DIMATIA Charles University and we see it as a way to fulfill one of the goals of DIMATIA - to foster the research in Discrete Mathematics and Graph Theory in our countries and to support the links among the Czech and Slovak groups. These meetings bring our researchers together, enable them to present their most recent results including work-in-progress reports and the informal atmosphere stimulates their collaboration.

This is the growing list of GTD's:

- Graph Theory Day I - Prague January 29, 1999
- Graph Theory Day II - Nečtiny October 8-10, 1999 (organized by DIMATIA partner institution West Bohemia University)
- Graph Theory Day III - Prague January 31, 2000 - in honor of Ivan Havel who passed away in the fall of 1999
- Graph Theory Day IV - Prague February 6, 2001
- Graph Theory Day V - Prague March 12 - celebrating 55th birthday of Jaroslav Nešetřil

Since 2001, the workshops are co-organized by ITI (Institute for Theoretical Computer Science Charles University, supported by the Ministry of Education of the Czech Republic as project LN00A056).

We are more than happy to see that DIMATIA Graph Theory Day's really bring our people together and hope that the so far short list of GTD's will keep expanding.

Jan Kratochvíl

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On the Cyclic Chromatic Number of 3–connected plane graphs

Mirko Horňák

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The *cyclic chromatic number* of a plane graph G, in symbol $\chi_c(G)$, is a minimum number of colours in such a vertex colouring of G that distinct vertices incident with a common face receive distinct colours. Clearly, $\chi_c(G) \geq \Delta^*(G)$, where $\Delta^*(G)$ is the maximum face degree of G. On the other hand, no 3-connected plane graph G is known with $\chi_c(G) > \Delta^*(G)+2$. Plummer and Toft in *Cyclic coloration of* 3*-polytopes*, J. Graph Theory 11 (1987), 507–515, proved that $\chi_c(G) \leq \Delta^*(G) + 9$ and conjectured (PTC) that $\chi_c(G) \leq \Delta^*(G) + 2$ for any 3-connected plane graph G. It is known that PTC is true for $\Delta^*(G) \leq 4$ and $\Delta^*(G) \geq 24$. A general upper bound has been lowered so far only to $\Delta^*(G) + 8$.

Theorem 1. *If G is a 3-connected plane graph, then $\chi_c(G) \leq \Delta^*(G) + 5$.*

Light Subgraphs of Planar Graphs

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The existence of subgraphs of low degree sum of their vertices in planar graphs is investigated. Let $K_{1,3}$, a subgraph of a graph G, be an $(x; a, b, c)$-star, a star with a central vertex of degree x and three leaves of degrees a, b, and c in G. The main results of the paper are:

1. Every planar graph G of minimum degree at least 3 contains an $(x; a, b, c)$ star with (i) $x = 3, a \leq 10$ and $a \leq b \leq c$, or (ii) $x = 4, a = 4, 4 \leq b \leq 9$, and $c \geq b$, or (iii) $x = 4, a = 5, 5 \leq b \leq 9$, and $c \geq b$, or (iv) $x = 4, a = 6 \leq b \leq 8$, and $c \geq b$, or (v) $x = 4, a = 7 \leq b \leq 8$, and $c \geq b$, or (vi) $x = 5, a = 5 \leq b \leq 6$, and $5 \leq c \leq 7$, or (vii) $x = 5$, and $a = b = c = 6$.
Moreover, the bounds are best possible except possibly for cases (iii) and (v).
In these exceptional cases the exact values differ at most by one from the mentioned bounds.

2. Every 3-connected planar graph G that contains a k-path, a path on k vertices, also contains a k-path P such that for its weight (the sum of degrees of its vertices) in G

$$w_G(P) := \sum_{A \in V(P)} \deg_G(A) \leq k^2 + 13k.$$

Moreover, there exists a 3-connected planar graph H in which every k-path P has weight

$$w_H(P) \geq k \log_2 k.$$

3. Let G be a connected planar graph of minimum vertex degree δ, minimum face size ρ and circumference $c(G)$. If $c(G) \geq \sigma|V(G)|$ for some constant $\sigma > 0$ then for any k, $1 \leq k \leq c(G)$, G contains a k-path P such that

$$w_G(P) = \sum_{A \in V(P)} \deg_G(A) < \left(\left(\frac{2\rho}{\rho - 2} - \delta\right)\frac{1}{\sigma} + \delta\right) k.$$

On Hamiltonian Cycles in Strong Products of Graphs

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Let the vertex set of the graph G be denoted by $V(G)$ and let its edge set be denoted by $E(G)$. The strong product of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$. Its two distinct vertices $[u_1, v_1]$ and $[u_2, v_2]$ are joined by an edge iff $u_1 = u_2 \vee u_1u_2 \in E(G)$ and $v_1 = v_2 \vee v_1v_2 \in E(H)$. A cycle containing all the vertices of the graph is called a hamiltonian cycle; each vertex is contained in such a cycle exactly once. We write G^k for the strong product of k copies of G and we call graphs containing a hamiltonian cycle hamiltonian graphs for the brevity.

Zaks asked in [3] whether there exists $k(G)$ for any connected graph G with at least two vertices such that $G^{k(G)}$ is hamiltonian. Bermond, Germa and Heydemann proved in [1] the existence of the number $k(G)$ and they proved that if G^k is hamiltonian, then also G^h is hamiltonian for all $k \leq h$ using some results of Rosehfeld and Barnette contained in [2]; let the smallest possible $k(G)$ be denoted as $h(G)$. Bermond et al. did not give any upper bound on $h(G)$ in terms of maximum degree of the graph G, but they conjectured that $h(G) \leq \Delta(G)$ holds for all connected graphs G with at least two vertices where $\Delta(G)$ is the maximum degree of G. We can prove this conjecture, but we focus here our attention to the problem of finding the smallest possible value of $h(G)$. Let $h_{\max}(\Delta)$ be $\max\{h(G)|\Delta(G) \leq \Delta\}$; we prove that $h_{\max}(\Delta) < \Delta$ for large value of Δ. On the other hand Zaks in [3] proved the following inequality (S_n is $K_{1,n}$, see the next section):

$$h(S_n) \geq \left\lceil \frac{\ln 2}{\ln\left(1 + \frac{1}{n}\right)} \right\rceil$$

Thus it is impossible to prove sublinear upper bounds for $h_{\max}(\Delta)$.

Theorem 1. *The product of any at least* $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 3$ *connected graphs of maximum degree at most* Δ *is hamiltonian for* $32 < \Delta$.

Theorem 2. *For each* $c > \ln \frac{25}{12} + \frac{1}{60}$ *there exists* c' *such that the product of any at least* $\lfloor c\Delta \rfloor + \lceil \log_2 \Delta \rceil + c'$ *connected graphs of maximum degree at most* Δ *is hamiltonian for* $32 < \Delta$.

The question of precise determining of $h_{\max}(\Delta)$ remains open; it seems that stars are in some sense the worst graphs and that the lower bound given by Zaks could be tight. The more interesting task is to describe the linear behaviour of $h_{\max}$. The lower bound proved by Zaks and our upper bound gives:

$$\liminf_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} \geq \ln 2 \approx 0.6931$$

$$\limsup_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \ln \frac{25}{12} + \frac{1}{60} \approx 0.7506$$

We conjecture that:

$$\lim_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} = \ln 2$$

References

[1] J. C. Bermond, A. Germa, M. C. Heydemann: Hamiltonian Cycles in Strong Products of Graphs, Canad. Math. Bull. Vol. 22 (3), 1979, pp. 305–309.

[2] M. Rosehfeld , D. Barnette: Hamiltonian Circuits in Certain Prisms, Discrete Math. 5, 1973, pp. 389–394.

[3] J. Zaks: Hamiltonian cycles in products of graphs, Canadian Math. Bull. vol. 17 (5), 1975, pp. 763–765.

Cayley Snarks and Almost Simple Groups

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A Cayley snark is a cubic Cayley graph which is not 3-edge-colourable. We discuss the problem of the existence of Cayley snarks. This problem is closely related to the problem of the existence of hamiltonian cycles in Cayley graphs and to the question whether every Cayley graph admits a nowhere-zero 4-flow.

So far, no Cayley snarks have been found. On the other hand, we prove that the smallest example of a Cayley snark, if it exists, comes either from a non-abelian finite simple group or from a group which has a single non-trivial proper normal subgroup. The subgroup must have index two and must be either non-abelian simple or the direct product of two isomorphic non-abelian simple groups. Several other properties of the group can also be derived.

The details can be found in

R. Nedela and M. Skoviera, *Cayley snarks and almost simple groups*, Combinatorica, to appear.

A Sufficient Condition for the Existence of Large Empty Convex Polygons

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Let P be a set of points in general position in the plane. We say that P is k-convex, if no triangle determined by P contains more than k points of P in the interior. We say that a subset A of P forms an *empty polygon* (in P), if the points of A are the vertices of a convex polygon containing no other points of P. Then for any k, n there is an $N = N(k, n)$ such that any k-convex set of at least N points in general position in the plane contains an empty n-gon. An analogous statement also holds in $\mathbb{R}^d$ for each odd $d \geq 3$. We also discuss some related questions, e.g. the so-called modular version of the Erdős–Szekeres theorem.

Presented Open Problems

Computational Complexity of k–subcoloring

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We consider coloring of vertices of a graph G s.t. each color class induces in G a disjoint union of complete graph, or — in other words — a graph without induced path of length two. Let $\chi_s(G)$ be the minimum k for which such coloring with k color classes exists.

We know that $\chi_s(G) \leq \frac{\Delta(G)}{2}$ and we know that for all fixed $k \geq 2$ the problem to decide whether $\chi_s(G) \leq k$ is NP-hard for graphs with $\Delta(G) \leq k^2$.

We pose the following problem: What is the computational complexity of deciding $\chi_s(G) \leq k$ for the class graphs of maximum degree l where l varies from $2k+1$ to k^2-1?

Observability of Q_n

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Problem (posed by **Mirko Horňák** and Roman Soták.) Let G be a graph with no component K_2 and with at most one component K_1. Given an edge colouring of G, a *colour set* of a vertex x of G is the set of colours of edges incident with x. An edge colouring of G is *vertex-distinguishing* if, for any two distinct vertices of G, their colour sets are distinct. The *observability* of G, denoted by $\operatorname{obs}(G)$, is a smallest number of colours in a proper vertex-distinguishing edge colouring of G. We proved in *Asymptotic behaviour of the observability of Q_n*, Discrete Math. 176 (1997), 139–148, that $\lim_{n\to\infty} \operatorname{obs}(Q_n)/n$ exists and is equal to $1+q^*$, where $q^* = 0.293815...$ is the unique solution of the equation $(x+1)^{(x+1)} = 2x^x$ in the interval $(0,\infty)$. Quite surprisingly, locally we have no result that could correspond to the above asymptotic statement.

Conjecture 1. *The sequence $\{\operatorname{obs}(Q_n)\}_{n=2}^{\infty}$ is non-decreasing.*

Tree Colorings with a Forbidden Pattern

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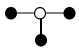
Let $T(abba, n)$ be the maximum number of vertices in a tree T that can be vertex-colored by at most n colors so that

1. the coloring is proper, which means that no edge is monochromatic,
2. no subgraph of T is a subdivision of the 2-colored 4-vertex path , and
3. no subgraph of T is a subdivision of the properly 2-colored 4-vertex star  .

Problem. Determine the function $T(abba, n)$.

In [M. Klazar, Combinatorial aspects of Davenport–Schinzel sequences, Ph.D. thesis, Charles University, Prague, 1995] and [M. Klazar, Combinatorial aspects of Davenport–Schinzel sequences, Discrete Math., 165/166 (1997), 431–445] I posed as a problem to show that $T(abba, n) = O(n)$. This was accomplished by P. Valtr [On an extremal problem for colored trees, Eur. J. Comb., 20 (1999), 115–121] who proved more generally that $T(a^i b^i a^i, n) \leq 24in$ (the power a^i denotes the sequence $aa \ldots a$ of length i and $T(a^i b^i a^i, n)$ is defined analogously to $T(abba, n)$). On the other hand it is not difficult to show that $T(abba, n) \geq 5n - 8$. So at present it is known only that

$$5n - 8 \leq T(abba, n) \leq 48n.$$

For $abba$ Valtr's upper bound can probably be improved but it would be interesting to know $T(abba, n)$ exactly.

Gap–freeness of Feasible Sets of Mixed Hypergraphs

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A mixed hypergraph is a triple $H = (V, C, D)$, where V is the vertex set of H and C and D are sets of subsets of V. A vertex coloring of H is proper if every edge $e \in C$ contains two distinct vertices of the same color and every edge $e \in D$ contains two veritces of different colors. A proper coloring is a strict k-coloring if it uses exactly k colors. The feasible set $\mathcal{F}(H)$ of H is the set of k's such that H has a strict k-coloring.

Examples of mixed hypergraphs whose feasible sets are not intervals (i.e., contain gaps) are known. On the other hand, large classes of mixed hypergraphs were identified, whose feasible sets are gap-free. We propose the following approach.

For a graph G, denote by $\mathcal{H}(G)$ the set of all mixed hypergraphs $H = (V, C, D)$ for which $V = V(G)$ and $G[e]$ (the subgraph of G induced by the vertices of e) is connected for every edge $e \in C \cup D$. Let $\mathcal{M}$ be the class of graphs G for which the feasible set $\mathcal{F}(H)$ is gap-free for every $H \in \mathcal{H}(G)$.

Problem Characterize graphs of $\mathcal{M}$.

It is known that $\mathcal{M}$ contains all forests and unicyclic graphs.

Sufficient Condition for Weak Pancyclicity

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Let G be a finite simple undirected graph and let $g(G)$ and $c(G)$ be the girth and the circumference of G (i.e. the length of a shortest cycle of G and the length of a longest cycle of G), respectively. We say that G is *weakly pancyclic* if G contains cycles of all lengths ℓ for $g(G) \le \ell \le c(G)$. The graph G is *locally connected* if the neighborhood of every vertex of G induces a connected graph.

Conjecture 1. *Every connected locally connected graph is weakly pancyclic.*

Conjecture on Snarks, Cyclic Edge Connectivity

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I. A *map* is a 2-cell embedding of a connected graph on a closed surface. An automorphism of a map is a permutation of darts (oriented edges) of the graph which can be accomplished by a self-homeomorphism of the surface. An orientable map is said to be *regular* if the group of all orientation-preserving map automorphisms acts transitively (and hence regularly) on the darts.

Problem 1. *Is the underlying graph of any orientable cubic regular map whose automorphism group is simple non-abelian necessarily* 3*-edge-colourable?*

The answer NO would imply the existence of somewhat strange Cayley snarks of girth 3 (cf. [1, Theorem 3 and Theorem 4]). The graph obtained by contacting the triangles in such a graph would still be a snark, in fact a dart-transitive one, but not necessarily Cayley. No example of this sort is known, however. On the other hand, a conjecture of B. Grünbaum claims that the dual of every orientable triangulation is 3-edge-colourable. If this is true, then the answer to Problem 1 is YES.

II. The cyclic edge-connectivity of a graph is the smallest number of edges whose removal from the graph produces at least two components containing cycles. If no such set exists, the cyclic edge-connectivity is set to be the cycle rank of the graph. (For cubic graphs, the largest graph where the latter part of the definition applies is $K_{3,3}$.)

Problem 2. Is there a polynomial time algorithm which determines the cyclic edge-connectivity of every cubic graph?

The expected answer is YES.

REFERENCES

[1] R. Nedela and M. Škoviera, *Which generalized Petersen graphs are Cayley graphs?*, J. Graph Theory **19** (1995), 1–11.

Drawings of Q_n in the Plane

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Is it possible to draw the graph of the n-dimensional cube Q_n (having 2^n vertices and $n2^{n-1}$ edges) in the plane so that vertices are represented by points and edges by segments (resp. by Jordan curves) so that there are no 100 (say) pairwise crossing edges? (Two edges meeting in a vertex are not considered as crossing edges.) This is trivially true for small values of n but we would like to find the drawing for any n. For large n, Q_n is not planar and therefore we cannot replace 100 by 2.

It is known that every drawing of a graph with n vertices and at least $c_k n \log n$ edges has k pairwise crossing edges. The graph Q_n seems to be a good candidate to show that the bound $c_k n \log n$ cannot be improved (up to the value of c_k), which motivates the above problem.

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