Arc Reversal in Nonhamiltonian Circulant Oriented Graphs

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8 March 2001

Abstract

Locke and Witte in [9] have described a class of nonhamiltonian circulant digraphs. We show that for infinitely many of them the reversal of any arc produces a hamiltonian cycle. This solves an open problem stated in [4]. We use these graphs to construct counterexamples to Ádám's conjecture. The smallest one $\mathrm{Cay}(\mathbb{Z}_{12}; 2, 3, 8)^4$ is the counterexample with the smallest known number of vertices.

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1 Introduction

An oriented graph is a directed simple graph, that is to say, a digraph without loops, multiple arcs, or cycles of length two. The problem 5.43, if there is a nonhamiltonian oriented graph in which the reversal of any arc results in a hamiltonian graph, was stated in [4]. In this paper we describe a class of circulant oriented graphs having this property. Moreover, using these graphs we construct counterexamples to Ádám's conjecture.

For any natural number n, we use $\mathbb{Z}_n$ to denote the additive cyclic group of integers modulo n. For any set A of integers, let $\mathrm{Cay}(\mathbb{Z}_n; A)$ be the Cayley digraph whose vertex set is $\mathbb{Z}_n$ and in which there is an arc from i to $i + a \pmod{n}$, for every $i \in \mathbb{Z}_n$ and every $a \in A$. Vertex aritmetic is taken modulo n (in particular $-i$ is understood as $n - i$). A digraph isomorphic to $\mathrm{Cay}(\mathbb{Z}_n; A)$ for some n and A is called *circulant digraph*. For oriented case (to avoid symetric arcs i.e., cycles of length two) we use only such A, that $0 \notin A + A$.

Locke and Witte in [9] presented a class of circulant nonhamiltonian oriented graphs of type $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$.

Theorem 1 (Locke,Witte [9, Thm. 4.1]) *Connected circulant digraph* $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$ *has no hamiltonian cycle if and only if* $gcd(b-a, k) = 1$; $gcd(a, 2k) \neq 1$; $gcd(b, k) \neq 1$; *either* a *or* k *is odd;* a *is even, or both of* b *and* k *are even.*

In section 2 we use some methods from [5] and [9] to prove

- characterization Theorem 3 showing when there is a Hamilton path from 0 to a in some $\mathrm{Cay}(\mathbb{Z}_n; a, b)$
- Construction 4 describing how to get a Hamilton path from 0 to b using a Hamilton path from 0 to a with special properties in some $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$
- Construction 5 showing when the existence of a Hamilton path from 0 to b yields the existence of a Hamilton path from 0 to $b+k$.

In Theorem 6 we describe the subclass of connected nonhamiltonian digraphs of type $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$ from Theorem 1, namely $\mathrm{Cay}(\mathbb{Z}_{8k+4}; 2k+1, 2, 4k+4)$ (for $k \geq 1$), for which all the three claims hold. As a result, due to vertex-transitivity, the reversal of any arc in these oriented graphs creates a hamiltonian cycle, which solves the problem in [4] mentioned above.

Section 3 is devoted to Ádám's conjecture on arc reversal. The conjecture, formulated by A. Ádám in the early sixties [1–3] says, that any digraph containing a directed cycle has an arc whose reversal decreases the total number of directed cycles.

For simple oriented graphs (without multiple arcs) it seems to be a hard problem. The only known nontrivial result is by Reid [11]. He proved conjecture for 2-(arc)connected tournaments which are not 3-(arc)connected.

On the other hand Ádám's conjecture does not hold for multidigraphs. The first counterexamples were described independently by Thomassen and Grinberg. Thomassen in [12] showed that the conjecture does not hold for digraphs based on $C_5 \times C_{7+10r} (r \geq 0)$ having at least 35 vertices. Grinberg [6,7] used digraphs $G_{n\times n}$ created by drawing regular lattice $n \times n$ on a Klein bottle. The smallest such counterexample $G^2_{4\times 4}$ (Fig. 4) has 16 vertices (where G^p denote the multidigraph, obtained from G by replacing each arc by p parallel ones).

We use the class of nonhamiltonian digraphs $\mathrm{Cay}(\mathbb{Z}_{8k+4}; 2k+1, 2, 4k+4)$, $k \geq 1$, described in section 2 to construct further "nonadamian" multidigraphs. The smallest one, $\mathrm{Cay}(\mathbb{Z}_{12}; 2, 3, 8)^4$, obtained by this construction has only 12 vertices.

2 Nonhamiltonian oriented graphs in which the reversal of any arc results in a hamiltonian graph

In this section we show that for some nonhamiltonian circulant digraphs with outdegree three (from the class of digraphs described in Theorem 1) the reversal of any arc creates a hamiltonian cycle. Since circulant digraphs $\mathrm{Cay}(\mathbb{Z}_n; A)$ are vertex-transitive it is sufficient to prove the existence of Hamilton paths from 0 to all $a \in A$.

First, we would like to obtain a characterization of the Hamilton paths from 0 to a in $\mathrm{Cay}(\mathbb{Z}_n; a, b)$. We use results of Curran and Witte in [5] about Hamilton paths obtained by studying a collection of spanning subdigraphs of $\mathrm{Cay}(\mathbb{Z}_n; a, b)$ with indegree and outdegree of any vertex at most 1.

Given a spanning subdigraph H of $\mathrm{Cay}(\mathbb{Z}_n; a, b)$, we will say that a vertex $v \in \mathbb{Z}_n$ *travels by* a in H if the arc from v to $v + a$ is in H and the arc from v to $v + b$ is not.

The next proposition is based on the following observation. Suppose that H is a Hamilton path from 0 to v in $\mathrm{Cay}(\mathbb{Z}_n; a, b)$. Clearly, vertex $-a$ cannot travel by a. So, $-a$ is either the terminal vertex v, or travels by b in H. If the vertex $i(b - a) - a$ travels by b then $(i + 1)(b - a) - a$ must travel by b or it is the terminal vertex v. Similarly, $-b = (2k - 1)(b - a) - a$ travels by a, and if $i(b - a) - a$ travels by a then $(i - 1)(b - a) - a$ must travel by a or it is v. If $gcd(n, b - a) = 1$, we obtain full 'arc-forcing' determination, because the element $b - a$ generate all vertices in $\mathbb{Z}_n$.

Proposition 2 (due to [5]) *Let $G = \mathrm{Cay}(\mathbb{Z}_n; a, b)$, $gcd(n, b - a) = 1$ and let $H(d)$ be the spanning subdigraph of G in which $i(b - a) - a$ travels by b if $0 \le i < d$ and travels by a if $d < i < n$. Then both indegree and outdegree of any vertex of $H(d)$ are 1, except 0 has indegree 0 and $d(b - a) - a$ has outdegree 0. Components of $H(d)$ are (mutually disjoint) elementary cycles and one path from 0 to $d(b - a) - a$.*

Moreover (by the preceeding paragraph) if there is a Hamilton path from 0 to v in G, then it is unique and is isomorphic to some $H(d)$ where $d(b - a) - a \equiv v \pmod{n}$. In this path d vertices travel by b and $n - d - 1$ vertices travel by a.

Theorem 3 *Let $G = \mathrm{Cay}(\mathbb{Z}_n; a, b)$, $gcd(n, b - a) = 1$, be a oriented graph. Suppose p is unique positive integer less than n for which $p(b - a) \equiv a \pmod{n}$. Let $B = gcd(n, a)$, $A = n/B$, and let $0 \le E < A$ with $Bb \equiv Ea \pmod{n}$. Then G contains a Hamilton path from 0 to a if and only if $2p+1 < n$ and there is a pair of relatively prime positive integers r and s, such that $sB = 2p$ and $rA - sE = n - 2p - 2$.*

Proof. ($\Rightarrow$) Let H is a Hamilton path from 0 to a in G. Since $p(b-a)-a \equiv 0$ we have $2p(b-a)-a \equiv a \pmod{n}$, so by Proposition 2 H is isomorphic to $H(2p)$.

If $2p \geq n$, then $H(2p) = H(2p-n)$. Therefore, $2p-n < p$ and 0 must travel by a in H, which contradicts the hamiltonicity of H.

Because $(2p+1)(b-a)-a \equiv b \pmod{n}$, also $2p+1 \neq n$ (otherwise $b = -a$, which creates a cycle of length two) and b travels by a in $H(2p+1)$. Graph $H(2p+1)$ contains exactly the same arcs as $H(2p)$, except instead of arc from b to $b+a$ there is arc from a to $b+a$. So, $H(2p+1)$ consists of one arc from 0 to b and one elementary cycle. In this cycle exactly $2p$ vertices travel by b and $n-2p-2$ vertices travel by a. Then, by [5] there are unique relatively prime r and s, such that $sB = 2p$ and $rA - sE = n-2p-2$.

($\Leftarrow$) Let $H(2p+1)$ be the spanning subdigraph of G from the Proposition 2. Because $(2p+1)(b-a)-a \equiv b \pmod{n}$ and $p < 2p+1 < n$, 0 must travel by b. So, $H(2p+1)$ consists of one arc from 0 to b and a collection of disjoint elementary cycles. By [5] there are r' and s', such that in these cycles $s'B = 2p$ vertices travel by b and $r'A - s'E = n-2p-2$ vertices travel by a. Moreover, $gcd(r', s')$ is the number of these cycles. By the assumption $(r', s') = (r, s)$, where r and s are relatively prime, so $H(2p+1)$ contains only one cycle. Then we take $H(2p)$, it consists of the same arcs as $H(2p+1)$, except instead of arc from a to $b+a$ there is arc from b to $b+a$. Clearly, this will be a Hamilton path from 0 to a. $\square$

Remark. By the assumption of Theorem 3 we have $p(b-a) \equiv a \pmod{n}$ and $gcd(n, b-a) = 1$, so p is a multiple of B. Let $p_0 = p/B$. From the definition of E, $B(b-a) \equiv a(E-B) \pmod{n}$, then we can conclude $p_0(E-B) \equiv 1 \pmod{A}$. Therefore $s = 2p_0$ and $r = B + 2(p_0(E-B)-1)/A$. Then G contains a Hamilton path from 0 to a if and only if $0 < 2p_0 < A$ and $gcd(2p_0, B + 2(p_0(E-B)-1)/A) = 1$. In this path $2p_0B$ vertices travel by b and $n - 2p_0B - 1$ vertices travel by a.

We will consider connected nonhamiltonian digraphs $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$, where a is odd. By the Theorem 1 also k and b is even, $gcd(2k, b-a) = 1$ and $gcd(a, k) \neq 1$. We present two constructions to obtain a Hamilton path from 0 to b and from 0 to $b+k$ using Hamilton path from 0 to a.

Construction 4 *Let* $G = \mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$*, a is odd, be a connected oriented graph without hamiltonian cycles. If its spanning subdigraph* $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b)$ *contains a Hamilton path H from 0 to a such that at least k vertices travel by b in H, then there is also a Hamilton path from 0 to b in G.*

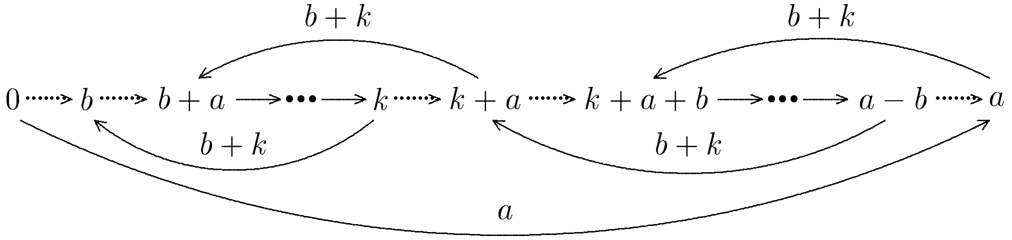

Fig. 1.

Proof. Let p is such that $p(b-a)-a \equiv 0 \pmod{2k}$. Then $2p(b-a)-a \equiv p(b-a) \equiv a \pmod{2k}$ and by Proposition 2 H is isomorphic to $H(2p)$. Thus p must be less than k, otherwise 0 must travel by a in $H(2p)$, which contradicts the hamiltonicity. By the assumption at least k vertices travel by b in $H(2p)$, so $p < k \leq 2p$.

We will need the following properties of H:

- clearly, 0 travels by b in H
- b travels by a
  because $(2p+1)(b-a)-a \equiv a+b-a \equiv b \pmod{2k}$ and $2p+1 > 2p$
- k travels by a
  because $(p+k)(b-a)-a \equiv k(b-a) \equiv k \pmod{2k}$ and $2p < p+k < 2k$
- $k+a$ travels by b
  because $(2p+k)(b-a)-a \equiv a+k(b-a) \equiv a+k \pmod{2k}$ and $2k+2p > 2p+k \geq 2k$
- finally, $a-b$ travels by b entering a in H

We start the construction in G with H and continue by the next three steps.

- First, we remove arc from b to $b+a$ and add arc from a to $b+a$. We obtain the simple arc from 0 to b and one elementary cycle of length $2k-2$.
- In the second step we consider vertices 0 (travels by b), k (travels by a) and $a-b$ (travels by b). Instead these we use 0 travels by a, k travels by $b+k$ and $a-b$ travels by $b+k$. Now we have the path from 0 through a, $a+b$, $\ldots$, k to b, and one cycle $k+a$, $k+a+b$, $\ldots$, $a-b$, $k+a$.
- Finally, we replace arcs from a to $a+b$ and from $k+a$ to $k+a+b$ by arcs from a to $k+a+b$ and from $k+a$ to $a+b$.

The result is a Hamilton path from 0 through a, $k+a+b$, $\ldots$, $a-b$, $k+a$, $a+b$, $\ldots$, k to b (see Fig. 1). $\square$

Construction 5 *If a nonhamiltonian connected digraph* $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$, *$a$ is odd, contains a Hamilton path from* 0 *to b, then it also contains a Hamilton path from* 0 *to* $b+k$ *(and vice versa).*

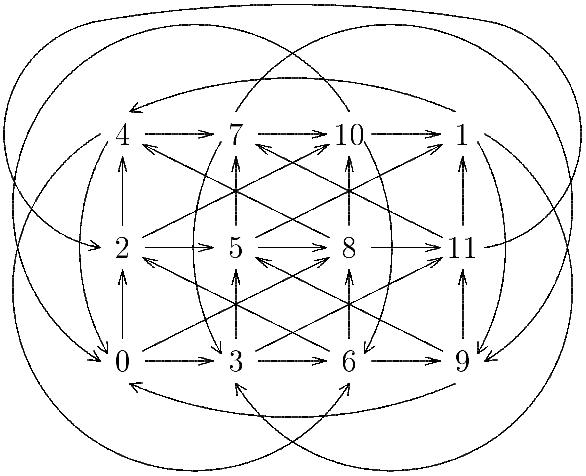

Fig. 2. $\mathrm{Cay}(\mathbb{Z}_{12}; 2, 3, 8)$

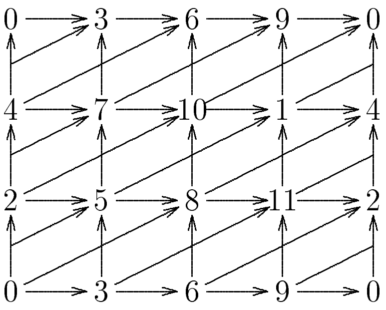

Fig. 3. $\mathrm{Cay}(\mathbb{Z}_{12}; 2, 3, 8)$ written on the torus

Proof. Let $B = gcd(2k, a)$ and $A = 2k/B$ be the order of the element a in the cyclic group $\mathbb{Z}_{2k}$. By oddness of a also B is odd and thus A is even. Therefore $k \equiv (A/2)a \pmod{2k}$ and every vertex $v \in \mathbb{Z}_{2k}$ can be uniquely written in the form $x_v a + y_v b$ with $0 \leq x_v < A$ and $0 \leq y_v < B$. This form also determine embedding of $\mathrm{Cay}(\mathbb{Z}_{2k}; a, b, b+k)$ on the torus by identifying a vertex $v = x_v a + y_v b$ with the point $[\frac{x_v}{A} + \frac{y_v E}{2k}, \frac{y_v}{B}]$ $(0 \leq E < A;\ Bb \equiv Ea \pmod{n})$ in the torus, and the arcs in the natural way (for $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 2, 8)$ see Fig. 3).

Choose a row y $(0 \leq y < B)$ and consider automorphism, where $v = x_v a + y_v b$ is mapped to $v' = (x_v + A/2)a + y_v b$ when $y_v = y$, and $v' = v$ otherwise (i.e., shifting row y by k longitudinally). Specially, if $y = 1$ this automorphism define one to one correspodence between Hamilton paths from 0 to b and Hamilton paths from 0 to $b + k$. □

Theorem 6 *For $t \geq 1$ the circulant oriented graph* $\mathrm{Cay}(\mathbb{Z}_{8t+4}; 2t+1, 2, 4t+4)$ *has no hamiltonian cycle and the reversal of any its arc results in a hamiltonian graph (for $t = 1$ see Fig. 2).*

Proof. Let $G = \mathrm{Cay}(\mathbb{Z}_{8t+4}; 2t + 1, 2, 4t + 4)$ for some $t \geq 1$.

- by the Theorem 1 this digraph has no hamiltonian cycle
- we have (by the notation in Theorem 3 and the Remark after it) $B = a = 2t + 1$, $A = 4$. Thus, by necessery condition of existence a Hamilton path from 0 to a in G, $0 < 2p_0 < 4$, so p_0 must be 1. Suppose $p_0 = 1$, then $E = (1+B) \bmod A = (2t+2) \bmod 4$, thus $s = 2$ and $r = a+2(E-a-1)/A = t + (t + 1) \bmod 2$ is odd, so gcd(r,s)=1 and required Hamilton path exists.

  The assumption $p_0 = 1$ is valid if and only if $a(b - a) \equiv a \pmod{n}$, i.e. $b - a \equiv 1 \pmod{A}$. Since $b - a - 1 = -2t$ and $b + k - a - 1 = 2t + 2$, then

either $b-a-1$ (if t is even) or $b+k-a-1$ (if t is odd) is a multiple of $A=4$. So by Theorem 3 G contains a Hamilton path H from 0 to a in its spanning subgraph $\mathrm{Cay}(\mathbb{Z}_{8t+4}; 2t+1, 2)$ when t is even and in $\mathrm{Cay}(\mathbb{Z}_{8t+4}; 2t+1, 4t+4)$ when t is odd.
- since in both cases in described Hamilton path $2a = 4t-2 = k$ vertices travel by b and $n-2a-1 = k-1$ vertices travel by a, by Construction 4 there is also a Hamilton path from 0 to b (or from 0 to $b+k$) in G.
- finally, by Construction 5 we can obtain both Hamilton paths, from 0 to b and from 0 to $b+k$ in G.

Since G is vertex-transitive, for any arc, it contains a Hamilton path from the initial vertex to the terminal vertex of this arc. Therefore, the reversal of any its arc results in a hamiltonian graph. $\square$

So, we have obtained an infinite class of nonhamiltonian oriented graphs such that the reversal of any arc creates a hamiltonian cycle, which solves the problem 5.43 in [4].

Remark. The same result can be proved for the class of nonhamiltonian oriented graphs $\mathrm{Cay}(\mathbb{Z}_{12k}; 3, 2, 6k+2)$ (for $k \geq 1$) using similar proof technics. On the contrary, $\mathrm{Cay}(\mathbb{Z}_{56}; 7, 2, 30)$ contains Hamilton path from 0 to 7 neither in $\mathrm{Cay}(\mathbb{Z}_{56}; 7, 2)$, nor in $\mathrm{Cay}(\mathbb{Z}_{56}; 7, 30)$.

3 Counterexamples to Ádám's conjecture

In 1963 A. Ádám made the conjecture than any digraph containing a directed cycle has an arc whose reversal decreases the total number of directed cycles [1–3]. It is easy to show that all digraphs containing cycle of length two (with symetric arcs) fulfil the conjecture. Also conjecture holds for each digraph containing a nontrivial strongly connected component which is not strongly 2-(arc)connected [8]. Thus we can restrict our considerations only to strongly 2-(arc)connected directed graphs.

The first counterexamples using multidigraphs were described independetly by Thomassen [12] and Grinberg [6]. Both classes of counterexamples use multidigraphs with regular number of parallel arcs. The problem than converses to finding the properties of the longest directed cycle in its base digraphs.

We denote G^p the multidigraph obtained from a simple oriented graph G (without loops and cycles of length two) by replacing each arc by p parallel arcs.

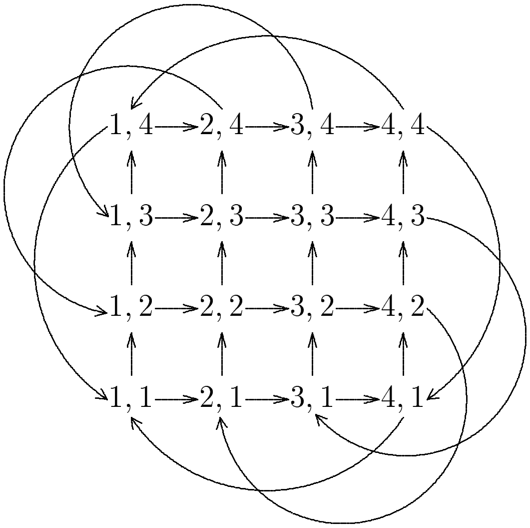

Fig. 4. $G_{4\times 4}$

Proposition 7 (by [12]) *Let G be an oriented graph such that the reversal of any its arc increases the length of a longest directed cycle. Then there exists a natural number t such that for* $p \geq t$ G^p *is a counterexample to Ádám's conjecture.*

Thomassen in [12] used results of Penn and Witte [10] about cycle lengths of the digraphs $C_a \times C_b$ (regular lattice embedded on a torus) and showed that $C_5 \times C_{7+10r} (r \geq 0)$ has no directed cycle of length $35 + 50k$ or $34 + 50k$. However, the reversal of any arc creates such a cycle. The minimal digraph of this type has 35 vertices and to be a counterexample each arc must be replaced by at least 11 parallel ones.

Counterexamples by Grinberg [6,7] use digraphs $G_{n\times n}$ created by drawing regular lattice $n \times n$ on a Klein bottle (Fig. 4). $G_{n\times n}$ with n^2 vertices has directed cycles of length n and $2n$ only. Howewer, the reversal of any arc (when $n \geq 4$) creates a cycle of length $2n+2$. The smallest such counterexample has 16 vertices and 64 arcs (every arc in double).

By Theorem 6 for all described nonhamiltonian digraphs the reversal of any its arc results in a hamiltonian digraph. So, by Proposition 7 these digraphs can be a base for counterexamples to Ádám's conjecture.

Theorem 8 *For* $k \geq 1$ *the circulant oriented graphs* $\mathrm{Cay}(\mathbb{Z}_{8k+4}; 2k+1, 2, 4k+4)^p$ *are, for sufficiently large p, counterexamples to Ádám's conjecture. Specially, in the case* $\mathrm{Cay}(\mathbb{Z}_{12}; 2, 3, 8)^4$ *we obtain the smallest known counterexample consisting of 12 vertices and 144 arcs.* □

On the other side, it can be simply proved, that Ádám's conjecture holds for all multidigraphs with at most 5 vertices. It remains open, whether there is a counterexample with less then 12 vertices. Also, the problem is open for simple oriented graphs.

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