# Improved Tree Decomposition Based Algorithms for Parameterized Planar Dominating Set*

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**Abstract**

We outline the main ideas behind an algorithm for computing the domination number of a planar graph that uses $O(c^{\sqrt{k}}n)$ time, where $k$ is the domination number of the given planar input graph, $n$ is the number of vertices in the graph and $c$ is a constant.

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*This short note is prepared for the Spring School on Combinatorics, of the Department of Applied Mathematics, Charles University Prague, held in Borová Lada, Czech Republic, and Finsterau, Fed. Rep. of Germany, April 17–25, 2001. It is meant as a reading guide and brief overview to our and some co-authors work on the improved parameterized complexity of PLANAR DOMINATING SET. An extended abstract of this work appeared in [1] and a long version in [2].

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# 1 Introduction

This text has been prepared as a guide through the paper "Fixed parameter algorithms for planar dominating set and related problems" by Hans L. Bodlaender, Henning Fernau, and ourselves [2]. This paper can be downloaded from either of the web pages

`http://www-fs.informatik.uni-tuebingen.de/`
`~niedermr/publications/index.html`, or
`~alber/publications.html`.

We propose to split the underlying text into basically two presentations, each of them may be prepared by one or two students. The text consists of six sections, where the focus of the first presentation should be on Section 2 and the focus of the second presentation should be on Section 3. Both topics are basically independent from each other, so the preparation can be done pretty autonomously from each other. We recommend everybody to read the introductory Section 1; Section 4 basically summarizes and combines the findings of the preceding sections (also showing that everything can be made constructive yielding an algorithm solving the PLANAR DOMINATING SET problem in the given time bound); Section 5 gives applications to problems similar to PLANAR DOMINATING SET, and Section 6 concludes the paper. The paper is considered to be self-contained, only assuming basic knowledge in graph theory and algorithms. Besides that no additional knowledge is necessary; however, some of the concepts and proofs require careful preparation and it might be helpful (also for the presentations) to think of illustrative examples. Although contained in the title, one has to know almost nothing about parameterized complexity theory in order to understand and present the results. One may consider the work simply as a graph algorithms paper.

In case of any problems, we strongly encourage to contact *both* authors (send email simultaneously to both of them in order to increase the likelihood of a small response time).

# 2 Presentation I: dominating set, outerplanarity, and tree decomposition

The outline of this presentation could be as follows:

## 2.1 Elementary notion

Introduce resp. recall the following notions (see Subsection 2.1 of the paper):

- Planar graphs, outerplanar graphs, and $r$-outerplanar ($\geq 1$) graphs;
- DOMINATING SET problem; mention its $NP$-completeness also for planar graphs and maybe also Baker's famous polynomial time approximation scheme result;
- Graph separators;
- Introduce layer decompositions of planar graphs and related notions.

## 2.2 Tree decompositions

Define the concepts of tree decomposition and treewidth for graphs and provide some illustrating example(s), not to be found in the paper.

## 2.3 Treewidth and outerplanarity

The main contribution of this presentation will be to prove the relationship

$$\mathrm{tw}(G) \leq 9\gamma(G) - 1 \tag{1}$$

for planar graphs (see Corollary 10 of the paper).[1] Here, $\gamma(G)$ denotes the domination number and $\mathrm{tw}(G)$ is the treewidth of the graph. Moreover, a corresponding tree decomposition can be obtained in linear time.

[1] In the second presentation this will be improved to $\mathrm{tw}(G) \leq d\sqrt{\gamma(G)}$.

- Convince yourself that there is no relation of the form

  $$\mathrm{tw}(G) \leq f(\gamma(G)),$$

  for some function $f$, that holds for *general* graphs.

Corollary 10 follows directly from Theorem 9 and the observation in Proposition 4.

The real challenge lies in showing the relation between treewidth and outerplanarity given in Theorem 9 (and more precisely in Theorem 12): An $r$-outerplanar graph has treewidth of at most $3r - 1$. The proof is given in Subsection 2.3 of the paper. It needs quite some care and a systematic presentation. This will probably be the hardest and most time consuming step of the presentation.

## 2.4 Dynamic programming using tree decompositions

Now, describe the use of having given the tree decomposition of a graph by describing briefly how to solve DOMINATING SET this way using dynamic programming. Present *the result* of Theorem 11, which then—together with Corollary 10— gives a direct $O(3^{9\gamma(G)}n)$ time algorithm for PLANAR DOMINATING SET.[2]

If there is time left to sketch the dynamic programming algorithm for Theorem 11, we strongly recommend to do this using the example of VERTEX COVER[3] rather than DOMINATING SET. For VERTEX COVER one can give a $2^{\ell}n$ time algorithm, when the graph is given together with a tree decomposition of width $\ell$. This is conceptually much easier to describe and understand than it is the case for DOMINATING SET. The algorithm has to be figured out by the student herself: Use the idea that, for each bag (i.e., each set $X_i$ of the corresponding tree decomposition), one keeps a table. These tables store, for every vertex in the bag, the information of whether that vertex is assumed to belong to the vertex cover we are seeking for or not. Since $|X_i| \leq \ell$, the table size for each bag is bounded by $2^{\ell}$.

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[2] In the second presentation this will be improved to $c^{\sqrt{\gamma(G)}}n$.

[3] VERTEX COVER is defined as follows: Given an undirected graph $G = (V, E)$ and a positive integer $k$, does $G$ have a vertex cover of size at most $k$? Herein, a vertex cover is a subset of vertices $C \subseteq V$ such that each edge in $E$ has at least one of its endpoints in $C$.

# 3 Presentation II: improved tree decomposition by separator techniques

In presentation II, the goal should be to further improve the relation (1) from presentation I to

$$\mathrm{tw}(G) \leq 6\sqrt{34\gamma(G)} + 8 \tag{2}$$

for planar graphs.

The main prerequisites needed for this presentation from the previous presentation are the statement of Theorem 12 (relating treewidth and outer-planarity), and the concept of layer decompositions from Definition 3. After recalling these, the outline of the presentation can be as follows:

## 3.1 Finding separators layerwisely

Refer to Subsection 3.2 of the paper. Here the main technical difficulty of this presentation arises and it should be prepared and presented with care. This is probably the most crucial and most time consuming part of this talk. Take your time! Based on the assumption of a (small) dominating set of size $k$, here we prove the existence of small graph separators. The separators lie over a constant number of layers each.

## 3.2 Bounding the size of the separators

Refer to Subsection 3.4 of the paper. We determine the total size of all graph separators constructed in Subsection 3.3.

## 3.3 Constructing tree decompositions for layerwisely separated graphs

Refer to Subsections 3.1 and 3.4 of the paper. Building on the results of Subsections 3.3 and 3.4, it is now relatively easy to conclude that the treewidth of the graph has to be of size $O(\sqrt{k})$.

### 3.4 An $O(c^{\sqrt{k}}n)$ time algorithm for PLANAR DOMINATING SET

The result of Theorem 27 of the paper, i.e. the relation (2) from above, together with Theorem 11 of the paper, basically yields the $O(c^{\sqrt{\gamma(G)}}n)$ time algorithm for PLANAR DOMINATING SET, where $c = 3^{6\sqrt{34}}$. Convince yourself that the tree decomposition of width $6\sqrt{34\gamma(G)} + 8$ can be constructed in linear time. This is outlined in Section 4 of the paper. Depending on how many students give presentation II, this step can be either omitted or worked out in detail.

## 4 Advertisement: Further reading and very recent related literature

We briefly want to give hints to further reading on related work.

- **Parameterized Complexity:** For an extensive introduction to parameterized complexity, we refer to the monograph of Downey and Fellows [6], the "fathers" of this theory. Recently, some survey papers appeared that also might serve for additional studies towards that field (see [5, 7]).
- **Related work on graph problems:** Besides the underlying paper of this reading guide we would like to point out two further papers, which also deal with the design of $O(c^{\sqrt{k}}n)$ time algorithms for various parameterized graph problems: Generalizations of the methods presented here can be found in [4], and further systematic approaches based on planar separator theorems are treated in [3].

Finally, we would like to point to the Web pages of our research projects "PEAL" (Parameterized Complexity and Exact ALgorithms) and "OPAL" (Optimal Solutions for hard Problems in Computational Biology), see

`http://www-fs.informatik.uni-tuebingen.de/peal/index.html.`

Here, in particular, one can find further literature on fixed parameter algorithms.

# References

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[7] M. R. Fellows. Parameterized complexity: new developments and research frontiers. In *Aspects of Complexity*. De Gruyter, 2001. To appear.