On Hamiltonian Cycles in Strong Products of Graphs*

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Abstract

We prove that the strong product of graphs $G_1 \times \cdots \times G_n$ contains a hamiltonian cycle for $n \approx c\Delta$ for any $c > \ln(25/12) + 1/60$ whenever all G_i are connected graphs of maximum degree at most Δ.

1 Introduction

Let the vertex set of the graph G be denoted by $V(G)$ and let its edge set be denoted by $E(G)$. The strong product of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$. Its two distinct vertices $[u_1, v_1]$ and $[u_2, v_2]$ are joined by an edge iff $u_1 = u_2 \vee u_1u_2 \in E(G)$ and $v_1 = v_2 \vee v_1v_2 \in E(H)$. A cycle containing all the vertices of the graph is called a hamiltonian cycle; each vertex is contained in such a cycle exactly once. We write G^k for the strong product of k copies of G and we call graphs containing a hamiltonian cycle hamiltonian graphs for the brevity.

Zaks asked in [4] whether there exists $k(G)$ for any connected graph G with at least two vertices such that $G^{k(G)}$ is hamiltonian. Bermond, Germa and Heydemann proved in [1] the existence of the number $k(G)$ and they proved that if G^k is hamiltonian, then also G^h is hamiltonian for all $k \leq h$ using

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some results of Rosenfeld and Barnette contained in [3]. They did not give any upper bound on $k(G)$ in terms of maximum degree of the graph G, but they conjectured that G^Δ should be hamiltonian for any connected graph G with at least two vertices and of maximum degree Δ (for $\Delta \geq 2$). Their conjecture was proved in [2]. We prove the statement that G^k is hamiltonian holds for $k \approx c\Delta$ for any $c \ln 25/12 + 1/60$, i.e. the upper bound on the order of power of G in terms of Δ given in [2] is not optimal.

We introduce notation used in the paper, we give basic properties of strong graph products and we summarize previous results in Section 2. We study structure of strong products of graphs in Section 3. We give the proofs of our hamiltonian results in Section 4.

2 Definitions and Basic Properties

All the graphs considered in this paper contain at least two vertices. Whenever we mention a product of graphs, we always mean the strong product of graphs; we are not going to use other definitions of the graph product in this paper. We write C_n for the cycle containing n vertices, P_n for the path containing n vertices and S_n for the star of order n, i.e. the star with n edges ($S_n = K_{1,n}$). We write $\leq k$ for the set of all the positive integer numbers less or equal to k; we use $\geq k$, $< k$ and $> k$ in a similar manner. Let I be a set of integer numbers, then S_I denotes the set of all the stars S_i such that $i \in I$; C_I and P_I are used in the similar manner. If A and B are sets of graphs, then we write $A \times B$ for $\{G \times H \mid G \in A \wedge H \in B\}$ and A^2 for $A \times A$, A^3 for $A \times A \times A$, etc.

We assign the vertices of P_n and S_n numbers and a star ($*$); this makes proofs dealing with paths or stars more clear. We understand the set $V(P_n)$ as $\{0, \ldots, n-1\}$; its two vertices are connected iff their difference is exactly one. We understand the set $V(S_k)$ as $\{*, 0, \ldots, k-1\}$; the edges of S_k are only between $*$ and the number vertices. Let S be a product of n stars; we write $V_i(S)$ for the set of vertices of S whose exactly i coordinates are equal to $*$. We mean by $x \bmod y$ the number between 0 and $y-1$ which is congruent to x modulo y, e.g. $9 \bmod 7$ is 2. Let $a = [a_0, \ldots, a_{n-1}]$ be any vertex of S. We use some useful notation in the paper: We write $\sum a$ for $\left(\sum_{i=0}^{n-1} a_i\right) \bmod n$; we consider $*$ to be counted as zero in this sum. We write $a[k \to s]$ for the vertex b obtained from a by substituting s for its k-th entry, i.e. $b_{k \bmod n} = s$ (we take $s \bmod k$ instead of s if necessary) and $b_i = a_i$ for $i \neq k \bmod n$.

We address the question of existence of a hamiltonian cycle in products of graphs. The key theorem proved in [1] using ideas from [3] is the following one:

Theorem 1 *Let G be a connected graph of maximum degree at most Δ. Then any graph of $C_{\geq\Delta} \times G$ is hamiltonian.*

The immediate corollary of this theorem is that if G^k is hamiltonian then G^h is hamiltonian for all $h \geq k$. We write $h(G)$ for the smallest value of h such that G^h is hamiltonian; the existence of this value follows from the results of [1].

The following theorem proved in [2] gives the proof of the conjecture of Bermond et al. and proves some more general version of this conjecture, namely that it is even true for products of graphs, not only for their powers:

Theorem 2 *Let $G_0, \ldots, G_{\Delta-1}$ be any connected graphs with maximum degree at most Δ. Then $G_0 \times \ldots \times G_{\Delta-1}$ is hamiltonian.*

Let $h_{\max}(\Delta)$ be $\max\{h(G)|\Delta(G) \leq \Delta\}$ where $\Delta(G)$ is maximum degree of G. The existence of $h_{\max}(\Delta)$ and the upper bound $h_{\max}(\Delta) \leq \Delta$ follows from Theorem 2 proved in [2]; Zaks addressed in [4] the question of lower bounds for $h_{\max}(\Delta)$, too (he used different notation in his paper). He proved the following theorem:

Theorem 3 *If S_n^k is hamiltonian, then the following inequality holds:*

$$k \geq \frac{\ln 2}{\ln(1+\frac{1}{n})} \approx n \ln 2$$

A covering of the graph G by the set of graphs I is a set J of subgraphs of G such that each vertex of G is contained exactly in one of the graphs of J and each graph of J is isomorphic to some of the graphs of I. Note that we cover the vertices of the graph, we do not cover its edges. The following theorem trivially holds (it is enough to consider all the subgraphs of $G \times H$ equal to the product of a subgraph covering G and each single vertex of H):

Theorem 4 *Let G and H be any graphs and let I be any set of graphs. If G can be covered by I, then $G \times H$ can be covered by I.*

The following lemma on covering graphs by stars was proved in [2]:

Lemma 1 *Each connected graph G of maximum degree at most Δ can be covered by $S_{\leq\Delta}$.*

3 Coverings Graph Products by Stars and Cycles

We first prove some theorems about covering products of graphs by small degree stars.

Theorem 5 *For all $k \geq 1$, $l \geq 2$, any graph of $S_{\leq kl}^k$ can be covered by $S_{\leq l}$.*

Proof: Let $S \in S^k_{\leq kl}$ be $S_{n_0} \times S_{n_1} \times \cdots \times S_{n_{k-1}}$ where $1 \leq n_i \leq kl$. We find a spanning subgraph T of S such that T has no isolated vertices and $\Delta(T) \leq l$. There exists a covering of T (and thus also of S) by $S_{\leq l}$ due to Lemma 1 used separately on each connected component of T.

Consider a vertex $a \in V(S) \setminus V_0(S)$. We say that coordinate i ($0 \leq i \leq k-1$) is substitutable iff there exists some $0 \leq b \leq n_i - 1$ such that $\sum a[i \to b] = i$. We call all b satisfying this condition possible substitutions for fixed i. Because $n_i \leq kl$ it is clear that there are at most l possible substitutions. Let $0 \leq i_1 < i_2 < \cdots < i_m \leq k-1$ be all indices such that $a_i = *$. Before constructing a spanning subgraph T of S we construct another spanning subgraph of S — we call it T'. We distinguish several cases when choosing edges adjacent to a in T':

- **At least one of the coordinates $i_1,\ i_2, \ldots,\ i_m$ of a is substitutable.** Let i be the smallest such substitutable coordinate and let $b_1,\ b_2, \ldots,\ b_p$ ($1 \leq p \leq l$) be all its possible substitutions. We join all the vertices $a[i \to b_j]$, $j = 1, \ldots,\ p$, to a in T'. We call the vertices $a[i \to b_j]$ for $j = 1, \ldots,\ p$ children of a and a their parent.

- **None of the coordinates $i_1,\ i_2, \ldots,\ i_m$ is substitutable.** If there exists some $p < i_1$ such that $a_p = 0$ and p is not substitutable, then we choose j to be the smallest such p and we join a and $b = a[j \to *]$ in T'. Otherwise, we join a and $b = a[i_1 \to 0]$ in T'. We call b a sibling of a. It is easy to see that if $b \notin V_0(S)$ then also a is a sibling of b. It is obvious that every vertex has at most one sibling and the siblings make pairs of the vertices which have no children.

All the edges of T' are determined by these two rules. It is easy to check that every vertex $a \in V_0(S)$ has exactly one parent, namely $a[\sum a \to *]$, and every vertex $b \in V(S) \setminus V_0(S)$ has at most one parent — the only possible candidate is $b[\sum b \to *]$. All the vertices of S have at most l children due to the upper bound on the maximum number of possible substitions. It is also easy to check that vertex $a \in V(S) \setminus V_0(S)$ has a sibling iff it has no children, T' has no isolated vertices and its maximum degree is at most $l+1$ (l children and a parent). We change some edges of T' to get the desired subgraph T in two consecutive steps:

- We remove one of the edges connecting $a \in V(S) \setminus (V_0(S) \cup V_1(S))$ to its children if the degree of a is $l+1$. The degree of the child of a remains at least one, since it has either some children or a sibling.

- Let $a \in V_1(S)$ be a vertex of degree $l+1$. If there is a child of a, say s, which has a sibling, we remove the edge sa. Otherwise let s be arbitrary child a and let f be the parent of a. We replace edges sa and af by the edge sf.

The degree of vertices $a \in V_0(S)$ can be increased during this procedure to at most 2 and the degree of all the other vertices is decreased or remains below l. The spanning subgraph T of S obtained from T' through these changes has clearly the desired properties. □

Theorem 6 *Let $G_0, \ldots, G_{n-1}$ be graphs of maximum degree at most Δ, $2 \leq l$ and $\Delta \leq nl$. Then $G_0 \times \cdots \times G_{n-1}$ can be covered by $S_{\leq l}$.*

Proof: By Lemma 1 each G_i can be covered by $S_{\leq nl}$ hence $G_0 \times \cdots \times G_{n-1}$ can be covered by $S^n_{\leq nl}$ and every graph of $S^n_{\leq nl}$ can be covered by $S_{\leq l}$ due to Theorem 5. □

The following technical lemmas will prove their value in Section 4:

Lemma 2 *The graph $S_k \times S_l$ can be covered by $S_{\{\lfloor \frac{l}{2} \rfloor, \lfloor \frac{l-1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor\}}$ for all $k \geq 2$, $l \geq 3$.*

Proof: We construct a covering of $S_k \times S_l$ with the desired properties. Centers of stars of the covering are the vertices $[a, b]$ such that at least one of a, b is $*$. The covering consists of several types of stars:

- The center vertex $[a, *]$ for odd a is joined to the following peripheral vertices $[a, 1]$, $[a, 3], \ldots,$ $[a, 2\left\lfloor \frac{l}{2} \right\rfloor - 1]$; order of this star is $\left\lfloor \frac{l}{2} \right\rfloor$.
- The center vertex $[a, *]$ for even a is joined to the following peripheral vertices $[a, 2]$, $[a, 4], \ldots,$ $[a, 2\left\lfloor \frac{l-1}{2} \right\rfloor]$; order of this star is $\left\lfloor \frac{l-1}{2} \right\rfloor$.
- The center vertex $[*, b]$ for odd b is joined to the following peripheral vertices $[0, b]$, $[2, b], \ldots,$ $[2\left\lfloor \frac{k-1}{2} \right\rfloor, b]$; order of this star is $\left\lfloor \frac{k+1}{2} \right\rfloor$.
- The center vertex $[*, b]$ for even b is joined to the following peripheral vertices $[1, b]$, $[3, b], \ldots,$ $[2\left\lfloor \frac{k}{2} \right\rfloor - 1]$; order of this star is $\left\lfloor \frac{k}{2} \right\rfloor$.
- The center vertex $[*, *]$ is joined to all the remaining peripheral vertices, namely $[0, 0]$, $[2, 0], \ldots,$ $[2\left\lfloor \frac{k-1}{2} \right\rfloor, 0]$; order of this star is $\left\lfloor \frac{k+1}{2} \right\rfloor$.

It is a routine to check that all the vertices of $S_k \times S_l$ are used exactly once and thus the construction gives a covering of $S_k \times S_l$ by $S_{\{\lfloor \frac{l}{2} \rfloor, \lfloor \frac{l-1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor\}}$.
□

Lemma 3 *Any graph of $S_{\leq 2n} \times S_{\leq 2n+1}$ can be covered by $S_{\leq n}$.*

Proof: It is clear due to Theorem 5 that all the graphs of $S_{\leq 2n} \times S_{\leq 2n}$ can be covered by $S_{\leq n}$. The graphs of $S_{\leq n} \times S_{\leq 2n+1}$ can be covered by $S_{\leq n}$ due to Theorem 4. Thus it is enough to prove that for every $n < k \leq 2n$ $S = S_k \times S_{2n+1}$ can be covered by $S_{\leq n}$. This is a direct corollary of Lemma 2 when setting $l = 2n + 1$. □

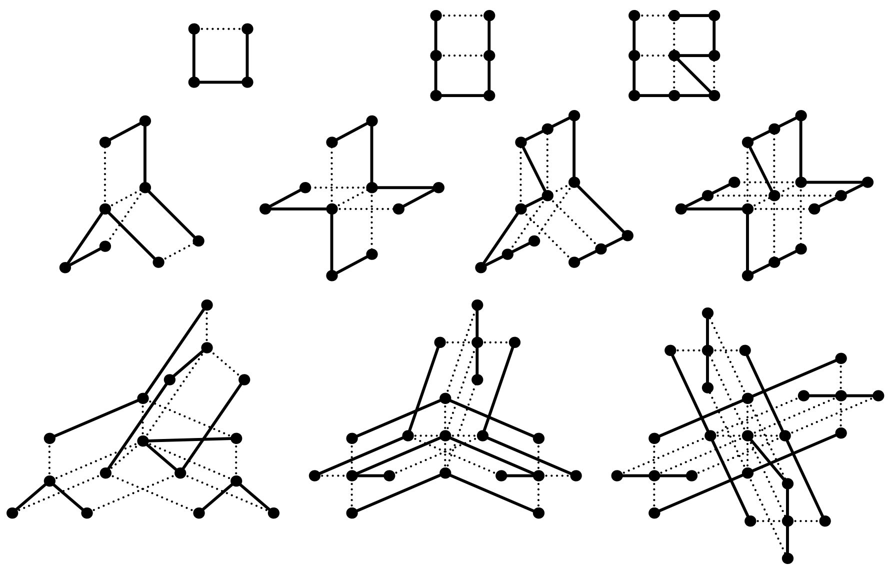

Figure 1: Path coverings of $S_1 \times S_1$, $S_1 \times S_2$, $S_2 \times S_2$, $S_1 \times S_3$, $S_1 \times S_4$, $S_2 \times S_3$, $S_2 \times S_4$, $S_3 \times S_3$, $S_3 \times S_4$ and $S_4 \times S_4$

Lemma 4 *Any graph of* $S_{\{4,5,6,7,8\}} \times S_{\{5,6,7,8,9\}}$ *can be covered by* $S_{\{2,3,4\}}$.

Proof: The statement immediately follows from Lemma 2. □

Lemma 5 *Any graph of* $S_{\leq 4} \times S_{\leq 4}$ *can be covered by* $P_{\geq 3}$.

Proof: It is straightfoward to check that the following coverings exists (see Figure 1):

- $S_1 \times S_1$ can be covered by one copy of P_4.
- $S_1 \times S_2$ can be covered by one copy of P_6.
- $S_1 \times S_3$ can be covered by two copies of P_4.
- $S_1 \times S_4$ can be covered by two copies of P_5.
- $S_2 \times S_2$ can be covered by one copy of P_9.
- $S_2 \times S_3$ can be covered by one copy of P_{12}.
- $S_2 \times S_4$ can be covered by one copy of P_7 and one copy of P_8.
- $S_3 \times S_3$ can be covered by four copies of P_3 and one copy of P_4.

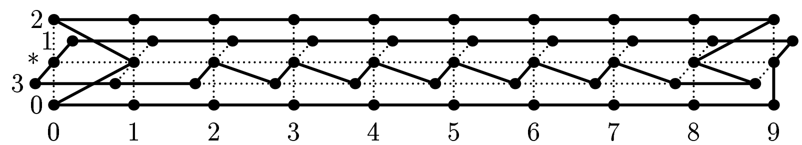

Figure 2: Hamiltonian cycle in $P_{10} \times S_4$

- $S_3 \times S_4$ can be covered by five copies of P_3 and one copy of P_5.
- $S_4 \times S_4$ can be covered by seven copies of P_3 and one copy of P_4.

□

Lemma 6 *Any graph of $P_{\geq 4} \times S_4$ is hamiltonian.*

Proof: We just construct a hamiltonian cycle in $P_k \times S_4$ for $4 \leq k$. The following sequence of the vertices of $P_k \times S_4$ is a hamiltonian cycle (see Figure 2):

$$[0,0],[1,0],\ldots,[k-2,0],[k-1,0],[k-1,*],[k-1,1],[k-2,1],\ldots,[1,1],[0,1],[0,*],$$

$$[0,3],[1,3],[2,3],[2,*],[3,3],[3,*],[4,3],[4,*],\ldots,[k-3,3],[k-3,*],[k-2,3],$$

$$[k-1,3],[k-2,*],[k-1,2],[k-2,2],\ldots,[1,2],[0,2],[1,*],[0,0]$$

□

We use results of [2] to prove the following lemma; we can use Theorem 2, but we use weaker version of that theorem, also proved in [2]:

Theorem 7 *Any graph of $S^{\Delta}_{\leq \Delta}$ is hamiltonian for $\Delta \geq 2$.*

Lemma 7 *Let n_0, n_1, n_2 be integers between 1 and 4 (inclusively) and let k be an integer such that $k \leq \min\{(n_0+1)(n_1+1)(n_2+1), 15\}$. Then $S_{n_0} \times S_{n_1} \times S_{n_2} \times S_k$ can be covered by $C_{\geq l}$ where l is equal to $\min\{(n_0+1)(n_1+1)(n_2+1), 15\} \cdot k$.*

Proof: Let us assume $n_0 \leq n_1 \leq n_2$. If $n_2 \leq 3$ then $S_{n_0} \times S_{n_1} \times S_{n_2}$ is hamiltonian due to Theorem 7 and $S_{n_0} \times S_{n_1} \times S_{n_2} \times S_k$ is hamiltonian due to Theorem 1 since $k \leq (n_0+1)(n_1+1)(n_2+1)$. Simple calculation gives that the length of this hamiltonian cycle is sufficient.

Let us consider the case that $n_2 = 4$. Due to Lemma 5 $S_{n_0} \times S_{n_1}$ can be covered by $P_{\geq 3}$. The product of $P_{\geq 4}$ and S_{n_2} is hamiltonian due to Lemma 6 and its hamiltonian cycle contains at least 20 vertices. The product of cycle of length at least 20 and S_k is hamiltonian due to Theorem 1 and thus the product of any of $P_{\geq 4}$ with $S_{n_2} = S_4$ and S_k can be covered by $C_{\geq 20(k+1)}$. The proof that $P_3 \times S_4 \times S_k$ is hamiltonian would finish the proof of the whole lemma. Simple calculation gives that the lengths of the found cycles are sufficient.

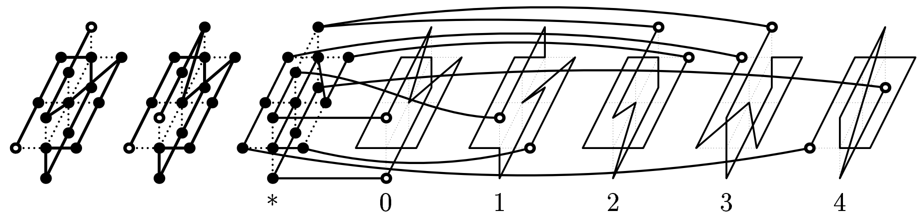

Figure 3: Hamiltonian paths in $P_3 \times S_4$ and a hamiltonian cycle in $P_3 \times S_4 \times S_5$

We prove now that $P_3 \times S_4 \times S_k$ is hamiltonian. Consider two possible types of hamiltonian paths in $P_3 \times S_4$ shown at the left in Figure 3. These paths in the different copies of $P_3 \times S_4$ corresponding to the numbered vertices of S_k can be joined together through the copy of $P_3 \times S_4$ corresponding to the vertex $*$ of S_k as shown in Figure 3 for $k = 5$; it is a straightforward work to check that the hamiltonian paths can be joined for all $1 \leq k \leq 15$. Since $k \leq 15$ due to the assumption of the lemma, we have found a hamiltonian cycle of $P_3 \times S_4 \times S_k$. $\square$

4 Hamiltonian Cycles in Products of Graphs

Lemma 8 *Any graph of $S^3_{\leq 8} \times S^3_{\leq 9} \times S_{\leq 15}$ can be covered by $C_{\geq 120}$.*

Proof: Let $G = S_{n_0} \times \ldots \times S_{n_6}$ be a graph in $S^3_{\leq 8} \times S^3_{\leq 9} \times S_{\leq 15}$; we assume that $n_0 \leq \ldots \leq n_6$. From the assumption of the lemma follows that $n_0, n_1, n_2 \leq 8$, $n_3, n_4, n_5 \leq 9$ and $n_6 \leq 15$. We distinguish several cases:

- **All n_i are at most 7 ($n_6 \leq 7$).**

  In this case G is hamiltonian due to Theorem 7. Since G contains at least $2^7 = 128$ vertices, the length of its hamiltonian cycle is at least 128 and this cycle covers G.

- **At least two of n_i are at most 3 and at least one of n_i is at least 8 ($n_1 \leq 3 \wedge n_6 \geq 8$).**

  Due to Theorem 6 $S_{n_3} \times S_{n_4} \times S_{n_5}$ can be covered by $S_{\leq 3}$. The product of the first six stars can be covered by $C_{\geq 16}$ due to Theorem 1 applied to S_{n_2} and the hamiltonian cycles in $S_{n_0} \times S_{n_1} \times S_{\leq 3}$. We conclude by application of Theorem 1 to the products of these cycles and S_{n_6} that G can be covered by $C_{\geq 128}$.

- **At most one of n_i is at most 3 and at least one of n_i is at least 8 ($n_1 \geq 4 \wedge n_6 \geq 8$).**

If $n_3 \leq 4$, then $S_{n_0} \times S_{n_1} \times S_{n_2} \times S_{n_3}$ is hamiltonian due to Theorem 7. We conclude by repeated application of Theorem 1 in the manner similar to the previous case that G can be covered by $C_{\geq 128}$.

If $n_3 \geq 5$, then $S_{n_0} \times S_{n_3}$ can be covered by $S_{\{1,2,3,4\}}$ due to Lemma 3 and both $S_{n_1} \times S_{n_4}$ and $S_{n_2} \times S_{n_5}$ can be covered by $S_{\{2,3,4\}}$ due to Lemma 4. Thus due to Lemma 7 applied to the product of this covering by $S_{\{1,2,3,4\}} \times S_{\{2,3,4\}} \times S_{\{2,3,4\}}$ and S_{n_6} we conclude that G can be covered by $C_{\geq 120}$.

□

Corollary 1 *The product of any at least* $\lfloor \frac{93}{120}\Delta \rfloor + 7$ *connected graphs of maximum degree at most* Δ *can be covered by* $C_{\geq 120}$.

Proof: The product of any $\lceil \frac{1}{8}\Delta \rceil$ (respectively $\lceil \frac{1}{9}\Delta \rceil$, $\lceil \frac{1}{15}\Delta \rceil$) connected graphs of maximum degree at most Δ can be covered by $S_{\leq 8}$ (respectively $S_{\leq 9}$, $S_{\leq 15}$) due to Theorem 6. Thus due to Lemma 8 the product of any $3\lceil \frac{1}{8}\Delta \rceil + 3\lceil \frac{1}{9}\Delta \rceil + \lceil \frac{1}{15}\Delta \rceil \leq \lfloor \frac{93}{120}\Delta \rfloor + 7$ such graphs can be covered by $C_{\geq 120}$. In case that the product consists of more graphs than necessary, we use Theorem 4 to get its covering. □

Corollary 2 *For each* $c > \ln \frac{25}{12}$ *there exists* c' *such that the product of any at least* $\lfloor c\Delta \rfloor + c'$ *connected graphs of maximum degree at most* Δ *can be covered by* $C_{\geq 120}$.

Proof: Consider the product of $\sum_{i=2^{k+3}}^{2^{k+3}+2^k-1} \lceil \frac{1}{i}\Delta \rceil$ connected graphs of maximum degree at most Δ for $k \geq 0$. The product of $\lceil \frac{1}{i}\Delta \rceil$ such graphs can be covered by $S_{\leq i}$ due to Theorem 6. Thus there exists a covering of the considered product by $S_{\leq 2^{k+3}} \times \cdots \times S_{\leq 2^{k+3}+2^k-1}$. We can cover the considered product by $S_{\leq 2^{k+2}} \times \cdots \times S_{\leq 2^{k+2}+2^{k-1}-1}$ due to Lemma 3 applied to the pairs of the consecutive members of the product $S_{\leq 2^{k+3}} \times \cdots \times S_{\leq 2^{k+3}+2^k-1}$; we conclude by repeated application of Lemma 3 in this manner that the considered product can be covered by $S_{\leq 8}$. We obtain by the same reasoning that the product of $\sum_{i=2^{k+3}+2^k}^{2^{k+3}+2^{k+1}-1} \lceil \frac{1}{i}\Delta \rceil$, $\sum_{i=2^{k+4}-2^k}^{2^{k+4}-1} \lceil \frac{1}{i}\Delta \rceil$, connected graphs of maximum degree at most Δ for $k \geq 0$ can be covered by $S_{\leq 9}$, $S_{\leq 15}$. Thus we conclude in the same manner as in the proof of Corollary 1 that the following number of graphs in the product is enough:

$$3 \sum_{i=2^{k+3}}^{2^{k+3}+2^k-1} \left\lceil \frac{1}{i}\Delta \right\rceil + 3 \sum_{i=2^{k+3}+2^k}^{2^{k+3}+2^{k+1}-1} \left\lceil \frac{1}{i}\Delta \right\rceil + \sum_{i=2^{k+4}-2^k}^{2^{k+4}-1} \left\lceil \frac{1}{i}\Delta \right\rceil \leq$$

$$\leq \left\lfloor \left(3 \sum_{i=2^{k+3}}^{2^{k+3}+2^{k+1}-1} \frac{1}{i} + \sum_{i=2^{k+4}-2^k}^{2^{k+4}-1} \frac{1}{i} \right) \Delta \right\rfloor + 3 \cdot 2^{k+1} + 2^k$$

We examine the behaviour of the linear coefficient in the expression for large value of k:

$$\lim_{k\to\infty} \left(3 \sum_{i=2^{k+3}}^{2^{k+3}+2^{k+1}-1} \frac{1}{i} + \sum_{i=2^{k+4}-2^k}^{2^{k+4}-1} \frac{1}{i} \right) = 3 \ln \frac{5}{4} + \ln \frac{16}{15} = \ln \frac{25}{12}$$

We can choose for any $c > \ln \frac{25}{12}$ the value of k large enough to be the linear coefficient smaller than c; the value of c' is bounded by $3 \cdot 2^{k+1} + 2^k$. □

Lemma 9 *For any $\Delta > 32$, the product of a cycle of length at least 120 and at least $\lfloor \frac{1}{60}\Delta \rfloor + \lceil \log_2 \Delta \rceil - 6$ graphs of maximum degree at most Δ can be covered by $C_{\geq \Delta}$.*

Proof: Let k be $\lceil \log_2 \Delta \rceil - 6$; $k \geq 0$, since $\Delta > 32$. The theorem holds trivially due to Theorem 4 for $k = 0$ $(32 < \Delta \leq 64)$. Otherwise, consider the product of $\sum_{i=1}^{k} \lceil \frac{1}{60 \cdot 2^i} \Delta \rceil$ graphs of maximum degree at most Δ. For all $1 \leq i \leq k$ the product of $\lceil \frac{1}{60 \cdot 2^i} \Delta \rceil$ graphs can be covered by $S_{\leq 60 \cdot 2^i}$ due to Theorem 6. We conclude by repeated application of Theorem 1 to the product of the cycle and the stars of the coverings that the product of $\sum_{i=1}^{k} \lceil \frac{1}{60 \cdot 2^i} \Delta \rceil$ graphs of maximum degree at most Δ and a cycle of length at least 120 can be covered by cycles of length at least $120 \cdot 2^k \geq \Delta$. The lemma follows now from the simple upper estimate of the number of graphs needed:

$$\sum_{i=1}^{k} \left\lceil \frac{1}{60 \cdot 2^i} \Delta \right\rceil \leq \left\lfloor \sum_{i=1}^{k} \frac{1}{60 \cdot 2^i} \Delta \right\rfloor + k \leq$$

$$\leq \left\lfloor \sum_{i=1}^{\infty} \frac{1}{60 \cdot 2^i} \Delta \right\rfloor + \lceil \log_2 \Delta \rceil - 6 = \left\lfloor \frac{1}{60} \Delta \right\rfloor + \lceil \log_2 \Delta \rceil - 6$$

□

The immediate corollaries of the previous lemma are the following:

Corollary 3 *The product of any at least $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 1$ connected graphs of maximum degree at most Δ can be covered by $C_{\geq \Delta}$ whenever $\Delta > 32$.*

Proof: Due to Corollary 1 the product of $\lfloor \frac{93}{120}\Delta \rfloor + 7$ of them can be covered by $C_{\geq 120}$; due to Lemma 9 the product of a cycle of length at least 120 and $\lfloor \frac{1}{60}\Delta \rfloor + \lceil \log_2 \Delta \rceil - 6$ connected graphs can be covered by $C_{\geq \Delta}$. The corollary immediately follows from these two facts and from Theorem 4 (the product may consist of more graphs than necessary). □

If we use Corollary 2 instead of Corollary 1, we obtain the following:

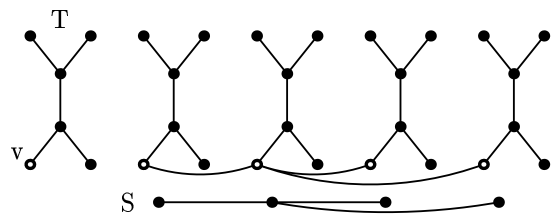

Figure 4: Construction of a small maximum degree spanning tree in a product of two trees

Corollary 4 *For each* $c > \ln\frac{25}{12} + \frac{1}{60}$ *there exists* c' *such that the product of any at least* $\lfloor c\Delta\rfloor + c'$ *connected graphs of maximum degree at most* Δ *can be covered by* $C_{\geq 120}$ *for* $\Delta > 32$*.*

Lemma 10 *The product of any number of connected graphs of maximum degree at most* Δ *contains a spanning tree of maximum degree at most* $\Delta + 1$*.*

Proof: We prove that the product of a tree of maximum degree at most Δ and a tree of maximum degree at most $\Delta + 1$ contains a spanning tree of maximum degree at most $\Delta+1$; the lemma easily follows from this statement by induction on the number of graphs in the product.

Let S be the tree of maximum degree at most Δ and let T be the tree of maximum degree at most $\Delta+1$. Let v be any leaf of T. We describe a spanning tree of $S \times T$ of maximum degree at most $\Delta + 1$. The desired spanning tree consists of edges in copies of T (all the edges with the same S–vertex coordinate) and the edges of S which are in the copy of S with the T–vertex coordinate equal to v (see the Figure 4). The maximum degree of this tree is clearly at most $\Delta+1$. □

Theorem 8 *The product of any at least* $\lfloor\frac{19}{24}\Delta\rfloor + \lceil\log_2\Delta\rceil + 3$ *connected graphs of maximum degree at most* Δ *is hamiltonian for* $\Delta > 32$*.*

Proof: We can suppose w.l.o.g. that one of the graphs in the product has maximum degree exactly Δ — call this graph H. Let G be any other of the graphs and let I be the product of the rest (at least $\lfloor\frac{19}{24}\Delta\rfloor + \lceil\log_2\Delta\rceil + 1$) graphs. The graph I can be covered by $C_{\geq\Delta}$ due to Corollary 3 — let F_C be the edges of the cycles in this covering. I contains a spanning tree of maximum degree at most $\Delta + 1$ due to Lemma 10; let us call this spanning tree T. Let F_T be the minimal set of edges of T such that the subgraph of I induced by the edges of $F_C \cup F_T$ is connected; clearly each vertex of I is adjacent to at most $\Delta + 1$ edges of F_T and the only cycles of the subgraph of I induced by the edges of $F_C \cup F_T$ are those of the covering of I. We can colour the edges of F_T by colours

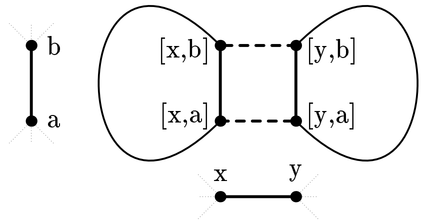

Figure 5: Concatenation procedure applied to two different cycles

$\{1, \ldots, \Delta + 1\}$ such that no vertex of I is adjacent to two edges of the same colour; this is due to the fact that the subgraph of I induced by the edges of F_T is a forest.

The product of H and each cycle of the covering of I is hamiltonian due to Theorem 1; this gives us a covering of $H \times I$ by $C_{\geq 2\Delta}$; let E_C be the set of edges of cycles in this covering. Let $v_1, \ldots, v_{d+1}$ be any distinct vertices of H; it contains at least $\Delta + 1$ vertices since the maximum degree of H is exactly Δ. Let E_T be the set of all the edges $[v_i, x][v_i, y]$ such that $xy \in F_T$ and it is coloured by colour i. Clearly, the subgraph of $H \times I$ induced by edges of $E_T \cup E_C$ is connected and its only cycles are those of the covering of $H \times I$. Each vertex of $H \times I$ is adjacent to at most one edge of E_T due to the choice of the edges of E_T — there is no vertex in I adjacent to two edges of F_T coloured with the same colour.

We are now ready to prove that $G \times H \times I$ is hamiltonian. We assume that G is a tree; let u be any of its leaves and let v be its (only) neighbour. First, consider a product of a cycle of length at least 2Δ and G. We cover this product by the copies of the cycle corresponding to each vertex except for u and v; the product of the cycle and the edge uv we cover by a comb graph as shown in Figure 6. We introduce a concatenation procedure: Let $[x, a][x, b]$ and $[y, a][y, b]$ be edges of two distinct cycles in a covering and let xy be an edge, then we can concatenate these two cycles by replacing $[x, a][x, b]$ and $[y, a][y, b]$ with $[x, a][y, a]$ and $[x, b][y, b]$ (see Figure 5). We can apply the concatenation procedure to all the copies of one cycle, since there are at least Δ edges with the same G-coordinate in each such a copy (see Figure 6); actually the same idea was used in [1] to prove Theorem 1.

We now concatenate all the cycles covering $G \times H \times I$ together. Let xy be an edge of E_T. The edges $[u, x][v, x]$ and $[u, y][v, y]$ are in different cycles covering $G \times H \times I$ and we can apply the concatenation procedure to them — see Figure 6. Each vertex of $H \times I$ is adjacent to at most one edge of E_T and thus we can apply the concatenation procedure to all the edges of E_T simultaneously. Since the subgraph of $H \times I$ induced by $E_C \cup E_T$ is connected and its only cycles are those of the covering of $H \times I$, we obtain the hamiltonian cycle of $G \times H \times I$.
□

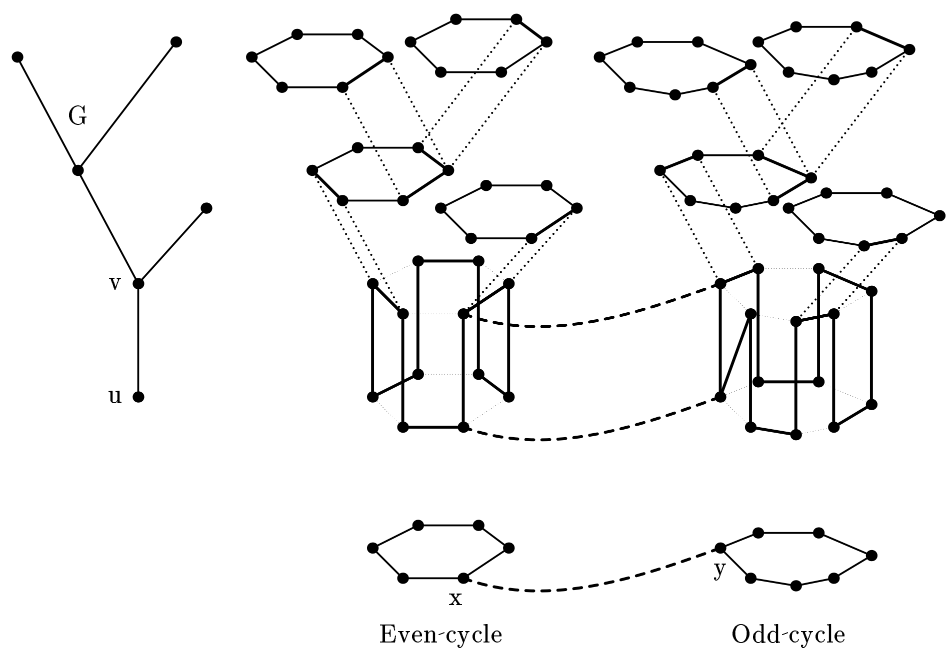

Figure 6: Covering the product of cycles and G by comb graphs and cycles and their concatenation

If we use Corollary 4 instead of Corollary 3 in the proof of Theorem 8, we obtain the following theorem:

Theorem 9 *For each* $c > \ln \frac{25}{12} + \frac{1}{60}$ *there exists* c' *such that the product of any at least* $\lfloor c\Delta \rfloor + \lceil \log_2 \Delta \rceil + c'$ *connected graphs of maximum degree at most* Δ *is hamiltonian for* $32 < \Delta$.

5 Conclusion

The question of precise determining of $h_{\max}(\Delta)$ remains open. Theorem 8 and Theorem 9 get better upper bounds on $h_{\max}(\Delta)$ for large values of Δ then Theorem 2 — their linear (in Δ) upper bounds for $h_{\max}(\Delta)$ grow slower but their constant parts are large. It seems to be an easier (but not less attractive) question to describe the linear behaviour of $h_{\max}(\Delta)$ for large Δ. Theorem 3 states that:

$$\liminf_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} \geq \ln 2 \approx 0.6931$$

On the other hand Theorem 9 states that:

$$\limsup_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \ln \frac{25}{12} + \frac{1}{60} \approx 0.7506$$

We conjecture that:

$$\lim_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} = \ln 2$$

The products of stars are in the just suggested question the worst case: Let $h^{\mathrm{S}}_{\max}(\Delta)$ be the minimal k such that all the graphs $S^k_{\leq \Delta}$ are hamiltonian, then it follows from the proof of Theorem 8 (note that the hamiltonian cycle of any graph of $S^{h^{\mathrm{S}}_{\max}(\Delta)}_{\leq \Delta}$ has the length at least $2^{\ln 2\Delta} \geq \Delta$, since $h^{\mathrm{S}}_{\max}(\Delta) \geq h(S_\Delta) \geq \Delta \ln 2$) that:

$$h_{\max}(\Delta) \leq h^{\mathrm{S}}_{\max}(\Delta) + 2$$

Thus it holds that:

$$\limsup_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \limsup_{\Delta \to \infty} \frac{h^{\mathrm{S}}_{\max}(\Delta)}{\Delta}$$

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