On Bermond, Germa and Heydemann's Conjecture*

Daniel Král'† Jana Maxová‡ Pavel Podbrdský§

Robert Šámal¶

Abstract

We prove that the strong product of graphs $G_1 \times \cdots \times G_n$ contains a hamiltonian cycle for $n \geq \Delta$ whenever all G_i are connected graphs of maximum degree at most Δ; in particular $G^{\Delta(G)}$ contains a hamiltonian cycle.

1 Introduction

Let the vertex set of the graph G be denoted by $V(G)$ and let its edge set be denoted by $E(G)$. The strong product of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$. Its two distinct vertices $[u_1, v_1]$ and $[u_2, v_2]$ are joined by an edge iff $u_1 = u_2 \vee u_1u_2 \in E(G)$ and $v_1 = v_2 \vee v_1v_2 \in E(H)$. A cycle containing all the vertices of the graph is called a hamiltonian cycle; each vertex is contained in such a cycle exactly once. We write G^k for the strong product of k copies of G and we call graphs containing a hamiltonian cycle hamiltonian graphs for the brevity.

Zaks asked in [3] whether there exists $k(G)$ for any connected graph G with at least two vertices such that $G^{k(G)}$ is hamiltonian. Bermond, Germa and Heydemann proved in [1] the existence of the number $k(G)$ and they proved that if G^k is hamiltonian, then also G^h is hamiltonian for all $k \leq h$ using

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†Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic, e-mail: `kral@kam.ms.mff.cuni.cz`, supported in part by NSF grant DMS-9900969 and in part by Institute for Theoretical Computer Science

‡Institute for Theoretical Computer Science (supported by Ministry of Education of Czech Republic as project LN00A056), Malostranské nám. 25, 118 00 Prague, Czech Republic, e-mail: `jana@kam.ms.mff.cuni.cz`

§Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic, e-mail: `podbrdsk@karlin.mff.cuni.cz`

¶Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic, e-mail: `samal@kam.ms.mff.cuni.cz`

some results of Rosenfeld and Barnette contained in [2]. They did not give any upper bound on $k(G)$ in terms of maximum degree of the graph G, but they conjectured that G^{Δ} should be hamiltonian for any connected graph G with at least two vertices and of maximum degree Δ (for $\Delta \geq 2$). We prove their conjecture in this paper; we prove even more, namely that the product of arbitrary Δ connected graphs of maximum degree Δ is hamiltonian.

We introduce notation used in the paper and we give basic properties of strong graph products in Section 2. We prove that graphs with bounded maximum degrees can be covered with small stars in Section 3. We prove the original conjecture for stars in Section 4 and we use the results of Section 4 to prove the conjecture for general graphs in Section 5.

2 Definitions and Basic Properties

All the graphs considered in this paper contain at least two vertices. Whenever we mention a product of graphs, we always mean the strong product of graphs; we are not going to use other definitions of the graph product in this paper. We write C_n for the cycle containing n vertices, P_n for the path containing n vertices and S_n for the star of order n, i.e. the star with n edges ($S_n = K_{1,n}$). We write $\leq k$ for the set of all the positive integer numbers less or equal to k; we use $\geq k$, $< k$ and $> k$ in a similar manner. Let I be a set of integer numbers, then S_I denotes the set of all the stars S_i such that $i \in I$; C_I and P_I are used in the similar manner. If A and B are sets of graphs, then we write $A \times B$ for $\{G \times H \mid G \in A \wedge H \in B\}$ and A^2 for $A \times A$, A^3 for $A \times A \times A$, etc.

We assign the vertices of P_n and S_n numbers and a star ($*$); this makes proofs dealing with paths or stars more clear. We understand the set $V(P_n)$ as $\{0, \ldots, n-1\}$; its two vertices are connected iff their difference is exactly one. We understand the set $V(S_k)$ as $\{*, 0, \ldots, k-1\}$; the edges of S_k are only between $*$ and the number vertices. Let S be a product of n stars; we write $V_i(S)$ for the set of vertices of S whose exactly i coordinates are equal to $*$. We mean by $x \bmod y$ the number between 0 and $y-1$ which is congruent to x modulo y, e.g. $9 \bmod 7$ is 2. Let $a = [a_0, \ldots, a_{n-1}]$ be any vertex of S. We use some useful notation in the paper: We write $\sum a$ for $\left(\sum_{i=0}^{n-1} a_i\right) \bmod n$; we consider $*$ to be counted as zero in this sum. We write $a[k \to s]$ for the vertex b obtained from a by substituting s for its k-th entry, i.e. $b_{k \bmod n} = s$ (we take $s \bmod k$ instead of s if necessary) and $b_i = a_i$ for $i \neq k \bmod n$.

We address the question of existence of a hamiltonian cycle in products of graphs. The key theorem proved in [1] using ideas from [2] is the following one:

Theorem 1 *Let G be a connected graph of maximum degree at most Δ. Then any graph of $C_{\geq \Delta} \times G$ is hamiltonian.*

The immediate corollary of this theorem is that if G^k is hamiltonian then G^h is hamiltonian for all $h \geq k$. We write $h(G)$ for the smallest value of h such that G^h is hamiltonian; the existence of this value follows from the results of [1].

There are also some lower bounds on $h(G)$ in term of the maximum degree of G due to Zaks (see [3]).

Theorem 2 *If S_n^k is hamiltonian, then the following inequality holds:*

$$k \geq \frac{\ln 2}{\ln(1+\frac{1}{n})} \approx n \ln 2$$

A covering of the graph G by the set of graphs I is a set J of subgraphs of G such that each vertex of G is contained exactly in one of the graphs of J and each graph of J is isomorphic to some of the graphs of I. Note that we cover the vertices of the graph, we do not cover its edges. The following theorem trivially holds (it is enough to consider all the subgraphs of $G \times H$ equal to the product of a subgraph covering G and each single vertex of H):

Theorem 3 *Let G and H be any graphs and let I be any set of graphs. If G can be covered by I, then $G \times H$ can be covered by I.*

We are going to deal with coverings of undirected graphs consisting of directed cycles — we simply take undirected cycles and consistently orient their edges, i.e. we orient the edges such that the in-degrees and out-degrees of all the vertices of the cycles are one; we do not demand that the orientation has to satisfy any additional conditions. We write u^+ for the successor and u^- for the predecessor of the vertex u in the cycle.

If G is $G_0 \times \cdots \times G_{m-1}$, we understand the vertex $v \in V(G)$ as $[v_0, \ldots, v_{m-1}]$ where $v_i \in G_i$ for $0 \leq i \leq m-1$. We say that v is k–constant iff $v_k = v_k^+$. If all G_i are stars, we say that v is simple, if $v_i \in \{*, 0\}$ for all i. We write $*v$ for the maximal i such that $v_i = *$. We write $*^m$ for a vertex of G whose all coordinates are stars. If $r = [r_0, \ldots, r_{k-1}, r_{k+1}, \ldots, r_{m-1}] \in G_0 \times \cdots G_{k-1} \times G_{k+1} \times \cdots \times G_{m-1}$ and $w \in G_k$, then we write $r[w]$ for $[r_0, \ldots, r_{k-1}, w, r_{k+1}, \ldots, r_{m-1}] \in G_0 \times \cdots \times G_{m-1}$; the value of k will be clear from the context.

3 Coverings Graphs by Stars

We prove a fundamental lemma on coverings graphs by stars in this section.

Lemma 1 *Each connected graph G of maximum degree at most Δ can be covered by $S_{\leq\Delta}$.*

Proof: We assume w.l.o.g. that G is a tree. The proof proceeds by induction on the number of vertices of G. The statement is trivial for $|V(G)| = 2$. Let us consider a leaf v of G for $|V(G)| \geq 3$. There exists a covering of $G \setminus \{v\}$ by

$S_{\leq\Delta}$ due to the induction hypothesis; let us consider this covering. Let w be the only neighbour of v in G (remember that G is a tree and v is its leaf). We distinguish several cases:

- **The vertex w is covered by S_1.**
  We extend the star covering w by v to S_2.
- **The vertex w is covered by S_k ($k \geq 2$) and w is its center.**
  We extend the star covering w by v to S_{k+1} (note that $k+1 \leq \Delta$).
- **The vertex w is covered by S_k ($k \geq 2$) and w is its peripheral vertex.**
  We remove w from that star (obtaining S_{k-1}) and we create a new star S_1 consisting only of v and w.

It is easy to check that the obtained covering of G is a covering of G by $S_{\leq\Delta}$. □

4 Strong Products of Stars

We first prove the conjecture for stars. Stars seem to be the worst case and we use later Theorem 4 for stars to prove the conjecture for general graphs.

Theorem 4 *Any graph of $S^{\Delta}_{\leq\Delta}$ is hamiltonian for $\Delta \geq 2$.*

Proof: The proof proceeds by induction on Δ. The statement is clear for $\Delta = 2$, since the product of two paths is certainly hamiltonian. Let $S = S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ be any graph of $S^{\Delta}_{\leq\Delta}$ and Δ be at least 3. We assume w.l.o.g. that $n_0 \geq n_1 \geq n_2, \ldots, n_{\Delta-1}$, i.e. we assume that n_0 and n_1 are the two largest integers between $n_0, \ldots, n_{\Delta-1}$. If $n_1 < \Delta$, then $S_{n_1} \times \cdots \times S_{n_{\Delta-1}}$ is hamiltonian due to the induction hypothesis and S is hamiltonian due to Theorem 1 used for S_{n_0} and the hamiltonian cycle of $S_{n_1} \times \cdots \times S_{n_{\Delta-1}}$. Let n_0 and n_1 be equal to Δ in the rest of the proof.

The construction of the hamiltonian cycle consists of several steps. First, we find a set $\mathcal{Z}$ of directed cycles covering all the vertices of $V_0(S)$. Next, we construct a long cycle L in S and we use it to join all the small cycles of $\mathcal{Z}$ covering $V_0(S)$. Finally, we insert the remaining vertices of S to the obtained cycle.

The edge xy is a directed edge of a cycle of $\mathcal{Z}$ iff it satisfies one of the following rules:

- $x \in V_0(S)$, $y \in V_1(S)$ and $y = x[\sum x+1 \to *]$ ($\Leftrightarrow x = y[*y \to *y - \sum y - 1]$)
- $x \in V_1(S)$, $y \in V_0(S)$ and $x = y[\sum y \to *]$ ($\Leftrightarrow y = x[*x \to *x - \sum x]$)

- $x \in V_1(S)$, $y \in V_2(S)$, $y = x[0 \to *]$ and x has a predecessor but no successor defined by the first two conditions ($\Leftrightarrow n_{*x} < \Delta$ and $(*x - \sum x) \bmod \Delta = n_{*x}$)

- $x \in V_2(S)$, $y \in V_1(S)$, $x = y[0 \to *]$ and y has an successor but no predecessor defined by the first two conditions ($\Leftrightarrow n_{*y} < \Delta$ and $*y - \sum y = 0$)

Before we prove that these rules define directed cycles, we give (as an example) a part of a cycle defined by these rules in $S_3 \times S_3 \times S_2$:

$$[0,0,1] \to [0,0,*] \to [*,0,*] \to [2,0,*] \to [2,0,0] \to [*,0,0] \to [0,0,0] \to \dots$$

Step 1: Every vertex of S has either both in-degree and out-degree one or both zero, i.e. $\mathcal{Z}$ consists of directed cycles covering $V_0(S)$.
It is clear that the in-degrees and out-degrees for vertices of $V_0(S)$, $V_1(S)$ and $V_{\geq 3}(S)$ are equal and at most 1 and that all the vertices of $V_0(S)$ are covered by $\mathcal{Z}$. Let $x \in V_2(S)$. If $x_0 \neq *$ or $x_{*x} = \Delta$ then x has both in-degree and out-degree zero. Otherwise it must hold that $x^- = x[0 \to *x - n_{*x} - \sum x]$ and $x^+ = x[0 \to *x - \sum x]$, thus the both in-degree and out-degree of x is exactly one.

Step 2: For each directed cycle $C \in \mathcal{Z}$ there is a vertex $u \in C \cap V_0(S)$ such that $u_1 = 0$ and $\sum u = 0$. We choose any such u to be a representant of C and we write $r(C)$ for it.
Let C be fixed for the proof of this statement. Let a be a vertex of $C \cap V_0(S)$; we call a vertex b the first non–star successor of a if $b \in C \cap V_0(S)$ and there is no vertex of $C \cap V_0(S)$ between a and b in the cycle C; note that either $b = a^{++}$ or $b = a^{++++}$. We call a vertex c the second non–star successor of a, if it is the first non–star successor of the first non–star successor of a; we define in this fashion the i–th non–star successor of a. Let a be any vertex of $C \cap V_0(S)$ and b its first non–star successor; the following three claims are true for a and b:

- $\sum b = (\sum a + 1) \bmod \Delta$
- $b_i = a_i$ for $i \neq 0$ and $i \neq (\sum a + 1) \bmod \Delta$
- $b_i = (a_i + 1) \bmod n_i$ for $i = (\sum a + 1) \bmod \Delta$

If $b = a^{++}$, then $b = a[\sum a + 1 \to a_{(\sum a+1) \bmod \Delta} + 1]$ and the claims are true. If $b = a^{++++}$, then $b = a[\sum a + 1 \to 0][0 \to a_0 + a_{(\sum a+1) \bmod \Delta} + 1]$ and the claims are true, too.

Let a be any vertex of $C \cap V_0(S)$; let b be its $(\Delta - \sum a)$-th non–star successor; $\sum b$ is zero due to the first claim. Let $b^0 = b$ and b^i be the $(i\Delta)$-th non–star successor of b for $1 \leq i < \Delta$. The sum of entries of any b^i is zero due to the first claim. Due to the second and the third claim, it holds that $b_1^{i+1} =$

$(b_1^i + 1) \bmod \Delta$. Thus exactly one of b^i $(0 \leq i < \Delta)$ satisfies the above conditions and can be choosen as a representant of the cycle C.

Step 3: There is a directed cycle L in S which contains all the vertices of the set $\{u \in V_2(S) | u_0 = u_1 = *\}$ and no vertices of cycles of $\mathcal{Z}$. Let W be the set $\{u \in V_2(S) | u_0 = u_1 = *\}$ and let $u^0, u^1, \dots, u^{m-1}$ be a sequence of its vertices sorted lexicographically; consider u^m to be u^0. We first deal separately with the case $|W| = 1$, i.e. $n_2 = \dots = n_{\Delta-1} = 1$. We L to be a cycle consisting of two vertices: $[*, *, 0, \dots, 0], [0, *, \dots, *], [*, *, 0, \dots, 0]$; this is not actually a cycle, but this does not affect the proof. If $|W| > 1$, fix i an integer between 0 and $m-1$ and let k be the first coordinate in which the vertex u^i and the vertex u^{i+1} differ. Let $\tilde{u}^i = [-\sum u^{i+1}, *, u_2^i, \dots, u_{k-1}^i, *, \dots, *] = [-\sum u^{i+1}, *, u_2^{i+1}, \dots, u_{k-1}^{i+1}, *, \dots, *]$ (note that all $\tilde{u}^i$ are mutually different). No vertex u^i or $\tilde{u}^i$ is contained in any cycle of $\mathcal{Z}$ and there are edges $u^i \tilde{u}^i$ and $\tilde{u}^i u^{i+1}$ in S. Thus we can set L to be $u^0, \tilde{u}^0, u^1, \tilde{u}^1, \dots, u^{m-1}, \tilde{u}^{m-1}, u^m = u^0$.

Step 4: There is a directed cycle L' in S which contains all the vertices of the cycles of $\mathcal{Z}$ and all the vertices of the cycle L. For each cycle C of $\mathcal{Z}$ let $r(C)$ be its representant and let $l(C)$ be $r(C)[0 \to *][1 \to *]$. We write in this step r for $r(C)$ and l for $l(C)$. We replace edges rr^+ and l^-l by edges l^-r^+ and rl. We need to check that l^-r^+ and rl are really edges of S and that l is a different vertex of W for different cycles of $\mathcal{Z}$:

- The entries of l and r are equal except for the first two entries which are in case of l equal to $*$; thus there is an edge rl in S.

- All the coordinates of l^- are either equal to the coresponding coordinates of r or they are equal to $*$ with a possible exception for the coordinate indexed by zero. $\sum r = 0$ due to the selection of the representant of C, $\sum l = (-r_0 - r_1 + \sum r) \bmod \Delta = (-r_0) \bmod \Delta$ due to the selection of l as $r[0 \to *][1 \to *]$. Since $l_0^- = -\sum l$ due to the construction of L, it must hold that $l_0^- = r_0$ and there is the edge l^-r in S. Since $\sum r = 0$, it holds that $r^+ = r[1 \to *]$ and thus (since there is the edge l^-r in S) there is the edge l^-r^+.

- Suppose that $l = l(C')$ for two different cycles C and C'. It follows that $r_i = r(C')_i$ for $i \geq 2$ from the equality $l = l(C')$, $r_1 = r(C')_1 = 0$ due to the selection of the representants of the cycles. But since $\sum r = \sum r(C') = 0$ it must be also $r_0 = r(C')_0$ and thus $r = r(C')$. Therefore the cycles C and C' are actually the same cycle. Thus the vertices l are different for different cycles $C \in \mathcal{Z}$.

Step 5: S is hamiltonian. The cycle L' contains all the vertices of $V_0(S)$. Let $u \notin L'$, we can insert this vertex to L' as follows: Let $\tilde{u}$ be any vertex of L' obtained from u by substituting some of the coordinates of u equal to $*$ by arbitrary numbers; note that if we substitute all the coordinates equal to $*$ by numbers, we get a vertex of $V_0(S)$, thus a vertex of L', i.e. it is always possible to get a vertex $\tilde{u} \in L'$. We can replace the edge $\tilde{u}\tilde{u}^+$ of L' with the edges $\tilde{u}u$ and $u\tilde{u}^+$ (it is easy to check that these are really edges of S). We can also insert u to L' by replacing the edge $\tilde{u}^-\tilde{u}$ with the edges $\tilde{u}^-u$ and $u\tilde{u}$. In this manner we can insert to L' all the vertices not contained in L'. The exact procedure of insertion of these vertices to the cycle L' is up to us and we will adjust this part of the proof of Theorem 4 in the proofs of Lemma 2 and Lemma 3. □

We describe the properties of the hamiltonian cycle constructed in the proof of Theorem 4 in the following three claims. Since we are going to deal with several modifications of the last step of the construction of the hamiltonian cycle later in this section, we get a precise description of all the vertices inserted to the cycle in the last step in the first of these claims. We also describe some basic properties of the cycle L' used in the proof. The proofs of the claims are straightforward and thus omitted.

Claim 1 *Any vertex v contained in the cycle L' in the proof of Theorem 4 satisfies one of the following conditions:*

- *$v \in V_0(S)$, i.e. v contains no coordinate equal to $*$*
- *$v \in V_2(S)$ and $v_0 = v_1 = *$*
- *$v \in V_2(S)$, $v_0 = *$ and $n_{*v} < \Delta$*
- *$v \in V_1(S)$, i.e. v contains one coordinate equal to $*$*
- *There exists $i_0 \geq 2$, such that $v_i = *$ iff $i \in \{1, i_0, i_0+1, \ldots, \Delta-2, \Delta-1\}$.*

Note that all the vertices satisfying any of the first three conditions are contained in L', i.e. the first three conditions are sufficient for vertices to be included in L'.

Claim 2 *If v is a vertex of L' containing at least one star and $v_1 \neq *$, then v^- and v are 1–constant in L'. (Note that such v has to be already contained in some of the cycles of $\mathcal{Z}$.)*

Claim 3 *If v is a vertex of L' containing at least two stars and $v_1 = *$, then v is 1–constant in L'.*

The last theorem states that the product of Δ stars of order at most Δ contains a hamiltonian cycle; the following two lemmas describe properties of these hamiltonian cycles in more depth. We will prove these lemmas together since their proofs are almost identical.

Lemma 2 *Let $\Delta \geq 3$, let $k \neq l$ be integers between 0 and $\Delta - 1$ and let $n_0, \ldots, n_{k-1}, n_{k+1}, \ldots, n_{\Delta-1}$ be integers between 1 and Δ. Let s be either $*^\Delta$*

or $*^{\Delta}[l \to 0]$. *Then there exists a subset* R *the vertices of* $S_{n_0} \times \cdots \times S_{n_{k-1}} \times S_{n_{k+1}} \times \cdots \times S_{n_{\Delta-1}}$ *such that* $\Delta - 1 \leq |R|$ *and for any choice of* n_k *between* 1 *and* Δ *there is a hamiltonian cycle* C *in* $S = S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ *satisfying the following conditions:*

- *The vertex* $r[t]$ *is* k*–constant for all* $r \in R$ *and* $t \in S_{n_k}$ *with* $\deg_{S_{n_k}} t < \Delta$.
- s *is* l*–constant,* s^+ *(the successor of* s *in* C*) is simple and* s *is different from* $r[*]$ *for all* $r \in R$.

Lemma 3 *Let* $\Delta \geq 3$*, let* $0 \leq k \leq \Delta - 1$ *and let* $n_0, \ldots, n_{k-1}, n_{k+1}, \ldots n_{\Delta-1}$ *be integers between* 1 *and* Δ. *Then there exists a subset* R *of the vertices of* $S_{n_0} \times \ldots \times S_{n_{k-1}} \times S_{n_{k+1}} \times \ldots \times S_{n_{\Delta-1}}$ *such that* $\Delta - 1 \leq |R|$ *and for any choice of* n_k *between* 1 *and* Δ *there is a hamiltonian cycle* C *in* $S = S_{n_0} \times \ldots \times S_{n_{\Delta-1}}$ *satisfying the following conditions:*

- *The vertex* $r[t]$ *is* k*–constant for all* $r \in R$ *and* $t \in S_{n_k}$ *with* $\deg_{S_{n_k}} t < \Delta$.
- R *contains at least two simple vertices.*

Proof: We prove the previous two lemmas together, since their proofs are very similar. The proof proceeds by induction on Δ. We prove even more, namely that there is always $r_0 \in R$ such that r_0 is simple, at least one of its coordinates is equal to $*$ and $r_0[t]$ is k–constant for all $t \in S_{n_k}$. We assume w.l.o.g. that $k = \Delta - 1$ and $n_0 \geq n_1 \geq \ldots \geq n_{\Delta-2}$. We distinguish several cases in the proof:

- **$n_0 < \Delta$, the proof of Lemma 2 and Lemma 3**
  We first consider the case $\Delta = 3$ and $n_0 = n_1 = 1$. We can assume w.l.o.g. that $l = 0$ in Lemma 2, thus s is either $[*, *, *]$ or $[0, *, *]$. We set R to $\{[*, 0], [0, 0]\}$ and r_0 to $[*, 0]$. The desired hamiltonian cycles in $S_1 \times S_1 \times S_1$, $S_1 \times S_1 \times S_2$ and $S_1 \times S_1 \times S_3$ for different choices of s are shown in Figure 1.

  There is either $\Delta \geq 4$ or $n_0 \geq 2$ in the remaining case. There is a hamiltonian cycle C' in $S_{n_0} \times \cdots \times S_{n_{\Delta-2}}$ due to Theorem 4. The length of this cycle is even at least 2Δ (since $\Delta \geq 4$ or $n_0 \geq 2$). Let $v^0, \ldots, v^m$ be the vertices of C' ($v^{i+1} = \left(v^i\right)^+$; note that $m \geq 2\Delta - 1$ and note that at least four of these vertices are simple) and let w.l.o.g. $v^0[*] = s$ (in case of Lemma 2). Let R be any set of vertices of C' of size $\Delta - 1$ which contains at least two simple vertices and which does not contain either v^0 or v^1 and let $r_0 \neq [0, \ldots, 0]$ be any simple vertex of R. We use a construction which is very similar to that used in the proof of Theorem 1 — see Figure 2. We use edges from and to vertices v^0 and v^1 to join the cycles corresponding to the vertices $*$ and to vertices 0 of $S_{n_{\Delta-1}}$ and we use edges from the vertices of C' different from v^0 and different from the vertices of R to join the cycles corresponding to the vertex $*$ and vertices $1, \ldots, n_{\Delta-1} - 1$

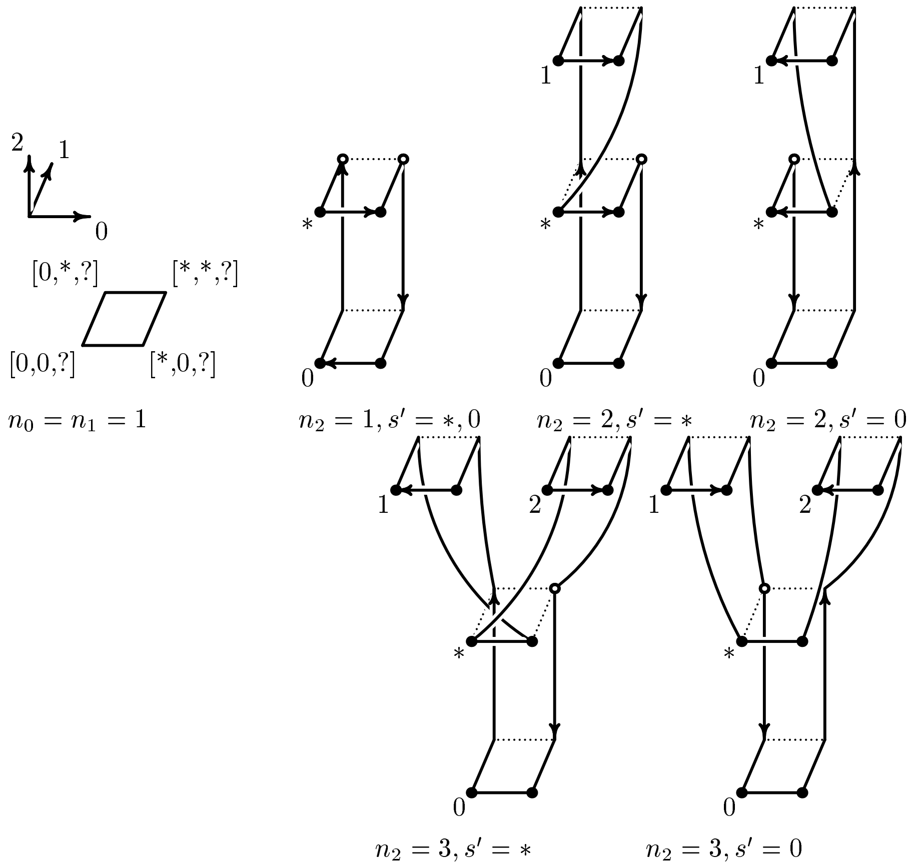


Figure 1: The desired hamiltonian cycles in $S_1 \times S_1 \times S_{n_2}$ for $n_2 = 1, 2, 3$ and for $s = [s', *, *], s' \in \{*, 0\}$. The set R is $\{[*, 0], [0, 0]\}$. The vertex s is drawn as an empty circle, other important vertices are drawn as full circles.

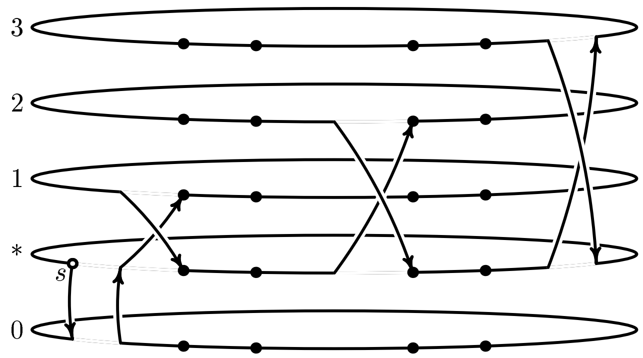


Figure 2: Joining different cycles in the product of a cycle C and a star $S_{n_{\Delta-1}}$ for $n_{\Delta-1} = 4$ in the proof of the case $n_0 < \Delta$; the vertex s and the vertices $r[t]$ for $r \in R$ and $t \in \{*, 0, \dots, 3\}$ are drawn as empty circles

of $S_{n_{\Delta-1}}$. It is clear that $s^+ = v^0[*]^+$ is simple, s is l–constant (for any $0 \le l \le \Delta - 2$) and it is also clear that all the vertices $r[t]$ for $r \in R$ and $t \in \{*, 0, \dots, n_{\Delta-1}\}$ are $(\Delta - 1)$–constant. Thus the just constructed hamiltonian cycle C satisfies the conditions of Lemma 2 and Lemma 3.

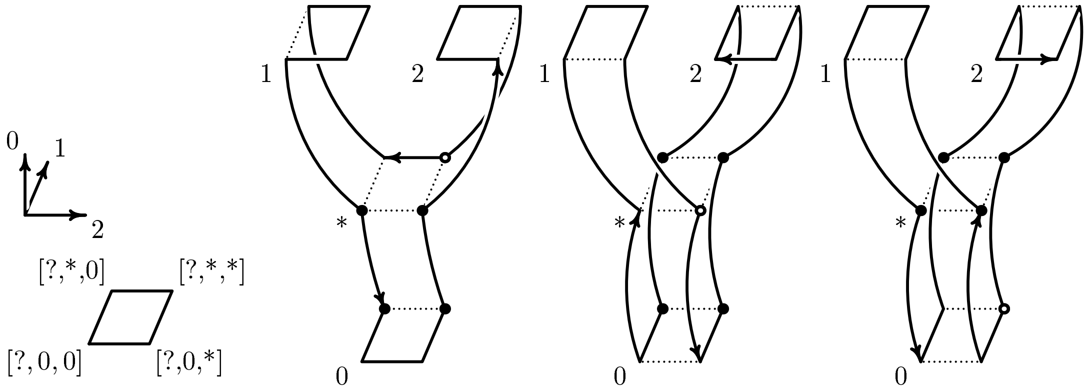
- $n_0 = \Delta > n_1$**,** $\Delta = 3$**, the proof of Lemma 2 and Lemma 3**
  We set R to a two–element subset of $\{[0,*], [*,0], [*,*]\}$ such that $r[*] \neq s$ for all $r \in R$ and we set r_0 to any member of R. If $n_2 \in \{1, 2\}$, then the hamiltonian cycles in $S_{n_0} \times S_{n_1} \times S_{n_2}$ are shown in Figure 3, in Figure 4, in Figure 5 and in Figure 6. The vertices of R are represented in the figures by full circles and the vertex s is represented by an empty circle.

  If $n_2 = 3 = \Delta$, we use a construction of a hamiltonian cycle based on the proof of Theorem 4. We use the construction of the proof for $n'_0 = n_0 = \Delta$, $n'_1 = n_2 = \Delta$ and $n'_2 = n_1 < \Delta$, i.e. for $S_3 \times S_3 \times S_1$ and $S_3 \times S_3 \times S_2$. We set k equal to 1 in the rest of the proof of this case to be consistent with the proof of Theorem 4. The cycle L' in the product $S_3 \times S_3 \times S_1$ looks as follows:

  $$[0,0,0], [*,*,0], [0,*,*], [0,*,0], [0,1,0], [0,1,*], [*,1,*], [1,1,*], [1,1,0],$$
  $$[*,1,0], [2,1,0], [2,*,0], [2,2,0], [2,2,*], [*,2,*], [0,2,*], [0,2,0], [*,2,0],$$
  $$[1,2,0], [1,*,0], [1,0,0], [1,0,*], [*,0,*], [2,0,*], [2,0,0], [*,0,0](, [0,0,0])$$

  The vertices $[*,*,*]$ and $[0,0,*]$ remain for the last (insertion) step of the construction (consult Claim 1 in case of $[*,*,*]$). The cycle L' in the product $S_3 \times S_3 \times S_2$ looks as follows (if we have choosen $[0,0,0]$ to be the representant):

  $$[0,0,0], [*,*,0], [1,*,*], [*,*,1], [0,*,*], [0,*,0], [0,1,0], [0,1,*],$$



$n_1 = n_2 = 1$ $\quad$ $s_l = *, l = 0/1$ $\quad$ $s_l = 0, l = 1$ $\quad$ $s_l = 0, l = 0$

Figure 3: The hamiltonian cycles in $S_3 \times S_1 \times S_1$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.

$$[0,1,1],[*,1,1],[1,1,1],[1,*,1],[1,2,1],[1,2,*],[*,2,*],[0,2,*],$$
$$[0,2,0],[*,2,0],[1,2,0],[1,*,0],[1,0,0],[1,0,*],[1,0,1],[*,0,1],$$
$$[2,0,1],[2,*,1],[2,1,1],[2,1,*],[*,1,*],[1,1,*],[1,1,0],[*,1,0],$$
$$[2,1,0],[2,*,0],[2,2,0],[2,2,*],[2,2,1],[*,2,1],[0,2,1],[0,*,1],$$
$$[0,0,1],[0,0,*],[*,0,*],[2,0,*],[2,0,0],[*,0,0](,[0,0,0])$$

The vertex $[*,*,*]$ remains to to the last (insertion) step of the construction (consult Claim 1). We reverse the orientation of the cycle L' in some of the cases. Check that $r[t]$ for $r \in \{[0,*],[*,0],[*,*]\}$ and $t \neq *$ are k–constant both before and after the reversion of the orientation of L'; this requires carefull inserton of $[0,0,*]$ in case of $S_3 \times S_3 \times S_1$. Check also that $[0,*][*] = [0,*,*]$ and $[*,0][*] = [*,*,0]$ are k–constant before the reversion of the cycle and that $[0,*][*] = [0,*,*]$ is k–constant after the reversion. We distinguish several cases:

– $s = [*,*,*]$, thus $R = \{[0,*],[*,0]\}$ and we set $r_0 = [0,*]$
  We considered the reversed cycle in this case. We insert in the last step the vertex $[*,*,*]$ just after the vertex $[*,0,*]$, i.e. it holds that $[*,*,*]^+ = [*,0,*]$ after the reversion of the cycle. Both s and s^+ are simple and s is l–constant (l is either 0 or 2, since it has to be different from k).

– $s = [0,*,*]$ (i.e. $l = 0$), thus $R = \{[*,*],[*,0]\}$ and we set $r_0 = [*,0]$
  We considered the original cycle in this case. Both s and $s^+ = [0,*,0]$ are simple and s is l–constant.

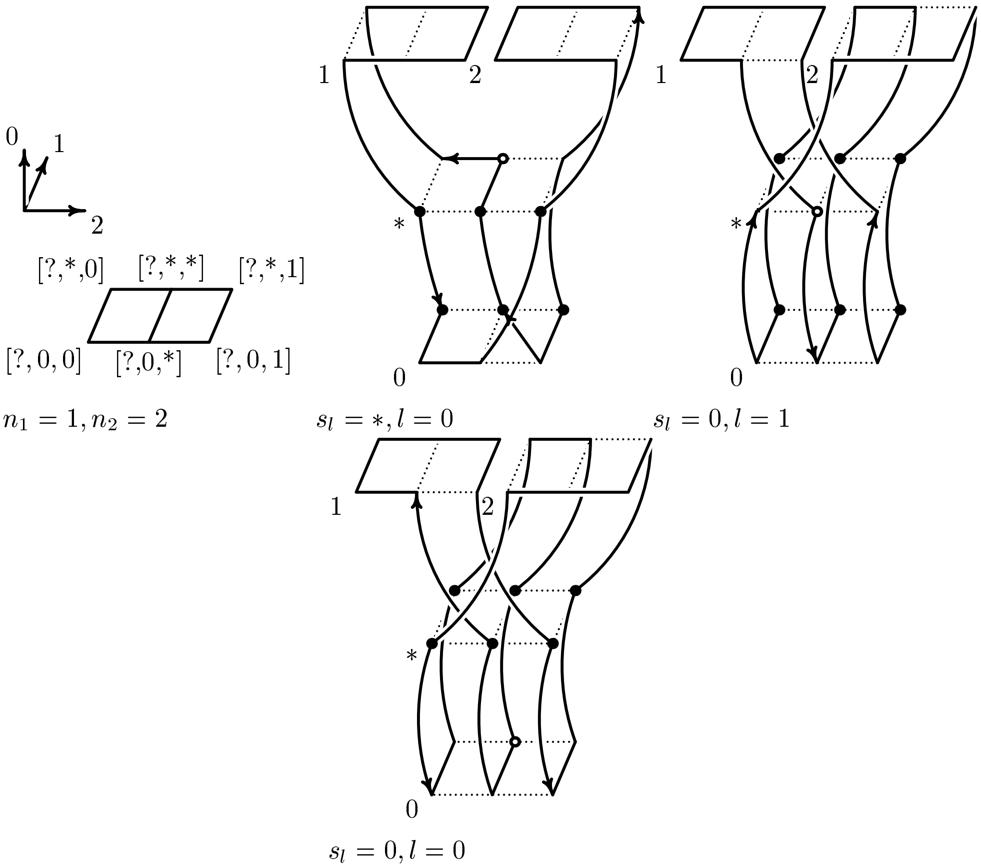
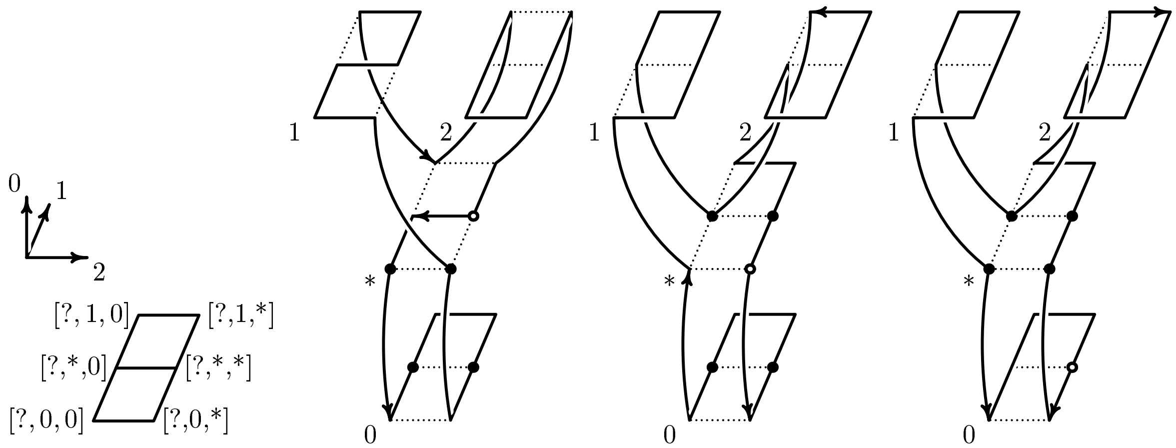


Figure 4: The hamiltonian cycles in $S_3 \times S_1 \times S_2$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.



$n_1 = 2, n_2 = 1$ $\quad$ $s_l = *, l = 0/1$ $\quad$ $s_l = 0, l = 1$ $\quad$ $s_l = 0, l = 0$

Figure 5: The hamiltonian cycles in $S_3 \times S_2 \times S_1$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.

- $s = [*, *, 0]$ (i.e. $l = 2$), thus $R = \{[*, *], [0, *]\}$ and we set $r_0 = [0, *]$ We consider the reversed cycle in this case. Both $s = [*, *, 0]$ and $s^+ = [0, 0, 0]$ are simple and s is l–constant. We insert $[*, *, *]$ just after $[*, *, 0]$, i.e. $[*, *, *]^+ = [*, *, 0]$ after the reversion of the cycle.

The detailed check that the constructed cycles satisfy the conditions of Lemma 2 and Lemma 3 is left to the reader, since it is very straightforward.

- **$n_0 = \Delta > n_1$, $\Delta \geq 4$, the proof of Lemma 2 and Lemma 3**
We are going to use the induction hypothesis for the product of $\Delta - 1$ stars: we set $n'_i = n_{i+1}$ for $0 \leq i \leq \Delta - 3$, $k' = \Delta - 2$ and in case of Lemma 2 we also set $s' = [s_1, \ldots, s_{\Delta-1}]$, $l' = l - 1$ for $l \geq 1$ and $l' = 0$ for $l = 0$. There exists a set $R' \subseteq S_{n'_0} \times \ldots \times S_{n'_{\Delta-3}} = S_{n_1} \times \ldots \times S_{n_{\Delta-2}}$ and a vertex $r'_0 \in R'$ satisfying the conditions of the lemmas for $n'_0, \ldots, n'_{\Delta-3}, k', s', l'$. We set R to $\{z \times r'_0 | z \in \{*, 0, \ldots, \Delta - 1\} = V(S_{n_0}) = V(S_\Delta\})$ and r_0 to be either $* \times r'_0$ or $0 \times r'_0$ in such manner that $r_0[t] \neq s$ for any choice of t. Note that there cannot be a vertex $r \in R$ such that $r[*] = s$, since this would imply that $r'_0[*] = s'$. We are going to show the existence of suitable hamiltonian cycles in $S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ for different values of $n_{\Delta-1}$. We distinguish two cases: $n_{\Delta-1} < \Delta$ and $n_{\Delta-1} = \Delta$.

If $n_{\Delta-1} < \Delta$, let C' be the hamiltonian cycle from the induction used for $n'_{\Delta-2} = n_{\Delta-1}$. The size of the cycle C' is at least $\Delta\ n_{\Delta-1} \geq 2\Delta$; let $v^1, \ldots, v^\Delta$ be any vertices of the cycle C' different from s'. We are going to use a construction which is very similar to that used in the proof of

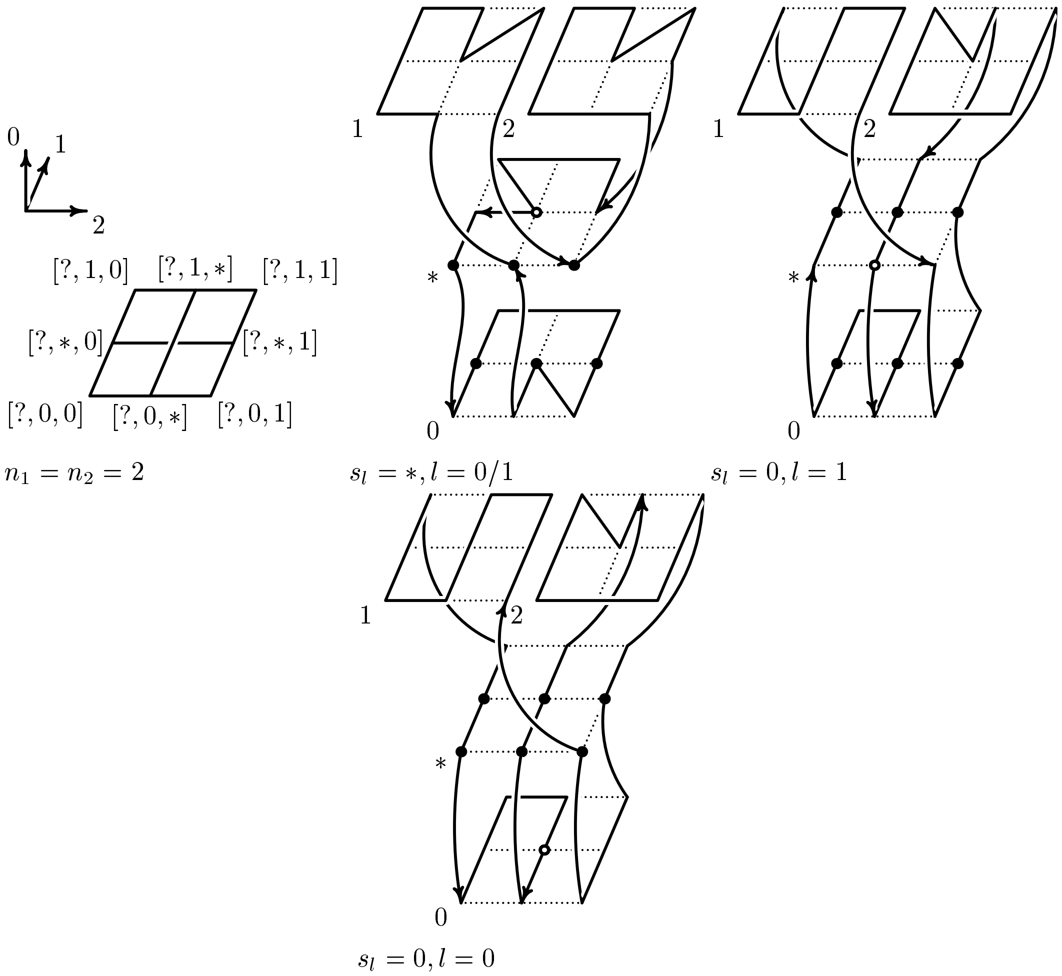


Figure 6: The hamiltonian cycles in $S_3 \times S_2 \times S_2$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.

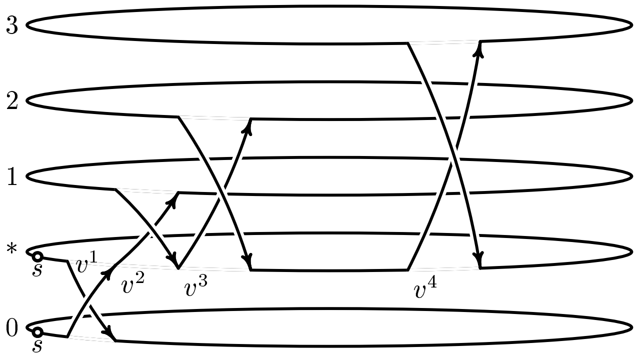


Figure 7: Joining different cycles in the product of a star S_{n_0} and a cycle C' for $n_0 = 4$ in the proof of the case $n_0 = \Delta > n_1$; the vertex s is drawn as an empty circle

Theorem 1 — see Figure 7. We use edges from the vertices $*\times v^1, \dots, *\times v^\Delta$ to join the copy of the cycle C' corresponding to the vertex $*$ of S_{n_0} to other copies of the cycle C'. We check that the just obtained hamiltonian cycle in $S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ satisfies all the conditions. Let t be any vertex of $S_{n_{\Delta-1}}$ and let r be any vertex of R; $r[t]$ is $(\Delta-1)$–constant, since r'_0 is constant in the corresponding entry in C' and the only entries of $r[t]$ and $r[t]^+$ possibly affected by joining the cycles are their first entries (i.e. $r[t]_0$ and $r[t]_0^+$). The vertex r_0 is simple and contains a star among its entries because the same holds for r'_0. The vertex s is l–constant: in case $l = 0$ it is 0–constant due to the joining procedure (s is different from all $* \times v^1, \dots, * \times v^\Delta$) and in case $l > 0$ it is l–constant since s' is constant in the corresponding entry. The vertex s^+ is simple, since it is equal to $s_0 \times s'^+$ and s'^+ is simple due to the induction.

If $n_{\Delta-1} = \Delta$, we cannot use the induction hypothesis, since $n_{\Delta-1} > \Delta-1$. We use the construction of the proof of Theorem 4 for the star product $S_{n_0} \times S_{n_{\Delta-1}} \times S_{n_1} \times \cdots \times S_{n_{\Delta-2}}$, thus $k = 1$ in the following discussion. We slightly modify the construction used in the proof. We choose the vertex $[0, \dots, 0]$ to be the representant of the cycle of $\mathcal{Z}$ in which it is contained; since the choice of representants was up to us (under some conditions which are clearly satisfied for the vertex $[0, \dots, 0]$), this can be done. We insert the vertex $*^\Delta = [*, \dots, *]$ in the last (insertion) step of the proof of Theorem 4 just before the vertex $[*, 0, *, \dots, *]$. If $r_0[*]$ is not contained in L', we insert $r_0[*]$ just before any vertex v such that $v_1 = *$. If $s \neq *^\Delta$ (i.e. $s = *^\Delta[l \to 0]$), and if $l > 0$ we insert the vertex $*^\Delta[l+1 \to 0]$ (note that the entries are shifted by one because $S_{n_{\Delta-1}}$ is the second star in the product) just before the vertex $[0, *, 0, \dots, 0]$. All the vertices to insert remain to the last step due to Claim 1. We check that this cycle has the desired properties in the next paragraph.

We need to check that all the vertices $r[t]$ for $r \in R$, $t \in \{0, \ldots, \Delta\}$, and the vertex $r_0[*]$ are k–constant. We adjust the insertion step: The vertex v not contained in L' is inserted in L' before the vertex w such that $v_1 = w_1$. All the vertices v containing $*$ such that $v_1 \neq *$ in the cycle L' constructed in the proof of Theorem 4 are 1–constant due to Claim 2; this property is kept through the insertion step (due to our adjustment). But $r[t]_1 = t \neq *$ (since r'_0 contains a star, r contains a star too) and thus all the vertices $r[t]$ contained in L' for all $r \in R$ and $t \in \{0, \ldots, \Delta - 1\}$ are k–constant ($k = 1$); the remaining vertices $r[t]$ (i.e. those not contained in L') are k–constant due to the adjusted insertion of them to the hamiltonian cycle in the last construction step. If $r_0[*]$ has not been contained in L', then it is k–constant due to the manner of its insertion. If $r_0[*]$ is contained in L', then it is k–constant in L' due to Claim 3 (note that r_0 contains at least one star) and this property is kept through the insertion step (due to our adjustment of it). In case of Lemma 2 we also need to check that the vertices s and s^+ have the desired properties. In case that $l = 0$ and $s = [0, *, *, \ldots, *]$, then $s^+ = [0, *, 0, \ldots, 0]$ is simple and s is l–constant. If $s = *^\Delta$ and $l > 0$, then $s^+ = [*, 0, *, \ldots, *]$ and thus s is l–constant. If $s = *^\Delta[l + 1 \to 0]$, then $s^+ = [0, *, 0, \ldots, 0]$ and thus s is l–constant.

- **$n_0 = n_1 = \Delta$, the proof of Lemma 2**
  We can assume w.l.o.g that $l \neq 1$. We set the vertex r_0 to be the vertex $[*, 0, 0, \ldots, 0]$ and R to the set $\{[*, 0, 0, \ldots, 0], [*, 1, 0, \ldots, 0], \ldots, [*, \Delta - 2, 0, \ldots, 0]\}$. We slightly modify the proof of Theorem 4 and then we use the construction described in it to get the desired hamiltonian cycle. We choose the vertex $[0, \ldots, 0]$ to be the representant of the cycle of $\mathcal{Z}$. In the last (insertion) step of the proof of Theorem 4, we insert the vertex $*^\Delta$ just before $*^\Delta[1 \to 0]$ and in case that $l \neq 0$ we insert the vertex $*^\Delta[l \to 0]$ just before $[0, \ldots, 0, *]$. Both the vertices $*^\Delta$ and $*^\Delta[l \to 0]$ (for $l \neq 0$; note that in this case Δ is at least 4, since $l \neq 1$, $l \neq k$ and $k = \Delta - 1$) remain to the last insertion step during the construction of the hamiltonian cycle (see the Claim 1). If $n_{\Delta-1} = \Delta$, then also the vertices $[*, t, 0, \ldots, 0, *]$ for all $t \in \{*, 0, \ldots, \Delta - 1\}$ remain to the last step and we insert them just before the vertices $[(-t) \bmod \Delta, t, 0, \ldots, 0, *]$.

  We now check that the just constructed hamiltonian cycle has the desired properties. Let $t' \in \{*, 0, \ldots, n_{\Delta-1} - 1\}$ be arbitrary. We need to check that for any $r \in R$ the vertex $r[t']$ is $(\Delta - 1)$-constant — this is clear, since $[*, t, 0, \ldots, 0, t']^+ = [(-t - t') \bmod \Delta, t, 0, \ldots, 0, t']$ for $t' \neq *$; if $t' = *$, then $[*, t, 0, \ldots, 0, t']^+ = [(-t - 1) \bmod \Delta, t, 0, \ldots, 0, *]$. Since we are proving Lemma 2, we need to check also that the vertices s and s^+ have also the desired properties. If $l = 0$ and $s = [0, *, *, \ldots, *]$, then $s^+ = [0, *, 0, \ldots, 0]$ is simple and s is l–constant. Otherwise the vertex s^+ is simple, since it is either $*^\Delta[1 \to 0] = [*, 0, *, \ldots, *]$ or $[0, \ldots, 0, *]$ and s

is l–constant due to the manner of insertion of s to the cycle in the last construction step.

- **$n_0 = n_1 = \Delta$, the proof of Lemma 3**
We set the vertex r_0 to be the vertex $[*, 0, 0, \ldots, 0]$ and the set R to be the set $\{[0, *, 0, \ldots, 0], [*, 0, 0, \ldots, 0], [*, 1, 0, \ldots, 0], \ldots, [*, \Delta-3, 0, \ldots, 0]\}$. We modify the last insertion step of the proof of Theorem 4 and then we use the construction of this proof to get the desired hamiltonian cycle. We insert the vertex $*^\Delta$ just after the vertex $[0, *, 0, \ldots, 0, *]$ and in case that $n_{\Delta-1} = \Delta$ we insert the vertices $[*, t, 0, \ldots, 0, *]$ for $t \in \{*, 0, \ldots, n_{\Delta-1}-1\}$ just before the vertices $[0, t, 0, \ldots, 0, *]$; all these vertices remain to the last insertion step due to Claim 1. We now check that the just constructed hamiltonian cycle has the desired properties. The size of R is $\Delta - 1$ and it contains two simple vertices. We finish the proof of this case by checking that the vertices $r[t']$ are $(\Delta - 1)$–constant for all $r \in R$ and for all $t' \in \{*, 0, \ldots, n_{\Delta-1} - 1\}$:
  - $t' \neq *$ and $r = [0, *, 0, \ldots, 0]$
  It holds that $r[t']^+ = r[t'][1 \to -\sum r - t' + 1]$ and thus the vertex $r[t']$ is $(\Delta - 1)$–constant.
  - $t' \neq *$ and $r = [*, t, 0, \ldots, 0]$
  It holds that $r[t']^+ = r[t'][0 \to -\sum r - t']$ and thus the vertex $r[t']$ is $(\Delta - 1)$–constant.
  - $t' = *$ and $r = [0, *, 0, \ldots, 0]$
  The vertex $r[t'] = [0, *, 0, \ldots, 0, *]$ is $(\Delta - 1)$–constant, since the vertex $*^\Delta$ has been inserted just after it.
  - $t' = *$, $r = [*, t, 0, \ldots, 0]$ for some t and $n_{\Delta-1} < \Delta$
  It holds that $r[t']^+ = [*, t, 0, \ldots, 0, *]^+ = [(-t) \bmod \Delta, t, 0, \ldots, 0, *]$ due to the construction of the cycles in the set $\mathcal{Z}$ (consult the proof of Theorem 4) and thus $r[t']$ is $(\Delta - 1)$–constant.
  - $t' = *$, $r = [*, t, 0, \ldots, 0]$ for some t and $n_{\Delta-1} = \Delta$
  The vertex $r[t'] = [*, t, 0, \ldots, 0, *]$ is $(\Delta - 1)$–constant, since it has been inserted just before the vertex $[0, t, 0, \ldots, 0, *]$.

□

5 The Proof of the Conjecture

We are now almost ready to prove the original conjecture of Bermond et al. We find a covering of each graph in the product by stars, then we find hamiltonian (directed) cycles in the products of these stars and finally, we join them together. Lemma 2 and Lemma 3 give us a tool for finding suitable hamiltonian directed cycles in the products of stars, but we need to develop some tools for joining them together. These missing tools are contained in the following two lemmas:

Lemma 4 *Let G be any graph, let $\mathcal{C}$ be a covering of G with directed cycles and let M be a set of edges of G satisfying the following conditions:*

- *No two edges of M have a common vertex.*
- *For each edge $xy \in M$, there are edges xy^+ and yx^+ in G.*
- *The spanning subgraph of G containing exactly edges of the cycles of $\mathcal{C}$ and the edges of the set M is connected.*

Then G contains a hamiltonian cycle.

Proof: Assume w.l.o.g. that no proper subset of M satisfies the conditions of the lemma; this means that in the spanning subgraph from the third condition, no edge of M is contained in any cycle of that spanning subgraph. We replace the edges xx^+ and yy^+ with xy^+ and yx^+ for all edges $xy \in M$; this is possible, since the edges of M are disjoint. We change the directed cycles of $\mathcal{C}$ to one connected oriented cycle, since the in–degree and out–degree of all the vertices is preserved and the resulting subgraph is really connected due to the third condition in the lemma and assumed minimality of M. $\Box$

We call a subgraph G' of $G = G_0 \times \cdots \times G_{\Delta-1}$ a copy of G_i in G if there exists $v_0 \in G_0, \ldots, v_{i-1} \in G_{i-1}, v_{i+1} \in G_{i+1}, \ldots, v_{\Delta-1} \in G_{\Delta-1}$ such that $G = v_0 \times \cdots \times v_{i-1} \times G_i \times v_{i+1} \times \cdots v_{\Delta-1}$.

Lemma 5 *Let $T_0, \ldots, T_{\Delta-1}$ be trees (possibly consisting of a single vertex), let a and b be two different leaves of T_0. Then there exists a spanning tree T in $T_0 \times \cdots \times T_{\Delta-1}$ satisfying the following conditions:*

- *T consists of copies of the trees $T_0, \ldots, T_{\Delta-1}$, i.e. for each edge e of T there exists T' a copy of T_i in T, $e \in T'$ and $T' \subseteq T$.*
- *If $v \in T$ and $v_0 \neq a, b$, then the edges incident to v in T are only edges of a copy of T_0.*
- *If $v \in T$ and $v_0 = a$ or $v_0 = b$, then the edges incident to v in T are edges of a copy of T_0 and of a copy of at most one another tree T_i.*

Proof: The proof proceeds by the induction on Δ. We prove a bit stronger statement, namely: There is T satisfying the conditions of the lemma and two vertices $v^1, v^2 \in T$ such that $v_0^1, v_0^2 \in \{a, b\}$ and the edges incident to the vertices v^1 and v^2 are only edges of copies of T_0. If $\Delta = 1$, the statement is trivial — let T be T_0, let v^1 be a and let v^2 be b. Let Δ be at least 2, now. We use the induction: Let T' be the spanning tree of $T_0 \times \ldots \times T_{\Delta-2}$ and let u^1 and u^2

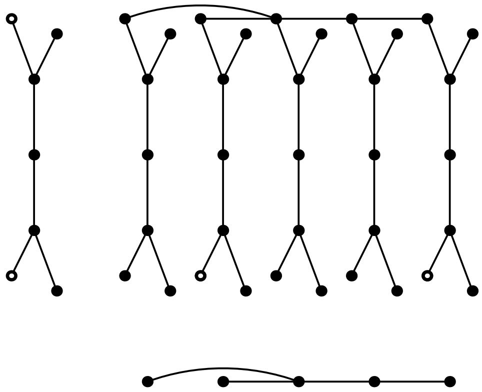


Figure 8: Creating T from T', ${v^1}'$, ${v^2}'$ and $T_{\Delta-1}$; the tree T' is in the left, the tree $T_{\Delta-1}$ is in the bottom; the vertices ${v^1}'$, ${v^2}'$, v^1 and v^2 are drawn as empty circles

be the two special vertices of T'. If $T_{\Delta-1}$ is a single vertex w, we simply set T to $T' \times w$, v^1 to $u^1 \times w$ and v^2 to $u^2 \times w$. Let w^1 and w^2 be any two vertices of $T_{\Delta-1}$. We set T to $\left(\cup_{w \in T_{\Delta-1}} T' \times w\right) \cup {v^1}' \times T_{\Delta-1}$, v^1 to $u^2 \times w^1$ and v^2 to $u^2 \times w^2$ — see Figure 8. It is routine to check that T has the desired properties. $\square$

We are now ready to prove the conjecture of Bermond et al.:

Theorem 5 *Let $G_0, \ldots, G_{\Delta-1}$ be connected graphs with maximum degree at most Δ. Then $G_0 \times \ldots \times G_{\Delta-1}$ is hamiltonian.*

Proof: The statement is trivial for $\Delta = 2$, since in this case both the graphs G_0 and G_1 are paths (with at least two vertices) and thus their product is hamiltonian. Let us suppose that Δ is at least 3, now. We can find coverings $\mathcal{C}_i$ of G_i by $S_{\leq\Delta}$ for $0 \leq i \leq \Delta - 1$ due to Lemma 1. If all the coverings consist of exactly one star each, then the product $G_0 \times \cdots \times G_{\Delta-1}$ is hamiltonian due to Theorem 4. Otherwise let us suppose w.l.o.g. that $\mathcal{C}_0$ contains at least two stars. Let E_i be the minimal set of edges of G_i such that the spanning subgraph of G_i with edge–set equal to the union of the edges used in $\mathcal{C}_i$ and the edges of E_i is connected — this set exists, since G_i is connected; call the graph with this edge–set $\tilde{G}_i$. Let $c_i : E_i \to \{0, \ldots, \Delta - 2\}$ be colouring of edges of E_i in $\tilde{G}_i$ by colours $\{0, \ldots, \Delta - 2\}$ such that no two incident edges have the same colour — such a colouring exists, since each vertex v of $\tilde{G}_i$ is incident to at most $\Delta - 1$ edges of E_i (the degree of v in G_i is at most Δ and at least one edge incident to v is a part of a star in a covering $\mathcal{C}_i$) and the subgraph of $\tilde{G}_i$ induced by edges E_i is a forest. Let T_i be the graph obtained from $\tilde{G}_i$ by contracting each star of the covering of $\mathcal{C}_i$ to one vertex; note that the edge–set of T_i is precisely E_i

and T_i is a tree (possibly consisting of a single vertex), since E_i is a minimal edge–set to make $\tilde{G}_i$ connected. Since $\mathcal{C}_0$ contains at least two stars, T_0 contains at least two vertices and thus also at least two leaves — call these leaves a and b. Let $\tilde{a}$ ($\tilde{b}$) be the vertex of G_0 incident with the only edge of E_0 incident with the star corresponding to a (b). Note that we can assume w.l.o.g. that the colours assigned by c_0 to the edges incident to $\tilde{a}$ and $\tilde{b}$ are only 0 or 1. Let $n_i(v)$ for $v \in T_i$ be the order of the star which has been contracted to the vertex v of T_i and let $f_i^v : V(G_i) \to V(S_{n_i(v)}) = \{*, 0, \ldots, n_i(v)-1\}$ be an isomorphism mapping the vertices of the star contracted to the vertex $v \in T_i$ to the original star $S_{n_i(v)}$. We assume w.l.o.g. that the vertices $\tilde{a}$ and $\tilde{b}$ in $\tilde{G}_0$ are mapped by f_0^a and f_0^b to $*$ or 0. Let T be a spanning tree of $T_0 \times \ldots \times T_{\Delta-1}$ with two special vertices of T_0 equal to a and b with properties described in Lemma 5. We call the subgraph of $G_0 \times \ldots \times G_{\Delta-1}$ corresponding to a vertex of T a cluster; the vertex $[v_0, \ldots, v_{\Delta-1}]$ of T corresponds to $S_{n_0(v_0)} \times \ldots \times S_{n_{\Delta-1}(v_{\Delta-1})}$ and there is a mapping[1] $f^{[v_0,\ldots,v_{\Delta-1}]} = f_0^{v_0} \times \ldots \times f_{\Delta-1}^{v_{\Delta-1}}$ of this cluster to the corresponding product of stars. We find a directed hamiltonian cycle in each cluster using Lemma 2 or Lemma 3 and we find a suitable set of edges M corresponding to edges of T between clusters. This procedure is described in more detail in the following two paragraphs.

We first find hamiltonian cycles in the clusters corresponding to vertices incident to edges of two different trees. We also find edges of M corresponding to edges between clusters of T in copies of $T_1, \ldots, T_{\Delta-1}$. Look at a fixed copy of T_k in T for $1 \le k \le \Delta-1$, i.e. we have fixed k and $v_0 \in T_0, \ldots, v_{k-1} \in T_{k-1}, v_{k+1} \in T_{k+1}, \ldots, v_{\Delta-1} \in T_{\Delta-1}$ such that $v_0 \times \ldots \times v_{k-1} \times T_k \times v_{k+1} \times \ldots \times v_{\Delta-1} \subseteq T$. We use Lemma 2 for $n_i = n_i(v_i)$ $(0 \le i \le \Delta - 1, i \ne k)$, $l = 0$ and $s = [t, *, \ldots, *]$, there t is $f_0^{v_0}(w)$ (either $*$ or 0) where w is the vertex ($\tilde{a}$ or $\tilde{b}$) connecting the star corresponding to v_0 in $\tilde{G}_0$ to the rest of the graph. There exists a set $R \subseteq S_{n_0} \times \ldots \times S_{n_{k-1}} \times S_{n_{k+1}} \times \ldots \times S_{n_{\Delta-1}}$ of size at least $\Delta - 1$ with properties described in the statement of the Lemma 2. Let $r_0, \ldots, r_{\Delta-2}$ be elements of R. We place a copy (under the inverse mapping to $f^{[v_0,\ldots,v_{\Delta-1}]}$) of a cycle C from Lemma 2 to each cluster corresponding to $[v_0, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_{\Delta-1}] \in T$ for $v_k \in T_k$. Let xy be an edge of E_k corresponding to an edge $v_k^x v_k^y$ in the fixed copy of T_k we deal with. We add to the set M an edge $\tilde{x}\tilde{y}$ such that $f^{[v_0,\ldots,v_{k-1},v_k^x,v_{k+1},\ldots,v_{\Delta-1}]}(\tilde{x}) = r_{c_k(xy)}[f_k^{v_k^x}(x)]$ and $f^{[v_0,\ldots,v_{k-1},v_k^y,v_{k+1},\ldots,v_{\Delta-1}]}(\tilde{y}) = r_{c_k(xy)}[f_k^{v_k^y}(y)]$. The existence of edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ is ensured by the k–constantness of $\tilde{x}$ and $\tilde{y}$: Since $\tilde{x}_k = \tilde{x}_k^+$, since $\tilde{x}_i = \tilde{y}_i$ for $i \ne k$ and since there is an edge between $\tilde{x}_i$ and $\tilde{x}_i^+$ in G_i (because $\tilde{x}\tilde{x}^+$ is an edge), there is an edge between $\tilde{y}$ and $\tilde{x}^+$ in $G_0 \times \ldots \times G_{\Delta-1}$. The proof of the existence of the edge $\tilde{x}\tilde{y}^+$ is actually the same. We do the above described procedure for all copies of $T_1, \ldots, T_{\Delta-1}$ in T.

[1]Let $f^1 : A_1 \to B_1$ and $f^2 : A_2 \to B_2$, then the mapping $f^1 \times f^2 : A_1 \times A_2 \to B_1 \times B_2$ is defined as $f^1 \times f^2([a_1, a_2]) = [f^1(a_1), f^2(a_2)]$.

We deal with the copies of T_0 in T now. Each vertex of $v \in T$ is contained in exactly one copy of T_0. The only edges incident to v are the edges of the copy of T_0 if $v_0 \notin \{a, b\}$. Let $v_1 \in T_1, \ldots, v_{\Delta-1} \in T_{\Delta-1}$ be fixed. We use Lemma 3 for $k = 0$ and $n_i = n_i(v_i)$ for $1 \leq i \leq \Delta - 1$. There exists a set $R \subseteq S_{n_1} \times \ldots \times S_{n_{\Delta-1}}$ of size at least $\Delta - 1$ with the properties described in the statement of the Lemma 3. Let $r_0, \ldots, r_{\Delta-2}$ be elements of R and assume that r_0 and r_1 are the two simple vertices which R has to contain. Let $v_0 \in T_0$; we place a copy (under the inverse mapping to $f^{[v_0,\ldots,v_{\Delta-1}]}$) of a cycle C from Lemma 3 (the cycle for $n_0 = n_0(v_0)$) to each cluster corresponding to $[v_0, \ldots, v_{\Delta-1}] \in T$ to which the hamiltonian cycle has not been placed in the previous paragraph; this happens iff the edges incident to $[v_0, \ldots, v_{\Delta-1}]$ in T are only the edges of the copy of T_0. Let xy be an edge of E_0 corresponding to an edge $v_0^x v_0^y$ in the fixed copy of T_0 we deal with, such that both x and y are different from $\tilde{a}$ and $\tilde{b}$. We add to the set M an edge $\tilde{x}\tilde{y}$ such that $f^{[v_0^x,v_1,\ldots,v_{\Delta-1}]}(\tilde{x}) = r_{c_0(xy)}[f_0^{v_0^x}(x)]$ and $f^{[v_0^y,v_1,\ldots,v_{\Delta-1}]}(\tilde{y}) = r_{c_0(xy)}[f_0^{v_0^y}(y)]$. The existence of edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ is ensured by the 0–constantness of $\tilde{x}$ and $\tilde{y}$ (for the same reason we used at the end of the previous paragraph).

It remains to add edges to M corresponding to the edges of T_0 incident to a and b:

- **ab is an edge of T_0.** We add to the set M the edge $\tilde{x}\tilde{y}$ such that $f^{[a,v_1,\ldots,v_{\Delta-1}]}(\tilde{x}) = [f_0^a(\tilde{a}), *, \ldots, *]$ and $f^{[b,v_1,\ldots,v_{\Delta-1}]}(\tilde{y}) = [f_0^b(\tilde{b}), *, \ldots, *]$. Both of the vertices $\tilde{x}$ and $\tilde{y}$ are 0–constant due to Lemma 2 and thus there exist edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ in $G_0 \times \ldots \times G_{\Delta-1}$ for the same reason we used at the end of the previous paragraph.

- **ab is not an edge of T_0, i.e. there are edges ay and xb in T_0.** Let us deal only with the edge ay; the case of the edge xb is actually same. Let $\tilde{a}z$ be the edge of E_0 corresponding to the edge ay of T_0. We add to the set M the edge $\tilde{x}\tilde{y}$ such that $f^{[a,v_1,\ldots,v_{\Delta-1}]}(\tilde{x}) = [f_0^a(\tilde{a}), *, \ldots, *]$ and $f^{[y,v_1,\ldots,v_{\Delta-1}]}(\tilde{y}) = r_{c_0(\tilde{a}z)}[f_0^y(z)]$. Both of the vertices $\tilde{x}$ and $\tilde{y}$ are 0–constant due to Lemma 2 and Lemma 3; the vertices $\tilde{x}$, $\tilde{x}^+$ and $r_{c_0(\tilde{a}z)}$ are simple. Thus there exist edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ in $G_0 \times \ldots \times G_{\Delta-1}$ for the similar reason we used at the end of the previous paragraph.

We do the above described procedure for all copies of T_0 in T.

We need to check that the set M has the properties demanded by Lemma 4. We have already checked that there are edges xy^+ and yx^+ for all the edges $xy \in M$. Since each edge in M corresponds to an edge of T, M is minimal set of edges such that the subgraph of $G_0 \times \ldots \times G_{\Delta-1}$ with the edge–set consisting of the edges of hamiltonian cycles of the clusters and the edges of M is connected. We are going to check that the edges of M are vertex disjoint.

Let xy and $x'y'$ be two non–disjoint edges of M corresponding to edges of a copy of T_i and $T_{i'}$; assume that $x = x'$ and $i \geq i'$. The edges of M corresponding to edges of T of two different copies of T_j and T_k for $j, k > 0$ (not necessarily $i \neq k$) are vertex disjoint, since the edges of T corresponding to them join different clusters; thus the same reason is true also for the edges corresponding to edges of T of different copies of T_0. It must hold that $i = i'$ or $i' = 0$.

We first deal with the case that $i = i'$. Let v be the cluster containing $x = x'$. The edges xy and $x'y'$ of G correspond to the edges $x_i y_i$ and $x'_i y'_i$ of E_i. Since $x_i = x'_i$, it holds that $c_i(x_i y_i) \neq c_i(x'_i y'_i)$. The vertices x and x' have been chosen to satisfy that $f^v(x) = r_{c_i(x_i y_i)}[f_i^{v_i}(x_i)]$ and $f^v(x') = r_{c_i(x'_i y'_i)}[f_i^{v_i}(x'_i)]$, but since $c_i(x_i y_i) \neq c_i(x'_i y'_i)$ it cannot be $x = x'$.

We deal with the remaining case that $i > i' = 0$. Let v be the cluster containing $x = x'$. It holds that $s = f^v(x')$ due to the construction of M and thus $f_j^{v_j}(x_j) = *$ for all $j > 0$, especially $f_i^{v_i}(x_i) = *$. Then it must hold $f^v(x) = r_c[f_i^{v_i}(x_i)]$ for some c due to the construction of M; but $r_c[*]$ is different from s for all choices of c — this contradicts our assumption that the edges xy and $x'y'$ are not disjoint.

We have just proved that the edges of M are vertex disjoint and thus we have finished the proof of the theorem. $\square$

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