On $\mathcal{P}$-universal graphs

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Abstract

An (n, m)-graph is a graph with both arcs, coloured from the set $\{1, 2, \dots, n\}$ and undirected edges coloured from the set $\{1', 2', \dots, m'\}$. An $(n, m, \mathcal{P})$-universal graph is an (n, m)-graph into which every (n, m)-graph whose underlying graph is planar maps homomorphically. Such graphs are known to exist for all (n, m). We find necessary conditions for an (n, m)-graph to be $(n, m, \mathcal{P})$-universal and prove that such a graph must have at least $1 + (2n + m) + \epsilon(2n + m)^2 + (2n + m)^3$ vertices, where ϵ is 1 when m is odd or zero and 2 when m is even and positive. In the case $(n, m) = (1, 0)$ (uncoloured oriented graphs) we obtain the better lower bound of 17.

1 Introduction

A homomorphism from a graph $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ is a function from V_1 to V_2 with the property that if two vertices are adjacent in G_1 then their images are adjacent in G_2 This notion has natural generalizations to digraphs and to edge coloured graphs, in which case homomorphisms are required to preserve orientation and edge colour respectively. Recently Nešetřil and Raspaud [7] have combined these cases by considering coloured mixed graphs.

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An (n,m)-mixed colored graph (for brevity an (n,m)-graph) contains both arcs, coloured from the set of colours $\{1,2,\ldots,n\}$ and undirected edges coloured from the set $\{1',2',\ldots,m'\}$ such that the underlying uncoloured undirected graph is simple.

A *homomorphism* between two such graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is defined to be a mapping $\phi : V_1 \longmapsto V_2$ with the property that if $\{u,v\}$ (resp. $(\phi(u),\phi(v))$) is an edge (resp. arc) of a given colour then $\{\phi(u),\phi(v)\}$ (resp. (u,v)) is an edge (resp. arc) of a the same colour.

Let $\mathcal{C}$ be a class of (uncoloured undirected simple) graphs. An $(n,m,\mathcal{C})$-*universal* graph is an (n,m)-graph G into which every (n,m)-graph whose underlying graph lies in $\mathcal{C}$ maps homomorphically (there is no requirement that G itself have underlying graph in $\mathcal{C}$). We may abbreviate to "$\mathcal{C}$-universal" or just "universal" when the parameters are clear from the context. A universal graph is *minimal* if none of its proper subgraphs are.

Given $\mathcal{C}$, n and m, the most obvious question about $(n,m,\mathcal{C})$-universal graphs is whether or not they exist (observe that the answer to this question would become trivially "yes" if we admitted infinite graphs or graphs with loops and would reduce to the uncoloured undirected case if we admitted multiple edges). In fact this question is completely solved by Kostochka, Sopena and Zhu [5] who prove that $(n,m,\mathcal{C})$- universal graphs exist only if there is a bound on the acyclic chromatic numbers of the graphs in $\mathcal{C}$ and by Nešetřil and Raspaud [7] who prove the converse.

Given that $(n,m,\mathcal{C})$-universal graphs exist we can then ask how small such graphs can be made. We define the *uniform* chromatic number $v(n,m,\mathcal{C})$ to be smallest possible order of an $(n,m,\mathcal{C})$-universal graph ($\chi(n,m,\mathcal{C}) = \infty$

if no such graph exists).

Closely related but distinct from this is the chromatic number $\chi(n, m, G)$ of an (n, m)-graph, defined to be the smallest possible order of a homomorphic image of G. In the case of uncoloured undirected graphs ($n = 0$, $m = 1$) this coincides with the familiar definition. The chromatic number $\chi(n, m, \mathcal{C})$ of a class $\mathcal{C}$ is then defined to be the (possibly infinite) supremum of the chromatic numbers of it members.

Of course

$$\chi(n, m, \mathcal{C}) \leq \upsilon(n, m, \mathcal{C}). \tag{1}$$

If $\mathcal{C}$ is a complete class of graphs, that is given any two graphs in $\mathcal{C}$, there is a third which contains them both as subgraphs, then equality holds above. We let $\mathcal{P}$ and $\mathcal{L}$ represent respectively the classes of planar graphs and paths. These classes are both complete.

Explicit upper bounds have been found for $\chi(1, 0, \mathcal{C})$, where $\mathcal{C}$ is the class of planar graphs with bounded girth [8], [3], k-trees [10], graphs with bounded degree [3], forests [7] and graphs with bounded acyclic chromatic number [7]. For the last two classes, bounds have been found for all $\chi(n, m, \mathcal{C})$.

The main subject of this paper is universal graphs and chromatic numbers for the class of planar graphs. A motivation for this work is the four colour theorem, which can be formulated as the statement that K_4 is $(0, 1, \mathcal{P})$-universal. Of course, since K_4 is also the unique minimal such graph, we have a complete characterization of $(0, 1, \mathcal{P})$-universal graphs. We can then ask how far it is possible to find analogues of the four colour theorem for (n, m)-graphs.

For $(n, m) \neq (0, 1)$ there seems to be no reason to believe that there should be a unique minimal $\mathcal{P}$-universal graph or even that there should be finitely many of them but we might at least hope to identify the $\mathcal{P}$-universal graph of smallest order, and so to determine $\chi(n, m, \mathcal{P})(= v(n, m, \mathcal{P}))$. Even this goal is presumably difficult since it still includes the four colour theorem. The best known bounds for $\chi(n, m, \mathcal{P})$ are [9]

$$(2n + m)^3 + 3 \leq \chi(n, m, \mathcal{P}) \leq 5(2n + m)^4,$$

with the sharper lower bound of 15 being known for $(n, m) = (1, 0)$ [10].

We derive some necessary conditions for $\mathcal{P}$-universality (Theorem 11) and use these to improve the lower bound above to

$$1 + (2n + m) + \epsilon(2n + m)^2 + (2n + m)^3, \tag{2}$$

where ϵ is 1 when m is odd or zero and 2 when m is even and positive.

From section 2 we deal exclusively with the case $(n, m) = (1, 0)$ that is uncoloured oriented graphs, the main theorem of this paper being

Theorem 1 $\chi(1, 0, \mathcal{P}) \geq 17$

In other words every $(1, 0, \mathcal{P})$-universal graph has order at least 17. The proof has two main steps. First we show that the only possible $(1, 0, \mathcal{P})$-universal of order 16 or less are the Tromp graph T_{16}, described in section 2, or extensions of T_{16} obtained by adding some new arcs. We complete the proof by exhibiting an oriented planar graph which does not map into T_{16} or any such extensions.

1.1 Notation

To avoid having to continually refer to "edges of the same colour or arcs of the same colour and orientation", we will use the following terminology. Let

$$\mathcal{E} = \mathcal{E}_{2n+m} = \{(k,-1)|\ 1 \le k \le n\} \cup \{(k,1)|\ 1 \le k \le n\} \cup \{k|\ 1 \le k \le m\}.$$

We say that an ordered pair (u,v) of adjacent vertices has *type* $(k,-1)$, $(k,1)$ or k according as (v,u) is an arc coloured k, (u,v) is an arc coloured k or $\{u,v\}$ is an edge coloured k.

If v is a vertex of an (n,m)-graph G and $c \in \mathcal{E}$ then $N_c(v)$, the set of c-neighbours of v is defined by $N_c(v) = \{w \in G|\ (v,w) \text{ is of type } c\}$. We let $V(G)$ denote the vertex set of an (n,m)-graph G.

By a slight abuse of notation we will say an (n,m)-graph has a property such as planarity, being a path, being in class $\mathcal{C}$ etc. when its underlying uncoloured undirected graph has this property.

2 Tromp graphs

For each $p \equiv 3 \bmod 4$ we define QR_p to be the tournament with vertex set Z_p and arcs $\{(a,b)|\ b-a \text{ is a nonzero quadratic residue}\}$. The condition $p \equiv 3 \bmod 4$ ensures that each pair of distinct vertices is joined by exactly one arc. The symmetry group of QR_p comprises the linear maps $u \to au+b$ where $a,b \in Z_7$ and a is a quadratic residue. It follows that QR_p is vertex transitive.

From these graphs we construct another family of highly symmetric graphs, using a construction initially due to Tromp [11] and generalized by Albiero and Sopena [2]. For each $p \equiv 3 \bmod 4$, define T_{2p+2} to be the oriented graph with vertex set comprising two copies of Z_p, which we denote by $\{0, 1, \ldots p-1\}$ and $\{0', 1', \ldots (p-1)'\}$ and two other vertices which we denote by V and V' and the following arcs: (a, b) and (b, a'), where $a, b \in Z_p$ and $b - a$ is a nonzero quadratic residue, (V, a), (a, V') (V', a') and (a', V) for all $a \in Z_p$.

Definition:- For each T_{2p+2}, τ is the involution of $V(T_{2p+2})$ defined by $\tau(x) = x'$, $\tau(x') = x$, whenever $x \in Z_p \cup \{V\}$. We refer to $\tau(x)$ as the *complement* of x. (We have thus used τ to refer to a whole family of different functions, but this should cause no confusion.)

It is easy to see that the arc between x to $\tau(y)$ is in the reverse direction to the arc between x and y and hence that τ is an automorphism. Observe that two vertices of T_{2p+2} are joined by an arc if and only if they are not complements of each other.

Theorem 2 *Each T_{2p+2} is vertex transitive.*

Proof:- Every automorphism ϕ of Z_p can be extended to a unique automorphism $\tilde{\phi}$ of T_{2p+2} which fixes V and V'. Explicitly $\tilde{\phi}(V) = V$ $\tilde{\phi}(V') = V'$, $\tilde{\phi}(m) = \phi(m)$ and $\tilde{\phi}(m') = (\phi(m))'$. These automorphisms, together with τ, act transitively on $V(T_{2p+2}) \setminus \{V, V'\}$ so that, to prove the theorem, we only need find an automorphism which maps V to vertex not in $\{V, V'\}$.

Let r be any fixed nonquadratic residue in Z_p, then the function ψ which maps 0 to V', V to 0, each nonzero quadratic residue n to $(rn^{-1})'$ and each non-quadratic residue n to rn^{-1} and whose values on the other vertices of

T_{2p+2} are determined by the requirement $\psi\tau = \tau\psi$ is easily checked to be an automorphism. □

3 P_k and Q_k graphs

The following is a natural generalization to mixed graphs of a definition introduced by Sopena [10] for oriented graphs.

Definition:- An (n, m)-graph G has property P_k if, for every sequence $v_1, v_2, \ldots v_l$ of $l \leq k$ vertices spanning a an l-clique in G and every sequence $c_1, c_2, \ldots c_l$ of l (not necessarily distinct) members of $\mathcal{E}$, there exists a vertex v of G such that for $1 \leq i \leq l$ $v \in N_{c_i}(v_i)$.

The smallest order $(0, m)$-graphs with property P_1 are the well known graphs of order $m + 1$ or $m + 2$ with a proper edge colouring. Nešetřil and Raspaud [7] have identified smallest order graphs with property P_1 for all values of (m, n).

For every k, plenty of (n, m)-graphs with property P_k exist (a sufficiently large complete graph with arc colours and orientations chosen randomly will almost surely be P_k) but finding "nice" small order graphs with property P_k for $k \geq 2$ seems much harder and little is known except in the directed case, which is discussed in [10]. In particular QR_7 and T_{16} are the smallest orientated graphs with properties P_2 and P_3 respectively. The following is an obvious adaption to mixed graphs of Theorem 3.7 of [10]

Lemma 3 *If G has property P_k then, for every $v \in V(G)$ and $c \in \mathcal{E}$, the subgraph induced by $N_c(v)$ has property $P_{(k-1)}$*

Proof:- Let $l \leq k$ and let $v_1, v_2, \ldots v_{l-1}$ and $c_1, c_2, \ldots c_{l-1}$ be sequences of distinct vertices in $N_c(v)$ and colours in $\mathcal{E}$. Let $v_l = v$ and $c_l = c$. Since G has property P_k, there is a vertex w such that for $1 \leq i \leq l$ $w \in N_{c_i}(v_i)$. In particular, $w \in N_c(v)$ so $N_c(v)$ has property $P_{(k-1)}$. $\square$

A similar generalization of Theorem 3.4 of [10] gives the result that every (n,m)-graph G with property P_k is (k-tree)-universal. When considering $\mathcal{P}$-universality the following weaker property is useful.

Definition:- Every (n,m)-graph has property Q_0. An (n,m)-graph G has property $Q_{(k+1)}$ if there exists a subgraph G' of G such that $\forall c \in \mathcal{E}, \forall v \in V(G')$ the subgraph of G' induced by $N_c(v)$ has property Q_k.

Note that it is an immediate consequence of the definition that every extension of a graph with property Q_k itself has property Q_k.

We can obtain a "sharp" version property of Q_k by requiring $G' = G$ in the above definition. In fact it will be useful to define a slightly weaker property $Q_k^{(m)}$ where sharpness is required only in the last m steps of the induction.

Definition:- An (n,m)-graph has property $Q_k^{(0)}$ if it has property Q_k. It has property $Q_k^{(m)}$ $(1 \leq m \leq k)$ if $\forall c \in \mathcal{E}, \forall v \in V(G)$ the subgraph of G induced by $N_c(v)$ has property $Q_k^{(m-1)}$.

Lemma 4 *Every (n,m)-graph with property P_k has property Q_k.*

Proof:- A straightforward induction using Lemma 3.

Proposition 5 *For uncoloured oriented graphs*

1. *There is no Q_1 graph of order less than 3. The only Q_1 graph of order 3 is the triangular cycle.*

2. *There is no Q_2 graph of order less than 7. The only Q_2 graph of order 7 is QR_7.*

3. *There is no Q_3 graph of order less than 16. The only Q_3 graphs of order 16 are T_{16} and its extensions.*

Proof:- The first statement is trivial and the second easy to prove. The proof of the last statement involves a long and tedious calculation and is relegated to the last section of this paper.

Remark:- In view of the aforementioned results from [10], the smallest order oriented graphs with property P_k and with property Q_k coincide for $k \leq 3$.

Definition:- Let G be an (n, m)-graph. G' is the (n, m)-graph by taking $2n + m$ disjoint copies of G $\{G_c | \ c \in \mathcal{E}\}$ indexed by $\mathcal{E}$, and one new vertex A and joining A to each vertex of each G_c by an edge or arc of type c. Let $G^{(k)}$ be defined inductively by $G^{(0)} = G$ and $G^{(k+1)} = (G^{(k)})'$.

Observation:- If G is a path then G' is outerplanar and $G^{(2)}$ is planar.

An easy induction shows that $G^{(k)}$ has property Q_k. The following result generalizes this observation slightly.

Proposition 6 *If G and H be (n, m)-graphs and $k \geq 0$ have the following properties.*

1. *Every homomorphic image of G into H has property Q_t,*

2. *If K and K' are k-cliques in $G^{(k)}$ and H respectively then every homomorphism from K to K' extends to homomorphism from $G^{(k)}$ to H (if*

$k = 0$, interpret this to mean that there is some homomorphism from $G^{(k)}$ to H),

then H has property $Q^{(k)}_{(k+t)}$

Proof:- The proof is by induction on k. If $k = 0$ the result is immediate. Suppose that the proposition is true for $k = l - 1$ and that G, H and $k = l$ satisfy the hypotheses of the proposition. Let $v \in V(G^{(l)})$, $w \in V(H)$ and $c \in \mathcal{E}$. Let K_1 and K_1' be an $(l-1)$-cliques in $N_c(v)$ and $N_c(w)$ respectively and suppose that there is an isomorphism ψ mapping K_1 to K_1'. This extends to an isomorphism from the subgraph of $G^{(l)}$ induced by $\{v\} \cup V(K_1)$ to the subgraph of H induced by $\{w\} \cup V(K_1')$, which must in turn, by hypothesis, extend to a homomorphism from $G^{(l)}$ to H. The restriction of this homomorphism to $N_c(v)$ maps into $N_c(w)$. Since $N_c(v)$ is isomorphic to $G^{(l-1)}$, the induction hypothesis gives that $N_c(w)$ has property $Q^{(k)}_{(l-1+t)}$ whence, since w and c are chosen arbitrarily, h has property $Q^{(k)}_{(l+t)}$. This completes the induction step. □

Definition:- A class $\mathcal{C}$ of (uncoloured undirected) graphs is *k-complete* if, for any $l \leq k$, given graphs $G_1, G_2 \in \mathcal{C}$ containing respectively l-cliques K_1 and K_2 and an isomorphism between K_1 and K_2, the graph G obtained from the disjoint union of G_1 and G_2 by identifying vertices of K_1 and K_2 which are paired by the isomorphism, is also in $\mathcal{C}$.

For example the class of k-trees is k-complete, and $\mathcal{P}$ is 2-complete.

Proposition 7 *Let $\mathcal{C}$ be a 2-complete class of graphs, and H a minimal $\mathcal{C}$-universal (n, m)-graph. If $G \in \mathcal{C}$, K_1 and K_2 are both 1-cliques or 2-cliques* *in G and H respectively and ϕ is an isomorphism from K_1 to K_2 then ϕ can be extended to a homomorphism from G to H.*

Proof:- We may assume that K_1 and K_2 are both 2-cliques (the other case follows easily), whence they are the sets of endpoints of edges or arcs, (u, v) and (u', v') respectively of the same type c. First note that there must be some $G_1 \in \mathcal{C}$ with the property that every homomorphism $G_1 \to H$ includes u and v in its range (otherwise the subgraph obtained by deleting (u, v) would be universal, contrary to the minimality hypothesis).

Let $\mathcal{U} = \{(a, b)|\ a, b \in V(G_1), (a, b) \text{ is of type } c\}$ (Note that if (u, v) is an edge then each pair of vertices appears twice, once in each order).

Now construct G_2 by taking a collection $\{G_{(a,b)}|(a, b) \in \mathcal{U}\}$ disjoint copies of G indexed by $\mathcal{U}$ and one copy of G_1 and then identify u and v in each $G_{(a,b)}$ with a and b respectively.

Since $\mathcal{C}$ is 2-complete, $G_2 \in \mathcal{C}$ so that there is a homomorphism $\phi : G_2 \to H$. For some type c edge or arc (a, b) of the imbedded G_1, $\phi(a) = u'$ and $\phi(b) = v'$. therefore ϕ restricted to $G_{(a,b)}$ must take u and v to u' and v' respectively, as required. $\square$

Lemma 8 *Let $\mathcal{C}$ be a complete class of graphs. For each n, m and k there exists an (n, m)-graph G in $\mathcal{C}$ with the property that every homomorphic image of G of order at most k is $(n, m, \mathcal{C})$-universal.*

Proof:- For each (n, m)-graph H of order at most k which is not $\mathcal{C}$-universal, there exists an (n, m)-graph in $\mathcal{C}$ which admits no homomorphism into H. By completeness of $\mathcal{C}$ there is a graph in $\mathcal{C}$ which contains all these graphs as subgraphs and this graph clearly has the property required by the lemma. □

Lemma 9 *If* $(n, m) \in \{(0,1), (0,2), (1,0)\}$ *then every* $(n, m, \mathcal{L})$*-universal graph* H *has property* Q_1*.*

Proof:- This is trivial when $(n, m) = (0, 1)$, since a single edge has property Q_1. For $(n, m) = (0, 2)$, H must contain a minimal subgraph H' with the property that every path with edges of alternating colour maps homomorphically into H'. For sufficiently long paths all homomorphisms into H' must be onto (otherwise, as in the proof of Proposition 7, minimality of H' would be contradicted). It follows that H', hence H, has property Q_1. A similar argument applies in the case $(n, m) = (1, 0)$, using directed paths in place of alternating ones. □

Remark:- This lemma is false in other cases [6].

Lemma 10 *Every* $(n, m, \mathcal{L})$*-universal graph* H *has order at least* $(2n+m)+\epsilon$*, where* ϵ *is 1 when* m *is odd or zero and 2 when* m *is even and positive.*

Proof:- [6].

Theorem 11 1. *Every* $(n, m, \mathcal{P})$*-universal graph has order at least*

$$1 + (2n + m) + \epsilon(2n + m)^2 + (2n + m)^3, \tag{3}$$

where ϵ *is 1 when* m *is odd or zero and 2 when* m *is even and positive.*

2. *If* $(n, m) \in \{(0,1), (0,2), (1,0)\}$ *then every minimal* $(n, m, \mathcal{P})$*-universal graph has property* $Q_3^{(2)}$*.*

Proof:- Let U be a $(n, m, \mathcal{P})$-universal graph. Using Lemma 8, there is a path L with the property that every homomorphic image of L of order at most $|U|$ is $\mathcal{L}$-universal The graph $L^{(2)}$ is planar and two points of $L^{(2)}$ can have the same homomorphic image only if they belong to the same imbedded copy of L. The lower bound (3) follows, using lemma 10. The second statement follows from Proposition 6 (with $k = 2$ and $t = 1$), using Proposition 7 and Lemmas 8 and 9. □

Remark:- The only properties of $\mathcal{P}$ used in the above theorem are the facts that it is 2-complete and contains $L^{(2)}$. Thus we may substitute any class with these properties for $\mathcal{P}$ above. For example, Theorem 11 holds for 3-trees and, together with Proposition 5 proves that the only possible (3-tree)-universal of order at most 16 are extensions of T_{16}. Sopena [10] has proved that conversely T_{16} is (3-tree)-universal.

In the same vein as Theorem 11 we have

Theorem 12 *If U is a minimal $\mathcal{P}$-universal graph then for every pair of adjacent vertices v_1 and v_2 in U and every $c_1, c_2 \in \mathcal{E}$, the subgraph induced by $N_{c_1}(v_1) \cap N_{c_2}(v_2)$ is $\mathcal{L}$-universal.*

Proof:- As in the proof of Theorem 11, there is a path L with the property that every homomorphic image of L of order at most $|U|$ is $\mathcal{L}$-universal. Let e be a single edge or arc with endpoints u_1 and u_2 such that (u_1, u_2) is of the same type as (v_1, v_2). Let H be the graph constructed by, for $i = 1, 2$, joining u_i to every vertex of L by an edge or arc of type c_i. By Proposition 7 there is a homomorphism from H to U which maps u_1 to v_1 and u_2 to v_2. Since the image of the imbedded path L must be $\mathcal{L}$-universal, U has the property required by the theorem. $\square$

Corollary 13 *Every minimal $\mathcal{P}$-universal graph has property P_2.*

4 Homomorphisms into T_{16}

In this section we construct a planar oriented graph which admits no homomorphisms into T_{16}. The basic method is to begin with an oriented 4-circuit and repeatedly subdivide it by adjoining a new point to three vertices of the "innermost" 4-circuit. By choosing the subdivisions carefully we eventually obtain a graph where the innermost circuit (and hence the graph) does not map into T_{16}.

Let ϕ and ψ be homomorphisms from an oriented graph H to T_{16}. Let $\phi \approx \psi$ if $\phi = \sigma\psi$ for some automorphism of T_{16}. Clearly $\approx$ is an equivalence relation and we denote the equivalence class of ϕ by $[\phi]$.

We identify some particular automorphisms of T_{16}. Let

$$\sigma = (0123456)(0'1'2'3'4'5'6')$$

$$\rho = (V0'V'0)(131'3')(252'5')(464'6')$$

These automorphisms give explicit maps from any point to V. Specifically, $\rho\sigma^{-n}$ maps n to V and $\rho\sigma^{-n}\tau$ maps n' to V.

Let μ be the automorphism which maps n to $2n$ and n' to $(2n)'$ and fixes V and V'

Observe that the automorphism subgroup which fixes V and 0 is the cyclic group of order 3 generated by μ.

It is routine to check that there are, up to isomorphism, four oriented 4-circuits. If we suppose the vertex sets of each to be $\{a, b, c, d\}$, then they may taken to be the graphs A, B, C and D, with edge sets as follows.

$$E(A) = \{(a, b), (b, c), (c, d), (d, a)\}$$

$$E(B) = \{(a, b), (b, c), (c, d), (a, d)\}$$

$$E(C) = \{(a, b), (b, c), (d, c), (a, d)\}$$

$$E(D) = \{(a, b), (c, b), (c, d), (a, d)\}$$

We now classify up to equivalence all the homomorphisms of these graphs into T_{16}.

Lemma 14 *Let $E \in \{A, B, C, D\}$. Every homomorphism $E \to T_{16}$ is equivalent to a unique homomorphism ϕ for which*

$$\phi(a) = V \quad \phi(b) = 0$$

$$\phi(c) \in \{1, 3, V, 1', 3', V'\}$$

$$\textit{if } \phi(c) \in \{V, V'\} \textit{ then } \phi(d) \in \{0, 1, 3, 0', 1', 3'\}$$

Proof:- Let $\phi : E \to T_{16}$. By applying to ϕ in turn an automorphism of the form $\rho\sigma^{-n}$ or $\rho\sigma^{-n}\tau$, a power of σ, and a power of μ, we can shift the value of $\phi(a)$ to V, of $\phi(b)$ to 0 and $\phi(c)$ to a member of the set $\{1, 3, V, 1', 3', V'\}$. If $\phi(c)$ is moved into $\{V, V'\}$ we may apply another power of μ to get $\phi(d)$ into $\{0, 1, 3, 0', 1', 3'\}$ Uniqueness is clear if one observes that no automorphism fixing V and 0 maps any member of $\{V, 0, 1, 3, V', 0', 1', 3'\}$ to another member of the same set. $\square$

Using this lemma we find nine equivalence classes of homomorphisms $D \to T_{16}$ and a canonical representative of each class. Since we have assumed that $\phi(a) = V$ and $\phi(b) = 0$, these homomorphisms are determined by their values at c and d. We tabulate and name (as D_i $(1 \le i \le 9)$) these nine homomorphisms below.

name	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
$\phi(c)$	$1'$	$1'$	$1'$	3	3	3	V	V	V
$\phi(d)$	0	4	6	0	4	5	0	1	3

A similar analysis finds six classes of homomorphism $B \to T_{16}$ (In this case $\phi(c) = V'$ is impossible), which we tabulate and name below.

name	B_1	B_2	B_3	B_4	B_5	B_6
$\phi(c)$	1	1	1	$3'$	$3'$	$3'$
$\phi(d)$	2	3	5	1	2	6

We do not use maps from A or C to T_{16} but it is not difficult to find that there are 9 classes in each of these cases too. In fact every homomorphism in $h(D, T_{16})$ can be changed to a homomorphism in $h(A, T_{16})$ (resp. homomorphism in $h(C, T_{16})$) by replacing the values at c and d (resp. at c) by their complements, and these correspondences are clearly bijective.

Let $H_D = \{D_i | \, 1 \le i \le 9\}$ with H_A, H_B and H_C being defined in a similar manner. We now define four relations between these sets which correspond to certain subdivisions of the graphs.

Let $S^{(i)}$ $(1 \le i \le 4)$ be extensions of B, B, B and D respectively obtained by adding in each case a new vertex u and three new arcs joining U to existing vertices. Specifically, let these arcs be $\{(u, a), (u, c)(u, d)\}$, $\{(u, b), (c, u)(d, u)\}$, $\{(u, b), (u, c)(u, d)\}$ and $\{(u, a), (c, u)(u, d)\}$ for $S^{(1)}$, $S^{(2)}$,

$S^{(3)}$ and $S^{(4)}$ respectively. Let Γ_i be the subgraph of $S^{(i)}$ induced by the vertex sets $\{a, b, c, u\}$ for $i \in \{1, 4\}$ and $\{a, b, d, u\}$ for $i \in \{2, 3\}$. Each Γ_i is a 4-circuit, isomorphic to D when $i = 3$ and to B in the other cases. Finally define $\nu_i : B \to \Gamma_i$ $(i \neq 3)$ and $\nu_3 : D \to \Gamma_3$ by $\nu_1(a, b, c, d) = (u, a, b, c)$, $\nu_2(a, b, c, d) = (a, d, u, b)$, $\nu_3(a, b, c, d) = (a, b, u, d)$ and $\nu_4(a, b, c, d) = (c, u, a, b)$ (where $\nu_1(a, b, c, d) = (u, a, b, c)$ abbreviates $\nu_1(a) = u$, $\nu_1(b) = a$ etc.). Now define relations R_1 and R_2 on B, R_3 from B to D and R_4 from D to B, by, for $k = 1, 2$,

$$R_k = \{(i, j) | \text{ there is an extension } \psi \text{ of } B_i \text{ to } S^{(k)} \text{ for which } (\psi|_{\Gamma_i}) \circ \nu_k \approx B_j\},$$

and for $k = 3, 4$, by the above, substituting D_j for B_j and D_i for B_i respectively.

Given i, we may determine which $(i, j) \in R_k$ as follows: Find all extensions of B_i (or D_i) to $S^{(k)}$, restrict to Γ_i, compose with ν_i (which amounts to remaming the vertices to conform with the canonical labelling of B) and finally apply automorphisms of T_{16} to put these homomorphisms into canonical form. The calculations are routine so we do only an illustrative example.

Suppose $k = i = 1$. To extend B_1 to $S^{(1)}$ we require that

$$\begin{aligned} \phi(u) &\in T_{16}^-(\phi(a)) \cap T_{16}^-(\phi(c)) \cap T_{16}^-(\phi(d)) \\ &= T_{16}^-(V) \cap T_{16}^-(1) \cap T_{16}^-(2) \\ &= \{3'\}, \end{aligned} \tag{4}$$

so that $\phi(u) = 3'$ gives the unique extension of B_1 to $S^{(1)}$. Restricting to Γ_1 and composing with ν_1 gives the homomorphism $\psi : B \to T_{16}$, which takes a, b, c and d to $3'$, V, 0 and 1 respectively. To get a mapped to V, let

$\psi_1 = \rho\sigma^{-3}\tau\psi$ which maps a, b, c and d to V, 0, $6'$ and 2 respectively. As it happens this also maps b to the required value of 0 (if it did not we could have composed with an appropriate power of σ), so it remains only to move $\psi_1(c)$ to the set $\{1, 3'\}$, which is done by applying μ^2. Finally $\psi_2 = \mu^2\psi_1$ maps a, b, c and d to V, 0, $3'$ and 1 respectively. This function is in the canonical form given by lemma 14 and inspection of the table shows that $\psi_2 = B_4$. That is $(1, j) \in R_1$ if and only if $j = 4$. Similar calculations give the rest of R_1 (which is in fact a permutation of H_B) and other relations as shown in table below.

The relations R_1-R_4 are summarized in the table below. (We omit braces when writing sets in this table.)

n	1	2	3	4	5	6	7	8	9
$R_1[n]$	4	5	1	3	6	2			
$R_2[n]$	3	1	4	6	2	5			
$R_3[n]$	8	6,9	3,9	2	5	3,6			
$R_4[n]$	1	3	2	4,5	4,6	5,6	$\emptyset$	$\emptyset$	$\emptyset$

Definition:- Let $\mathcal{G} = (G, \Gamma, \mu)$ be a triple where G is an oriented graph, Γ is an imbedded 4-circuit in G and μ is a homomorphism of $E \in \{A, B, C, D\}$ into G whose image is Γ. Let $\mathcal{G}'$ be the set of indices of equivalence classes of restrictions to Γ of homomorphisms from G to T_{16}, that is

$$\mathcal{G}' = \{j \mid \exists f \in h(G, T_{16}),\ f \circ \mu \approx E_j\}$$

If E can be subdivided into one of the $S^{(i)}$ defined previously (in which case E is B or D) then we define $\mathcal{G}_i = (\tilde{G}_i, \tilde{\mu}(\Gamma_i), \tilde{\mu} \circ \nu_i)$, where $\tilde{G}_i$ is obtained from G by adding a new vertex u' and three new arcs in such a way that μ

extends to a map $\tilde{\mu} : S^{(i)} \to \tilde{G}_i$ mapping u to u', Γ_i is the 4-circuit of $S^{(i)}$ and ν_i the mapping of B or D into Γ_i defined above.

Observation:- If G is planar and Γ is a face of G then $\tilde{G}_i$ is planar and $\tilde{\mu}(\Gamma_i)$ is a face of $\tilde{G}_i$.

The next lemma follows straightforwardly from the above definitions.

Lemma 15 $\mathcal{G}_i' = R_i[\mathcal{G}']$

Proof of theorem 1:- By Proposition 5 and Theorem 11 the only possible minimal $\mathcal{P}$-universal graphs of order 16 are graphs with an imbedded T_{16}. Every order 16 proper extension of T_{16} must contain an edge joining some vertex v to its complement v'. But such a graph does not have property P_2 and so, by Corollary 13 is not minimal $\mathcal{P}$-universal. Thus, to prove the theorem, it suffices to find an oriented planar admitting no homomorphisms into T_{16} itself.

Let G be the graph obtained from B by adding an extra arc from a to c, $\Gamma = B$ and $\mu : B \to G$ the inclusion map. Let $\mathcal{G}^{(0)} = (G, \Gamma, \mu)$ and define $\mathcal{G}^{(k)} = (G^{(k)}, \Gamma^{(k)}, \mu^{(k)})$ inductively by $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}_{\phi(k)}$, where $\phi(0) = \phi(3) = 2$, $\phi(1) = \phi(4) = \phi(7) = 3$, $\phi(2) = \phi(5) = \phi(8) = 4$ and $\phi(6) = 1$. It is routine to check that each of these operations is well defined and, by the above observation the $G^{(k)}$ are all planar. For $k = 2, 5, 8$, $\Gamma^{(k)}$ is isomorphic to D. For the rest it is isomorphic to B. Using Lemma 15 one finds that $(\mathcal{G}^{(0)})', (\mathcal{G}^{(1)})', (\mathcal{G}^{(2)})', \ldots (\mathcal{G}^{(9)})'$ are in turn $\{1,2,3\}$, $\{1,3,4\}$, $\{2,3,8,9\}$, $\{2,3\}$, $\{1,4\}$, $\{2,8\}$, $\{3\}$, $\{1\}$, $\{8\}$ and finally $\emptyset$. Thus $G^{(9)}$ is a planar graph admitting no homomorphisms into T_{16}. $\Box$

Table 1: Incomplete incidence table for G

	a	b	c	d	e	f	g
a	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
b	$-$	$\bullet$				$-$	
c	$-$	$+$	$\bullet$				
d	$-$			$\bullet$			
e	$+$				$\bullet$		
f	$+$					$\bullet$	
g	$+$				$+$		$\bullet$

5 Proof of theorem 5 (3)

Let Γ be a Q_3 oriented graph of order 16. We prove that Γ must contain an imbedded T_{16}. The proof, which is really just a computation, turns on the fact that for each $v \in \Gamma$ at least one of $\Gamma^+(v)$ and $\Gamma^-(v)$ is of order 7 and so, by part 2 of the theorem, is isomorphic to QR_7. Often just a few entries in the incidence matrix (shown here as a table) for a QR_7 are sufficient to determine it entirely. If G is isomorphic to QR_7, $a \in V(G)$, $\{b,c,d\} = G^+(a)$ and $\{e,f,g\} = G^-(a)$ then each of these sets is a triangular circuit and there is a bijection $\pi : G^+(a) \to G^-(a)$, with the property that, for each $v \in G^+(a)$, there is an arc from $\pi(x)$ to x and from x to the other two members of $G^-(a)$. Moreover π is orientation reversing, so that the cycle $\pi(a)$, $\pi(b)$, $\pi(c)$ is oriented the opposite way to a, b, c.

Thus for example if the an incomplete incidence table for G is as shown below then it can only be completed in one way, which is shown in the next table. (In this example we have $\pi(b) = f$, $\pi(c) = g$ and $\pi(d) = e$.)

It is not necessary to know every vertex in G to draw inferences about

Table 2: Completed incidence table for G

	a	b	c	d	e	f	g
a	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
b	$-$	$\bullet$	$-$	$+$	$+$	$-$	$+$
c	$-$	$+$	$\bullet$	$-$	$+$	$+$	$-$
d	$-$	$-$	$+$	$\bullet$	$-$	$+$	$+$
e	$+$	$-$	$-$	$+$	$\bullet$	$+$	$-$
f	$+$	$+$	$-$	$-$	$-$	$\bullet$	$+$
g	$+$	$-$	$+$	$-$	$+$	$-$	$\bullet$

the table. As few as four may suffice (for example $\{a, b, c, d\}$ above). It is also useful to be able to show show that a certain order 7 digraph is *not* QR_7. For example this would be the case above if there were an arc from f to c.

The following observation will also be useful

Lemma 16 *Let Γ be a Q_3 oriented graph of order 16.*

For any vertices $v, w \in V(\Gamma)$, if $|\Gamma^{\pm}(v) \cap \Gamma^{\mp}(w)| \geq 4$ and $w \in \Gamma^{\pm}(v)$, then $|\Gamma^{\pm}(v)| = 8$ whence $|\Gamma^{\mp}(v)| = 7$. If $|\Gamma^{\pm}(v) \cap \Gamma^{\mp}(w)| \geq 5$ then the subgraph induced by $\Gamma^{\pm}(v) \setminus \{w\}$, is isomorphic to QR_7.

Proof:- We must have $7 \leq |\Gamma^{\pm}(v)| \leq 8$. Since the subgraph induced by $\Gamma^{\pm}(v)$ must be Q_2, either $\Gamma^{\pm}(v) \setminus \{w\}$ is also Q_2 or else $\Gamma^{\pm}(v) \cap \Gamma^{+}(w)$ and $\Gamma^{\pm}(v) \cap \Gamma^{-}(w)$ are both Q_1, and so contain at least three vertices each. By assumption one of these sets actually contains at least four vertices. Both these cases entail $|\Gamma^{\pm}(v)| = 8$. If the stronger inequality holds then only the first alternative is possible. $\Box$

We abbreviate the statement " $|\Gamma^{+}(v) \cap \Gamma^{-}(v')| \geq 5$ and $|\Gamma^{-}(v) \cap \Gamma^{+}(v')| \geq$

5" to $(\star v)$. By lemma 16, $(\star v)$ implies that $|\Gamma^{\pm}(v) \setminus \{v'\}| = 7$ for both choices of sign, whence each of the corresponding induced subgraphs is QR_7. The incidence tables for these subgraphs can then yield information about Γ.

We now show that the general result reduces to the case where Γ is a tournament. Assume that the result holds in this case. Let Γ be any Q_3 oriented graph of order 16 and $\tilde{\Gamma}$ be a tournament obtained by adding arcs between vertices of Γ. By assumption $\tilde{\Gamma}$ is isomorphic to an extension of T_{16} so that for every vertex v of $\tilde{\Gamma}$, there is another vertex v' such that $|\tilde{\Gamma}^{\pm}(v) \cap \tilde{\Gamma}^{\mp}(v')| = 7$, whence $|\Gamma^{\pm}(v) \cap \Gamma^{\mp}(v')| \geq 5$. By the above lemma the subgraph of Γ induced by $\Gamma^{\pm}(v) \setminus \{v'\}$, is isomorphic to QR_7, whence v is joined to every other vertex of Γ except possibly v', so that Γ also contains an imbedded T_{16}.

We may thus assume that Γ is a tournament. We assume that $V(\Gamma) = V(T_{16})$. Let

$$A = \{v \in V(\Gamma) | \; |\Gamma^{+}(v)| = 7\}$$

$$B = \{v \in V(\Gamma) | \; |\Gamma^{-}(v)| = 7\}$$

By counting, $|A| = |B| = 8$ and there are 28 arcs directed from vertices in A to vertices in B. The vertices of A thus have a mean of 3.5 out neighbours in B, whence some vertex, which we may assume to be V, has at least four out-neighbours in B. We may assume that $\Gamma^{+}(V) = T_{16}^{+}(V) = \{0, 1, \ldots 6\}$ and that the subgraphs induced by this set is the same in both graphs. Since the vertices of $B \cap T_{16}^{+}(V)$ must include a transitive triangle, and any two transitive triangles in QR_7 can be mapped to each other by an automorphism of QR_7, we may assume that $\{0,3,5\} \subseteq B$. We have $\{V,3,5,6\} \subseteq \Gamma^-(0)$ and we may assume that the remaining three vertices are $\{4',2',1'\}$ so that $\Gamma^-(0) = T_{16}^-(0)$, and that this set induces the same subgraphs in Γ and T_{16} (Table 1).

Next $\Gamma^-(5)$ contains $\{V,1,3,4,2'\}$. Since $|\Gamma^-(5) \cap \Gamma^-(0)| = 3$, neither $1'$ nor $4'$ can be in $\Gamma^-(5)$ and we may assume by symmetry that the remaining members are $6'$ and $0'$ and further that $(6',0') \in \Gamma$. We may now partially fill the incidence matrix for $\Gamma^-(5)$ (Table 2) and incorporate this new information into the incidence table for Γ (Table 3).

Now $\Gamma^-(3)$ contains $\{V,1,2,6,4'\}$. Since $|\Gamma^-(3) \cap \Gamma^-(5)| = 3$, $\Gamma^-(3)$ must also contain exactly one of $\{0',6'\}$. By symmetry, we may assume that the remaining vertex in $\Gamma^-(3)$ is $5'$. There are thus two cases to consider:

- **Case 1:** $6' \in \Gamma^-(3)$
- **Case 2:** $0' \in \Gamma^-(3)$

There is now enough information in each case to fill the incidence tables for $\Gamma^-(5)$ (Tables 4 and 10) and then to partially fill incidence tables for $\Gamma^-(3)$ (using the tables for $\Gamma^-(5)$)

For each case there are three possible ways to fill the incidence table for $\Gamma^-(3)$, giving rise to the following three subcases for each

- **Case 1.1:** $(6,6') \in \Gamma$
- **Case 1.2:** $(6',6),(4',2) \in \Gamma$
- **Case 1.3:** $(6',6),(2,4') \in \Gamma$

- **Case 2.1:** $(1,4') \in \Gamma$
- **Case 2.2:** $(4',1),(0',2) \in \Gamma$
- **Case 2.3:** $(4',1),(2,0') \in \Gamma$.

See tables 7-9, 13-15. Note that in only one case (namely 2.1) does Γ as constructed so far coincide with T_{16} (with some additional edges between vertices not joined in T_{16}). We show first that every case but this one leads to a contradiction and then that the graph of case 2.1 must be an extension of T_{16}.

Case 1.1: If $|\Gamma^-(6')| = 7$ then it includes the set $\{0,1,4,6,2',5'\}$ and one other point, which must be from the set $\{V',3',1'\}$. Since $\{1,4,5'\} \subseteq \Gamma^+(0)$, this point must be in $\Gamma^-(0)$, whence it must be $1'$. But now

$$\Gamma^-(6') \cap \Gamma^-(1') = \{1,4,5'\},$$

which induces a transitive triangle. This is impossible if $|\Gamma^-(6')| = 7$. Therefore $|\Gamma^-(6')| = 8$ and $|\Gamma^+(6')| = 7$. By completing the incidence table for $\Gamma^+(6')$ we infer

$$(2,0'),(0',4') \in \Gamma \tag{5}$$

Using (5) above, we have that $\Gamma^-(4')$ contains $\{0',2,5,6,6',2'\}$. If $|\Gamma^-(4')| = 7$ then the incidence table for $\Gamma^-(4')$ forces $(0',6') \in \Gamma$, a contradiction. Therefore $|\Gamma^+(4')| = 7$ and the incidence table for this subgraph yields

$$(1',5'),(1,1') \in \Gamma \tag{6}$$

If $|\Gamma^-(1)| = 7$ then $(4',0') \in \Gamma$, contradicting (5) above. Therefore $|\Gamma^+(1)| = 7$, which gives $(5',1') \in \Gamma$, contradicting (6) above.

Case 1.2: $|\Gamma^-(4')| = 7$ is impossible, whence $|\Gamma^+(4')| = 7$ which gives

$$(2, 1'), (1', 5') \in \Gamma \tag{7}$$

$|\Gamma^+(6)| = 7$ is impossible, whence $|\Gamma^-(6)| = 7$ which gives

$$(1', 4), (6', 1') \in \Gamma \tag{8}$$

Now $(\star 1')$, whence $|\Gamma^+(1')| = 7$, but this is impossible so we have a contradiction.

Case 1.3:

$|\Gamma^-(6')| = 7$ is impossible, whence $|\Gamma^+(6')| = 7$ which gives

$$(0', 5'), (6, 0') \in \Gamma \tag{9}$$

$|\Gamma^-(2)| = 7$ is impossible, whence $|\Gamma^+(2)| = 7$ which gives

$$(4, 4') \in \Gamma \tag{10}$$

$|\Gamma^+(4)| = 7$ is impossible, whence $|\Gamma^-(4)| = 7$ which gives

$$(2, 2') \in \Gamma \tag{11}$$

$|\Gamma^+(1)| = 7$ and $|\Gamma^-(1)| = 7$ are both impossible, giving the required contradiction.

Case 2.2:

$\Gamma^-(6) \supseteq \{V, 2, 4, 5, 5', 1'\}$ and, if $|\Gamma^-(6)| = 7$ the remaining member must be in $\Gamma^-(5)$ whence it is $6'$. Now both $|\Gamma^-(6)| = 7$ and $|\Gamma^+(6)| = 7$ lead to contradictions.

Case 2.3: $|\Gamma^-(2)| = 7$ and $|\Gamma^-(0')| = 7$ are both impossible and the tables for $|\Gamma^+(2)|$ and $|\Gamma^+(0')|$ yield in turn

$$(4',4),(2',5') \in \Gamma \tag{12}$$

Now $|\Gamma^+(4')| = 7$ is also impossible and $\Gamma^-(4') \supseteq \{2,5,6,5',2'\}$. If $|\Gamma^-(4')| = 7$ then the remaining two members must be in $\Gamma^-(5)$ and so can only be 4 and $6'$. This case then also leads to a contradiction.

Case 2.1

Since $5 \in \Gamma^-(5')$ and $\{0,2,6,4'\} \subseteq \Gamma^-(5') \cap \Gamma^+(5)$, Lemma 16 gives $|\Gamma^-(5')| = 8$, whence $|\Gamma^+(5')| = 7$. $\Gamma^+(5') \cap \Gamma^+(V) \supseteq \{1,3\}$, and the only other vertex available to be the third is 4. Hence $(5',4) \in \Gamma$.

Now $(\star 4)$, whence the incidence tables for $\Gamma^-(4)$ and $\Gamma^+(4')$ yield

$$(5',6'),(6',2),(2,1'),(1',5') \in \Gamma$$

Now $(\star 6)$ and the tables for $\Gamma^\pm(6)$ give

$$(0',1'),(1',4),(5',2') \in \Gamma$$

Now $(\star 1)$. We finally break the symmetry between $3'$ and V' and assume

$$(1,V'),(3',1) \in \Gamma.$$

The tables for $\Gamma^\pm(1)$ now give

$$(4,3'),(3',6),(2',3'),(4',6'),(2,V'),(V',0'),(V',4'),(V',6'),(3',5') \in \Gamma$$

For the following values of v (taken in the order given) $(\star v)$ holds and we can fill the gaps, other than in positions (v, v') and (v', v), in row and column v of the incidence table of Γ using $|\Gamma^{\pm}(v)\backslash\{v'\}| = 7$: $v = 0', 4, 6, 5', V', 4', 2, 3', 1'$

The incidence table of Γ is now filled completely except for some of the positions (v, v') and Γ contains an imbedded T_{16}. $\square$

In the following tables circled entries denote new entries.

Table 3: Γ incorporating $\Gamma^{-}(V)$ and $\Gamma^{-}(0)$

	V	0	1	2	3	4	5	6	$6'$	$5'$	$4'$	$3'$	$2'$	$1'$	$0'$	V'
V	$\bullet$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
0	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
1	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$								
2	$-$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$								
3	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$-$			$-$		$+$	$+$		
4	$-$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$								
5	$-$	$+$	$-$	$+$	$-$	$-$	$\bullet$	$+$			$+$		$-$	$+$		
6	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$\bullet$			$+$		$+$	$-$		
$6'$	$+$	$-$							$\bullet$							
$5'$	$+$	$-$								$\bullet$						
$4'$	$+$	$+$			$+$		$-$	$-$			$\bullet$		$-$	$+$		
$3'$	$+$	$-$										$\bullet$				
$2'$	$+$	$+$			$-$		$+$	$-$			$+$		$\bullet$	$-$		
$1'$	$+$	$+$			$-$		$-$	$+$			$-$		$+$	$\bullet$		
$0'$	$+$	$-$													$\bullet$	
V'	$+$	$-$														$\bullet$

Table 4: $\Gamma^{-}(5)$

	V	1	3	4	$6'$	$2'$	$0'$
V	•	+	+	+	−	−	−
1	−	•	+	−			
3	−	−	•	+		+	
4	−	+	−	•			
$6'$	+				•	−	+
$2'$	+		−		+	•	−
$0'$	+				−	+	•

Table 5: Γ incorporating $\Gamma^{-}(5)$ and table 5

	V	0	1	2	3	4	5	6	$6'$	$5'$	$4'$	$3'$	$2'$	$1'$	$0'$	V'
V	•	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
0	−	•	+	+	−	+	−	−	+	+	−	+	−	−	+	+
1	−	−	•	+	+	−	+	−								
2	−	−	−	•	+	+	−	+								
3	−	+	−	−	•	+	+	−			−		+	+		
4	−	−	+	−	−	•	+	+								
5	−	+	−	+	−	−	•	+	−	+	+	+	−	+	−	+
6	−	+	+	−	+	−	−	•			+		+	−		
$6'$	+	−					+		•				−		+	
$5'$	+	−					−			•						
$4'$	+	+			+		−	−			•		−	+		
$3'$	+	−					−					•				
$2'$	+	+			−		+	−	+		+		•	−	−	
$1'$	+	+			−		−	+			−		+	•		
$0'$	+	−					+		−				+		•	
V'	+	−					−									•

Table 6: $\Gamma^{-}(3)$, Case 1

	V	1	2	6	$6'$	$4'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$+$		
2	$-$	$-$	$\bullet$	$+$			
6	$-$	$+$	$-$	$\bullet$		$+$	
$6'$	$+$	$-$			$\bullet$		
$4'$	$+$			$-$		$\bullet$	
$5'$	$+$						$\bullet$

Table 7: $\Gamma^{-}(5)$, Case 1

	V	1	3	4	$6'$	$2'$	$0'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$+$	$+$	$-$
3	$-$	$-$	$\bullet$	$+$	$-$	$+$	$+$
4	$-$	$+$	$-$	$\bullet$	$+$	$-$	$+$
$6'$	$+$	$-$	$+$	$-$	$\bullet$	$-$	$+$
$2'$	$+$	$-$	$-$	$+$	$+$	$\bullet$	$-$
$0'$	$+$	$+$	$-$	$-$	$-$	$+$	$\bullet$

Table 8: Γ, Case 1, incorporating $\Gamma^{-}(3)$ and table 5

	V	0	1	2	3	4	5	6	$6'$	$5'$	$4'$	$3'$	$2'$	$1'$	$0'$	V'
V	$\bullet$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
0	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
1	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$+$				$+$		$-$	
2	$-$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$								
3	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
4	$-$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$+$				$-$		$+$	
5	$-$	$+$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$-$	$+$
6	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$\bullet$			$+$		$+$	$-$		
$6'$	$+$	$-$	$-$		$+$	$-$	$+$		$\bullet$				$-$		$+$	
$5'$	$+$	$-$			$+$		$-$			$\bullet$						
$4'$	$+$	$+$			$+$		$-$	$-$			$\bullet$		$-$	$+$		
$3'$	$+$	$-$			$-$		$-$					$\bullet$				
$2'$	$+$	$+$	$-$		$-$	$+$	$+$	$-$	$+$		$+$		$\bullet$	$-$	$-$	
$1'$	$+$	$+$			$-$		$-$	$+$			$-$		$+$	$\bullet$		
$0'$	$+$	$-$	$+$		$-$	$-$	$+$		$-$				$+$		$\bullet$	
V'	$+$	$-$			$-$		$-$									$\bullet$

Table 9: $\Gamma^{-}(3)$, Case 1.1, $(6, 6') \in \Gamma$

	V	1	2	6	$6'$	$4'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$+$	$\ominus$	$\oplus$
2	$-$	$-$	$\bullet$	$+$	$\ominus$	$\oplus$	$\oplus$
6	$-$	$+$	$-$	$\bullet$	$\oplus$	$+$	$\ominus$
$6'$	$+$	$-$	$\oplus$	$\ominus$	$\bullet$	$\oplus$	$\ominus$
$4'$	$+$	$\oplus$	$\ominus$	$-$	$\ominus$	$\bullet$	$\oplus$
$5'$	$+$	$\ominus$	$\ominus$	$\oplus$	$\oplus$	$\ominus$	$\bullet$

Table 10: $\Gamma^{-}(3)$, Case 1.2 $(6', 6), (4', 2) \in \Gamma$

	V	1	2	6	$6'$	$4'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$+$	$\oplus$	$\ominus$
2	$-$	$-$	$\bullet$	$+$	$\oplus$	$\ominus$	$\oplus$
6	$-$	$+$	$-$	$\bullet$	$\ominus$	$+$	$\oplus$
$6'$	$+$	$-$	$\ominus$	$\oplus$	$\bullet$	$\oplus$	$\ominus$
$4'$	$+$	$\ominus$	$\oplus$	$-$	$\ominus$	$\bullet$	$\oplus$
$5'$	$+$	$\oplus$	$\ominus$	$\ominus$	$\oplus$	$\ominus$	$\bullet$

Table 11: $\Gamma^{-}(3)$, Case 1.3, $(6', 6), (2, 4') \in \Gamma$

	V	1	2	6	$6'$	$4'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$+$	$\ominus$	$\oplus$
2	$-$	$-$	$\bullet$	$+$	$\oplus$	$\oplus$	$\ominus$
6	$-$	$+$	$-$	$\bullet$	$\ominus$	$+$	$\oplus$
$6'$	$+$	$-$	$\ominus$	$\oplus$	$\bullet$	$\ominus$	$\oplus$
$4'$	$+$	$\oplus$	$\ominus$	$-$	$\oplus$	$\bullet$	$\ominus$
$5'$	$+$	$\ominus$	$\oplus$	$\ominus$	$\ominus$	$\oplus$	$\bullet$

Table 12: $\Gamma^{-}(3)$, Case 2

	V	1	2	6	$4'$	$0'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$		$+$	
2	$-$	$-$	$\bullet$	$+$			
6	$-$	$+$	$-$	$\bullet$	$+$		
$4'$	$+$			$-$	$\bullet$		
$0'$	$+$	$-$				$\bullet$	
$5'$	$+$						$\bullet$

Table 13: $\Gamma^-(5)$ Case 2

	V	1	3	4	$6'$	$2'$	$0'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$+$	$-$	$+$
3	$-$	$-$	$\bullet$	$+$	$+$	$+$	$-$
4	$-$	$+$	$-$	$\bullet$	$-$	$+$	$+$
$6'$	$+$	$-$	$-$	$+$	$\bullet$	$-$	$+$
$2'$	$+$	$+$	$-$	$-$	$+$	$\bullet$	$-$
$0'$	$+$	$-$	$+$	$-$	$-$	$+$	$\bullet$

Table 14: Γ Case 2 incorporating $\Gamma^-(5)$ and table 5

	V	0	1	2	3	4	5	6	$6'$	$5'$	$4'$	$3'$	$2'$	$1'$	$0'$	V'
V	$\bullet$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
0	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
1	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$\oplus$				$\ominus$		$\oplus$	
2	$-$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$								
3	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$\oplus$	$\ominus$	$-$	$\oplus$	$+$	$+$	$\ominus$	$\oplus$
4	$-$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$\ominus$				$\oplus$		$\oplus$	
5	$-$	$+$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$-$	$+$
6	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$\bullet$			$+$		$+$	$-$		
$6'$	$+$	$-$	$\ominus$		$\ominus$	$\oplus$	$+$		$\bullet$				$-$		$+$	
$5'$	$+$	$-$			$\oplus$		$-$			$\bullet$						
$4'$	$+$	$+$			$+$		$-$	$-$			$\bullet$		$-$	$+$		
$3'$	$+$	$-$			$\ominus$		$-$					$\bullet$				
$2'$	$+$	$+$	$\oplus$		$-$	$\ominus$	$+$	$-$	$+$		$+$		$\bullet$	$-$	$-$	
$1'$	$+$	$+$			$-$		$-$	$+$			$-$		$+$	$\bullet$		
$0'$	$+$	$-$	$\ominus$		$\oplus$	$\ominus$	$+$		$-$				$+$		$\bullet$	
V'	$+$	$-$			$\ominus$		$-$									$\bullet$

Table 15: $\Gamma^-(3)$, Case 2.1, $(1,4') \in \Gamma$

	V	1	2	6	$4'$	$0'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$\oplus$	$+$	$\ominus$
2	$-$	$-$	$\bullet$	$+$	$\ominus$	$\oplus$	$\oplus$
6	$-$	$+$	$-$	$\bullet$	$+$	$\ominus$	$\oplus$
$4'$	$+$	$\ominus$	$\oplus$	$-$	$\bullet$	$\ominus$	$\oplus$
$0'$	$+$	$-$	$\ominus$	$\oplus$	$\oplus$	$\bullet$	$\ominus$
$5'$	$+$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\oplus$	$\bullet$

Table 16: $\Gamma^-(3)$, Case 2.2, $(4',1),(0',2) \in \Gamma$

	V	1	2	6	$4'$	$0'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$\ominus$	$+$	$\oplus$
2	$-$	$-$	$\bullet$	$+$	$\oplus$	$\ominus$	$\oplus$
6	$-$	$+$	$-$	$\bullet$	$+$	$\oplus$	$\ominus$
$4'$	$+$	$\oplus$	$\ominus$	$-$	$\bullet$	$\ominus$	$\oplus$
$0'$	$+$	$-$	$\oplus$	$\ominus$	$\oplus$	$\bullet$	$\ominus$
$5'$	$+$	$\ominus$	$\ominus$	$\oplus$	$\ominus$	$\oplus$	$\bullet$

Table 17: $\Gamma^-(3)$, Case 2.3, $(4',1),(2,0') \in \Gamma$

	V	1	2	6	$4'$	$0'$	$5'$
V	$\bullet$	$+$	$+$	$+$	$-$	$-$	$-$
1	$-$	$\bullet$	$+$	$-$	$\ominus$	$+$	$\oplus$
2	$-$	$-$	$\bullet$	$+$	$\oplus$	$\oplus$	$\ominus$
6	$-$	$+$	$-$	$\bullet$	$+$	$\ominus$	$\oplus$
$4'$	$+$	$\oplus$	$\ominus$	$-$	$\bullet$	$\oplus$	$\ominus$
$0'$	$+$	$-$	$\ominus$	$\oplus$	$\ominus$	$\bullet$	$\oplus$
$5'$	$+$	$\ominus$	$\oplus$	$\ominus$	$\oplus$	$\ominus$	$\bullet$

Table 18: Γ Case 2.1 (incorporating $\Gamma^{-}(3)$)

	V	0	1	2	3	4	5	6	$6'$	$5'$	$4'$	$3'$	$2'$	$1'$	$0'$	V'
V	$\bullet$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
0	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
1	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$+$	$\ominus$	$\oplus$		$-$		$+$	
2	$-$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$		$\oplus$	$\ominus$				$\oplus$	
3	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$+$	$-$	$+$
4	$-$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$+$	$-$				$+$		$+$	
5	$-$	$+$	$-$	$+$	$-$	$-$	$\bullet$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$-$	$+$
6	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$\bullet$		$\oplus$	$+$		$+$	$-$	$\ominus$	
$6'$	$+$	$-$	$-$		$-$	$+$	$+$		$\bullet$				$-$		$+$	
$5'$	$+$	$-$	$\oplus$	$\ominus$	$+$		$-$	$\ominus$		$\bullet$	$\ominus$				$\oplus$	
$4'$	$+$	$+$	$\ominus$	$\oplus$	$+$		$-$	$-$		$\oplus$	$\bullet$		$-$	$+$	$\ominus$	
$3'$	$+$	$-$			$-$		$-$					$\bullet$				
$2'$	$+$	$+$	$+$		$-$	$-$	$+$	$-$	$+$		$+$		$\bullet$	$-$	$-$	
$1'$	$+$	$+$			$-$		$-$	$+$			$-$		$+$	$\bullet$		
$0'$	$+$	$-$	$-$	$\ominus$	$+$	$-$	$+$	$\oplus$	$-$	$\ominus$	$\oplus$		$+$		$\bullet$	
V'	$+$	$-$			$-$		$-$									$\bullet$

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